Wednesday 5 November 2014

Time allowed: 2 hours

For candidates applying for Physics, Physics and Philosophy, Engineering or Materials

---

There are two parts (A and B) to this test, carrying equal weight.

Part A: Mathematics for Physics Q1-Q9 [50 Marks]

Part B: Physics Q10-Q19 [50 Marks]

Answers should be written on the question sheet in the spaces provided and you should attempt as many questions as you can from each part.

Marks for each question are indicated in the right hand margin. There are a total of 100 marks available and total marks for each section are indicated at the start of a section. You are advised to divide your time according to the marks available, and to spend equal effort on parts A and B.

No calculators, tables or formula sheets may be used.

Answers in Part A should be given exactly unless indicated otherwise. Numeric answers in Part B should be calculated to 2 significant figures.

Use $g = 10\,\mathrm{m\,s^{-2}}$.

Do NOT turn over until told that you may do so.

Part A: Mathematics for Physics [50 Marks]

1. A jar contains buttons of four different colours. There are twice as many yellow as green, twice as many red as yellow and twice as many blue as red. What is the probability of taking from the jar:

    a) a blue button

    b) a red button

    c) a yellow button

    d) a green button

    (You may assume that you are only taking one button at a time and replacing it in the jar before selecting the next colour.) [4]

Y	G	R	B	Tot
2	1	4	8	15

a) $\frac{8}{15}$ b) $\frac{4}{15}$ c) $\frac{2}{15}$ d) $\frac{1}{15}$

2. What is the sum of the following terms:

$$1 + e^{-x} + e^{-2x} + \ldots$$

Over what range of x is the solution valid? [4]

G.P. with $a = 1$, $r = e^{-x}$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-e^{-x}}$$

Valid for $|r| < 1$

$$|e^{-x}| < 1$$

$$e^{-2x} < 1$$

$$-2x < \ln 1$$

$$x > 0$$

3. Evaluate the integrals:

a) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\mathrm{d}x$

b) $\displaystyle\int_0^{2} \frac{x}{x^2+6x+8}\mathrm{d}x.$

[6]

a) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin x}dx = \left[\ln|1+\sin x|\right]_0^{\pi/2}$

$= \ln 2 - \ln 1$

$= \underline{\ln 2}$

b) $\displaystyle\int_0^{2} \frac{x}{x^2+6x+8}dx = \int_0^2\left(\frac{-4/-2}{x+4} + \frac{-2/2}{x+2}\right)dx$

$\displaystyle = 2\int_0^2 \frac{1}{x+4}dx - \int_0^2 \frac{1}{x+2}dx$

$= \left[2\ln|x+4| - \ln|x+2|\right]_0^2$

$\displaystyle = \left[\ln\left|\frac{(x+4)^2}{x+2}\right|\right]_0^2 = \ln 9 - \ln 8$

$\displaystyle = \underline{\ln\left(\frac{9}{8}\right)}$

4. What is the coefficient of x^7 in the expansion of $(1+2x)^4(1-2x)^6$? [4]

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}
$$

$(1)\ 1^4(2x)^0$

$(4)\ 1^3(2x)^1$ *

$(6)\ 1^2(2x)^2$ □

$(4)\ 1^1(2x)^3$ △

$(1)\ 1^0(2x)^4$ ○

$(1)\ 1^6(-2x)^0$

$(6)\ 1^5(-2x)^1$

$(15)\ 1^4(-2x)^2$

$(20)\ 1^3(-2x)^3$ ○

$(15)\ 1^2(-2x)^4$ △

$(6)\ 1^1(-2x)^5$ □

$(1)\ 1^0(-2x)^6$ *

$$\text{Coeff} = 4(2)(-2)^6 + 36(2)^2(-2)^5 + 60(2)^3(-2)^4 + 20(2)^4(-2)^3$$

$$= 4(128) - 36(128) + 60(128) - 20(128)$$

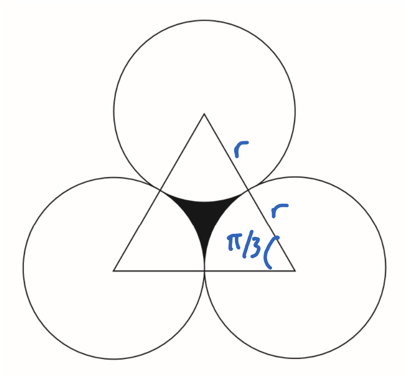
$$= 128(4 - 36 + 60 - 20)$$

$$= 128 \times 8$$

$$= \underline{1024}$$

5. Consider the shape shown below. What is the area of the equilateral triangle (with length of side $= 2r$) which is not enclosed within the circles (each with radius $= r$), and which is shown shaded black in the figure?

[5]



$$A = A_{triangle} - 3A_{sector}$$

$$A_{sector} = \frac{\pi}{3} \times \frac{1}{2} \times r^2 = \frac{\pi r^2}{6}$$

$$A_{triangle} = \frac{1}{2}(2r)^2 \times \sin\frac{\pi}{3}$$
$$= \frac{1}{2} \cdot 4r^2 \cdot \frac{\sqrt{3}}{2}$$
$$= \sqrt{3}\, r^2$$

$$\Rightarrow A = \sqrt{3}\, r^2 - 3 \times \frac{\pi r^2}{6}$$
$$= \left(\sqrt{3} - \frac{\pi}{2}\right) r^2$$

6. You want to make a snowman out of modelling clay. The snowman consists of 2 spheres, where one sphere has a radius r, the other has a radius $2r$. The modelling clay comes in the form of a cylinder with radius $r/2$. What length of modelling clay is required to make the snowman? [5]

$$V_{snow} = \frac{4}{3}\pi r^3 + \frac{4}{3}\pi (2r)^3$$

$$= 12\pi r^3$$

$$V_{clay} = \pi \left(\frac{r}{2}\right)^2 \cdot \ell = 12\pi r^3$$

$$\ell = \frac{12\pi r^3}{\pi r^2 / 4}$$

$$= \underline{48r}$$

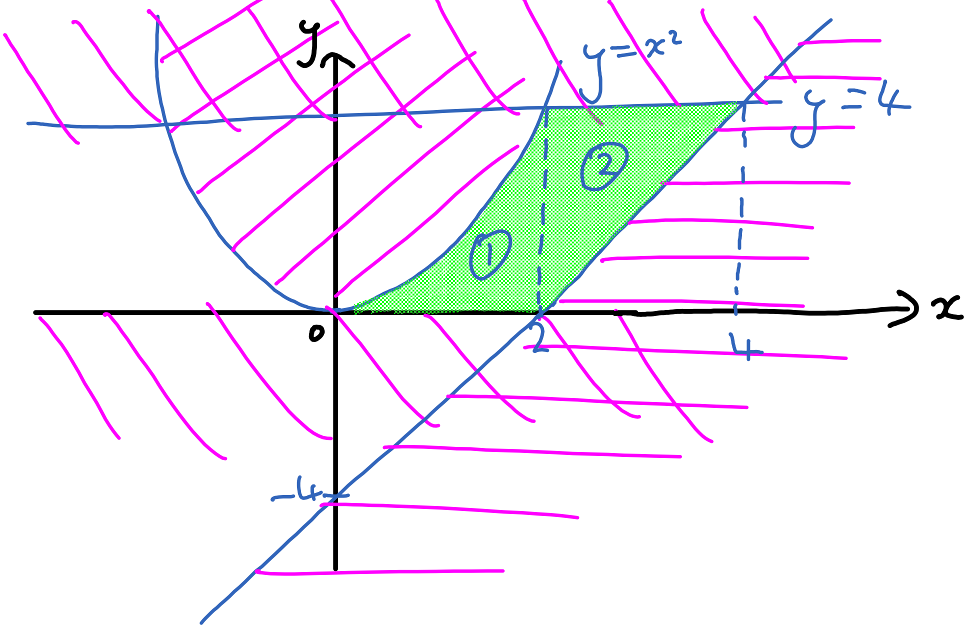
7. Sketch the region defined by:

$y \leq x^2$ and $4 \geq y \geq 0$ and $y \geq 2x - 4$.

Evaluate the area defined by the above inequalities. [7]

$4 = 2x - 4$

$x = 4$



$$1: \int_0^2 x^2 \, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

$$2: \frac{4 \times 2}{2} = 4$$

$$\Rightarrow A = 4 + \frac{8}{3} = \frac{20}{3}$$

8. If $f(x) = e^x$ for $x < 0$ and $f(x) = e^{-x} + 2x$ for $x \geq 0$, sketch the function $f(x)$ and its first, second and third derivatives. [8]

	$x < 0$	$x \geqslant 0$
$f(x)$	e^x	$e^x + 2x$
$f'(x)$	e^x	$-e^{-x} + 2$
$f''(x)$	e^x	e^{-x}
$f'''(x)$	e^x	$-e^{-x}$

$f'(x) = 0$

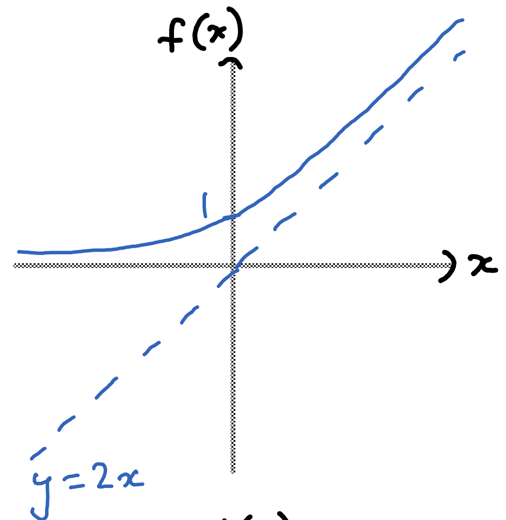
$-e^{-x} + 2 = 0$

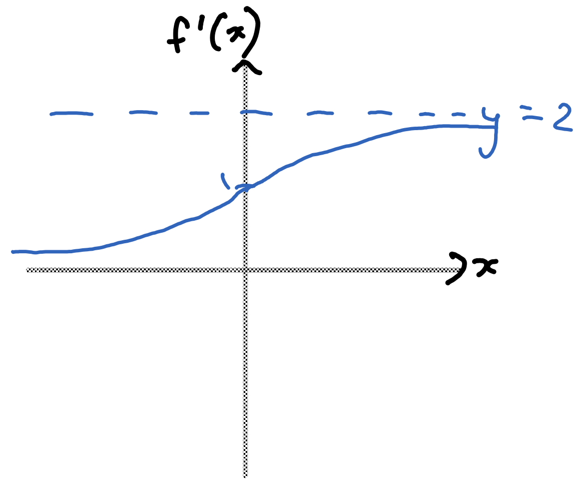
$e^{-x} = 2$

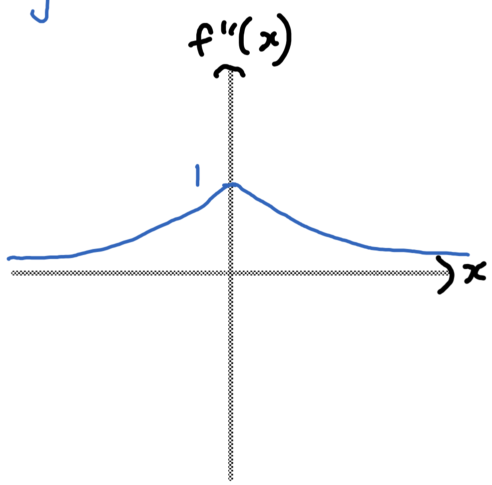
$-x = \ln 2$

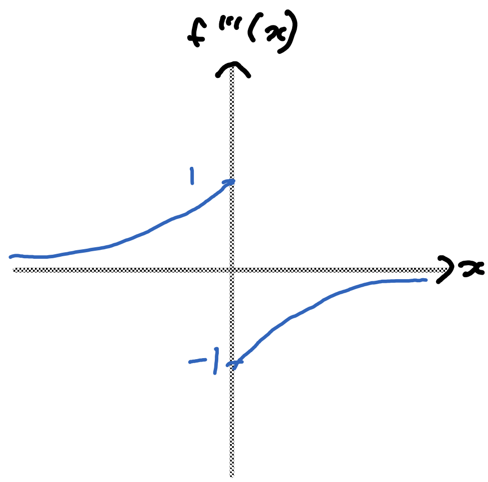
$x = \ln \frac{1}{2}$

$\Rightarrow$ min pt. -ve

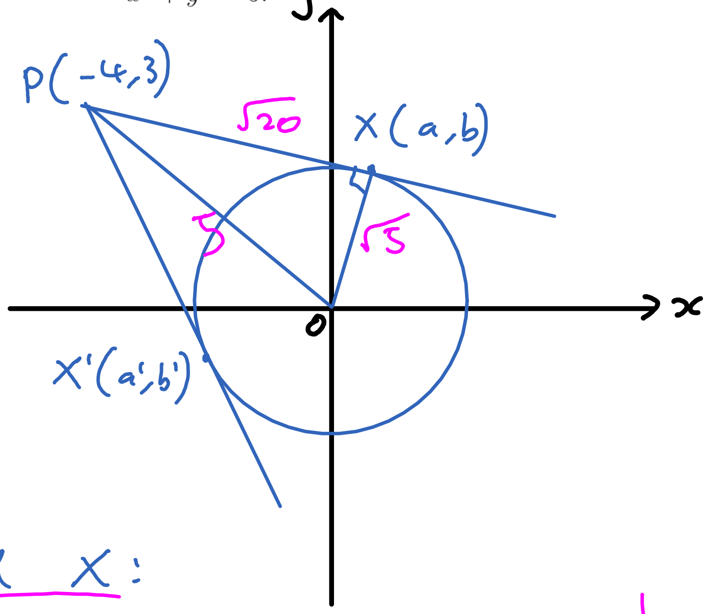








9. Find the equations of the tangents drawn from the point (-4, 3) to the circle $x^2 + y^2 = 5$. [7]



$OP^2 = 3^2 + (-4)^2 = 25$

$PX^2 = OP^2 - OX^2$

$= 25 - 5$

$= 20$

Find X:

PX: $(a+4)^2 + (b-3)^2 = 20$

$a^2 + 8a + 16 + b^2 - 6b + 9 = 20$

$a^2 + b^2 + 8a - 6b = -5 \quad (1)$

OX: $a^2 + b^2 = 5 \quad (2)$

(1) - (2): $8a - 6b = -10$

$b = \dfrac{4a+5}{3} \quad (3)$

(1): $a^2 + \left(\dfrac{4a+5}{3}\right)^2 + 8a - 6\left(\dfrac{4a+5}{3}\right) = -5$

$9a^2 + 16a^2 + 40a + 25 + 72a - 72a - 90 = -5$

$25a^2 + 40a - 20 = 0$

$5a^2 + 8a - 4 = 0$

$(5a - 2)(a + 2) = 0$

$a = \frac{2}{5}$ or $a' = -2$

(3): $b = \dfrac{4(2/5) + 5}{3} = \dfrac{11}{5}$

$b' = \dfrac{4(-2) + 5}{3} = -1$

Find eqn's:

PX: $m = \dfrac{11/5 - 3}{2/5 + 4} = \dfrac{11 - 15}{2 + 20}$

$= -\frac{2}{11}$

$y - 3 = -\frac{2}{11}(x + 4)$

$\underline{2x + 11y - 25 = 0}$

PX': $m = \dfrac{-1 - 3}{-2 + 4} = -2$

$y - 3 = -2(x + 4)$

$\underline{2x + y + 5 = 0}$

Part B: Physics [50 Marks]

Multiple choice (6 marks)

Please circle **one** answer to each question only.

10. Excluding Pluto, for the planets in our solar system, in order of increasing mean distance from the Sun, which of the following statements is/are correct?

X i) the duration of the day on each planet increases Own rotation

✓ ii) the duration of the year on each planet increases Kepler's 3rd

X iii) the size/volume of the planets increases Jupiter

X iv) the number of moons of each planet increases ~ mass

✓ v) the planets change from rocky to gas giants

A statements i) and ii) and v)
B statement ii) only
C statements iii) and iv)
D statements ii) and v) [2]

11. In which part of the Electromagnetic Spectrum do waves have a frequency of approx. 100 GHz?

A X rays
B visible light
C microwave
D radio wave

$100\,GHz = 10^{11}\,Hz$

$\lambda = \frac{c}{f} = \frac{10^8}{10^{11}} = 1\,mm$

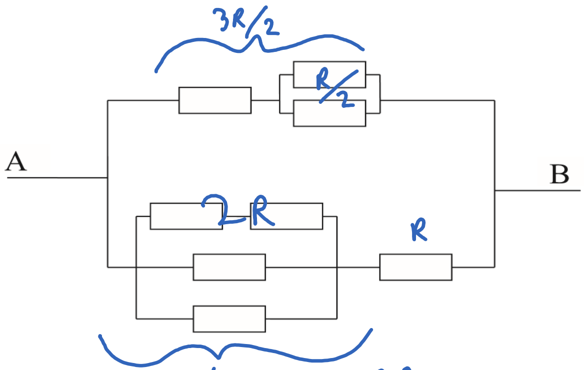
[2]

12. An object with small mass becomes detached from the International Space Station (ISS) while it orbits the Earth. Its relative velocity with respect to the ISS can be neglected. Would the object

A follow ISS in its orbit
B go straight along a direction tangential to ISS orbit at the point when it became detached
C fall straight down towards the Earth
D stay still with respect to the Earth [2]

Written answers (24 marks)

13. Given the circuit below, where all resistors have the same value ($= R$), what is the resistance between A and B? [5]



$$\frac{1}{1/R + 1/R + 1/2R} = \frac{2R}{5}$$

$$\frac{7R}{5}$$

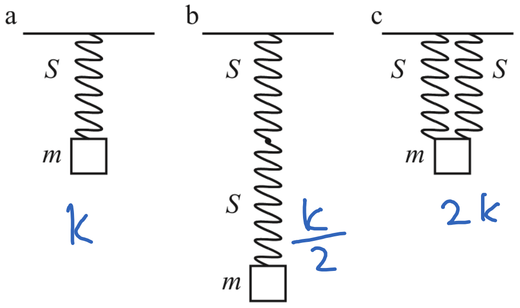
$$R_T = \frac{1}{\frac{2}{3R} + \frac{5}{7R}}$$

$$= \frac{1}{\frac{14 + 15}{21R}}$$

$$= \frac{21R}{29}$$

14. A mass m is attached to a spring S (as sketched in figure **a** below) and oscillates with a period T. What would be the period of the oscillation if two springs S are connected in series (figure **b**) or in parallel (figure **c**)?

What would be the period of the oscillations in case (a) on a planet with surface gravity $2g$? [4]



$T = 2\pi\sqrt{\frac{m}{k}}$

series: $T' = 2\pi\sqrt{\frac{m}{k/2}} = \sqrt{2}\,T$

parallel: $T' = 2\pi\sqrt{\frac{m}{2k}} = \frac{T}{\sqrt{2}}$

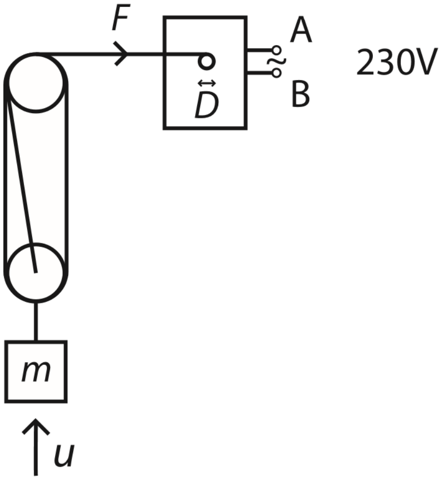
other planet: same because period doesn't depend on g

15. An electric motor is lifting a mass via a system of pulleys as sketched below. The motor is powered by a voltage source of 230 V. The diameter of the motor winding reel is $D = 5$ cm and a mass $m = 100$ kg is being lifted with a speed $u = 0.5$ m/s. The masses of the pulleys and the string can be neglected.

a) What is the electric current driving the motor?

b) What is the angular velocity of the motor's winding reel?

c) What is the force F with which the motor is pulling? [5]



a) $P = IV$

$P = \frac{mgh}{t} = \frac{100 \times 10 \times 0.5}{1}$

$= 500W$

$\Rightarrow I = \frac{500}{230} = \frac{50}{23} A$

b) $v = \omega r$

$v = 3u = 1.5 ms^{-1}$ $\quad r = D/2 = 0.025 m$

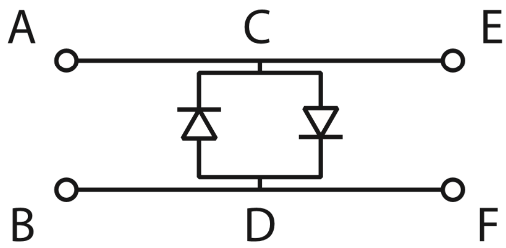
$\omega = \frac{1.5}{0.025} = \frac{15}{0.25} = 60 rads^{-1}$

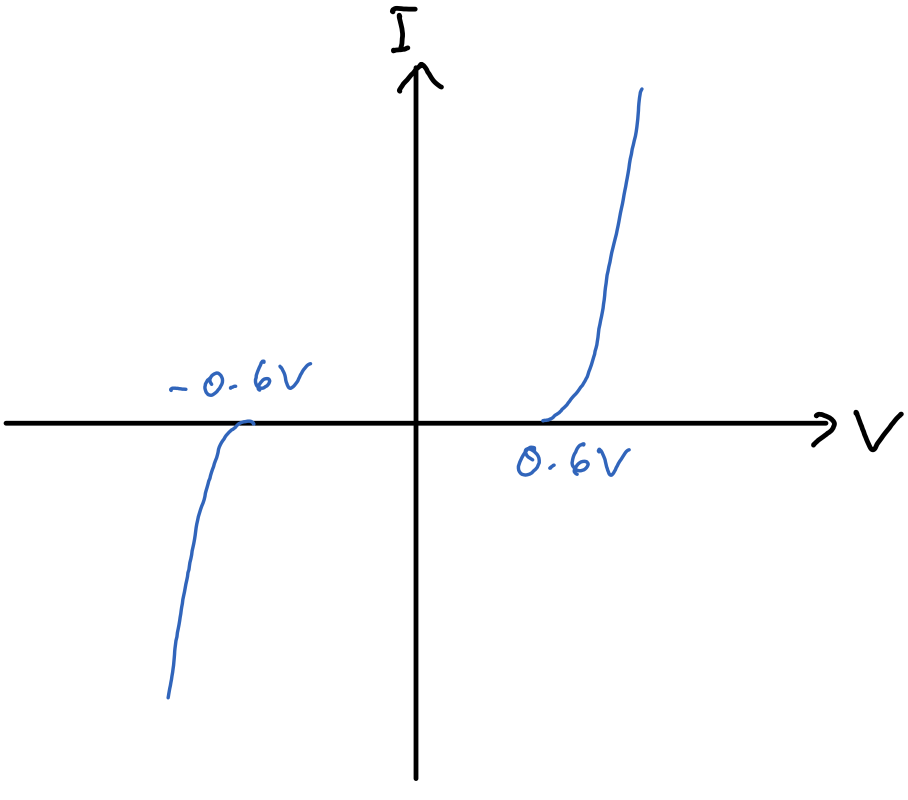
c) $P = Fv$

$F = \frac{P}{v} = \frac{500}{1.5} = 333 N$

16. Two diodes are connected as sketched below. Sketch the current flowing between points C and D as a function of voltage applied between points C and D.

A sensitive amplifier is connected to terminals E and F to measure small electric signals from an instrument connected to terminals A and B. From time to time there are discharges in the instrument which might destroy the amplifier if the amplifier is connected to the instrument directly, without the diodes. Explain briefly how the diodes protect the amplifier. [4]





For small signals (low voltage), current doesn't flow through C-D. Amplifier operates.

When there's a discharge (high voltage), current flows through C-D, not through the amplifier

17. For this question you may assume that the electrostatic potential energy of two positively charged particles (with charge $+Q_1$ and $+Q_2$) separated by a distance x is given by

$$k\frac{Q_1Q_2}{x}$$

where k is a constant.

Two charged particles are placed a distance d apart from each other. One has charge $= +Q$ and mass $= m$, whilst the other has charge $= +2Q$ and mass $= 2m$. The charges are initially held stationary, but are then released. Find an expression for the maximum speed of the particle with mass $= 2m$.

[6]

Particle 1: $2m$, $2Q$, v

Particle 2: m, Q, v'

CoM: $0 = 2mv - mv'$

$v' = 2v$

CoE: $\frac{1}{2}m(v')^2 + \frac{1}{2}(2m)v^2 = \frac{2kQ^2}{d}$

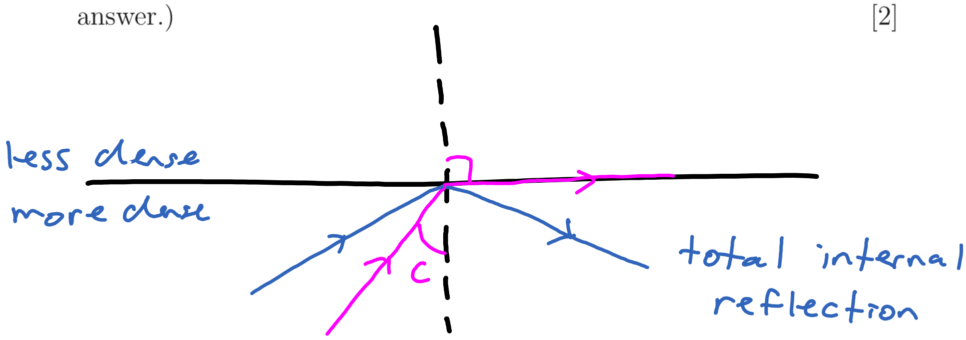
$\frac{1}{2}m4v^2 + mv^2 = \frac{2kQ^2}{d}$

$$\underline{v = \sqrt{\frac{2kQ^2}{3md}}}$$

Long questions (20 marks)

18. This question concerns total internal reflection, optical fibres and refraction. You may assume that the refractive index of glass is larger than that of water, and that the refractive index of water is larger than that of air.

a) Explain what is meant by the phrases "total internal reflection" and "critical angle". (You are encouraged to use a diagram to explain your answer.) [2]



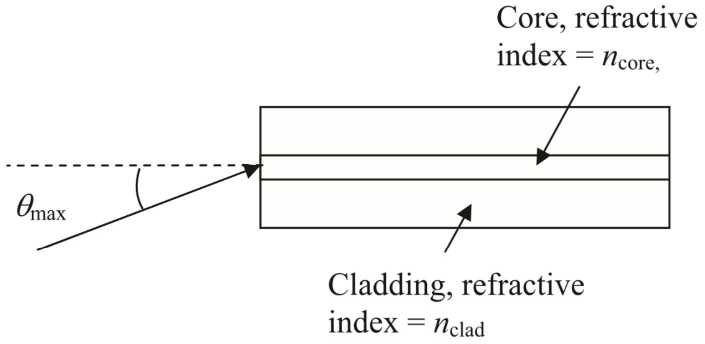
b) Derive an equation relating the critical angle and the refractive indices of two materials, n_1 and n_2 where $n_2 < n_1$. [2]

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

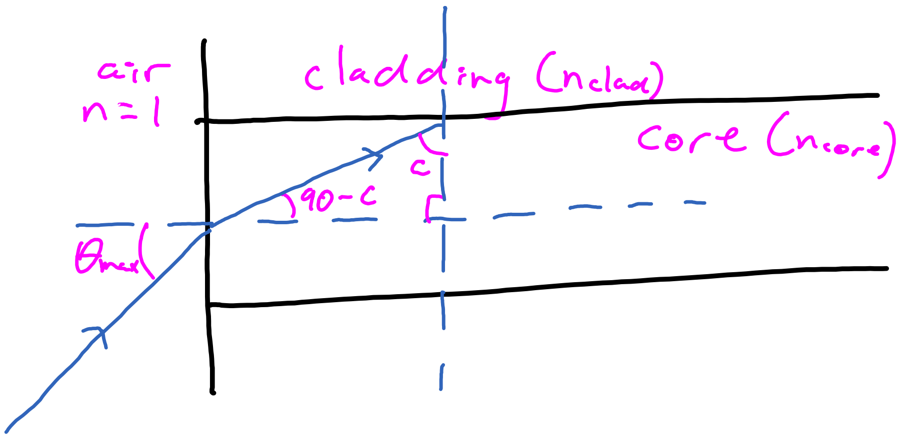
$$n_1 \sin c = n_2 \sin 90$$

$$\sin c = \frac{n_2}{n_1}$$

c) An optical fibre is usually made of two materials - a core and a cladding as shown in the diagram below (not drawn to scale).



Light may only be transmitted along the fibre if the incident angle of the light is less than a maximum angle θ_{max}. By using your expression from part b) and Snell's law, or otherwise, derive an expression for θ_{max} in terms of the core and cladding refractive indices only. [3]



part b: $\sin c = \frac{n_2}{n_1} = \frac{n_{core}}{n_{clad}}$

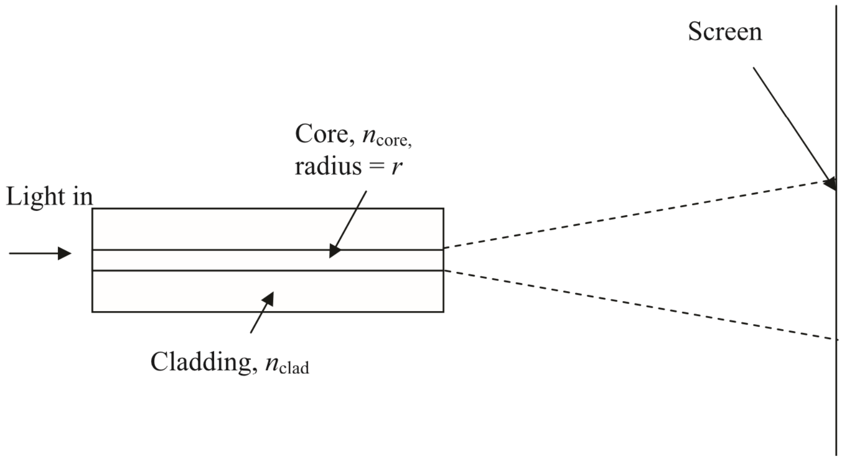
Snell's law: $\sin\theta_{max} = n_{core}\sin(90-c)$

$= n_{core}\cos c$

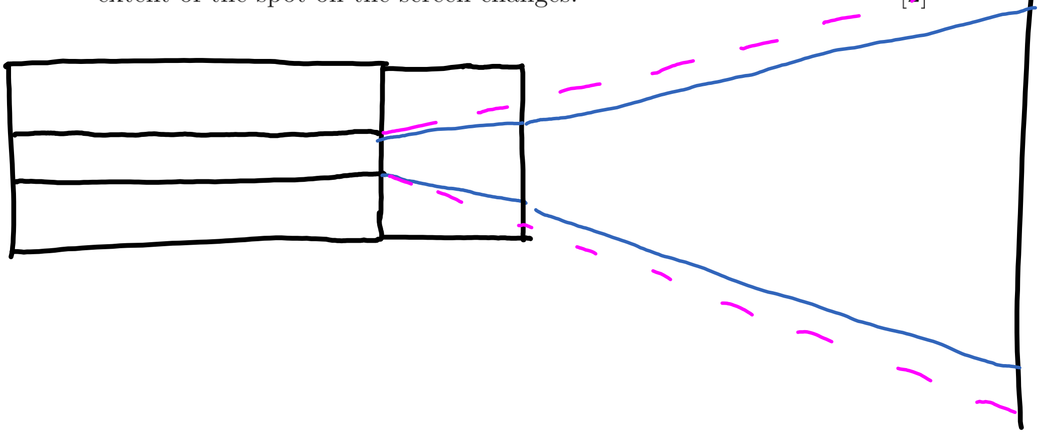
$= n_{core}\sqrt{1-\sin^2 c}$

$= n_{core}\sqrt{1-\left(\frac{n_{core}}{n_{clad}}\right)^2}$

$\theta_{max} = \arcsin\left[n_{core}\sqrt{1-\left(\frac{n_{core}}{n_{clad}}\right)^2}\right]$



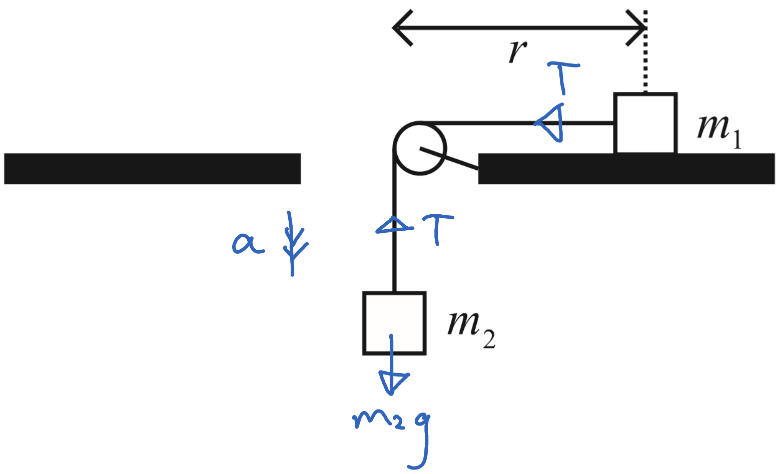
d) In an experiment, light is transmitted along the glass fibre before leaving at a perfectly vertical end. A screen is placed a long distance away from the end of the fibre and a uniform circular spot is seen as shown above. A small glass tank containing water is placed in front of the beam so it perfectly touches the end of the fibre. You may assume the tank is parallel to the screen. On a copy of the above diagram, sketch how the extent of the spot on the screen changes. [2]



e) The tank is kept in place and now white light is used instead of a laser. What will the image now look like? [1]

White central spot with rainbow around

19. Two masses m_1 and m_2 are connected by a massless, non-extensible string supported by a massless pulley attached to a table with a hole in the middle; see sketch below.



a) Assuming no friction, derive an expression for the acceleration of the masses and for the tension of the string.

[2]

1: $[F = ma] \leftarrow$

$T = m_1 a$

2: $[F = ma] \downarrow$

$m_2 g - T = m_2 a$

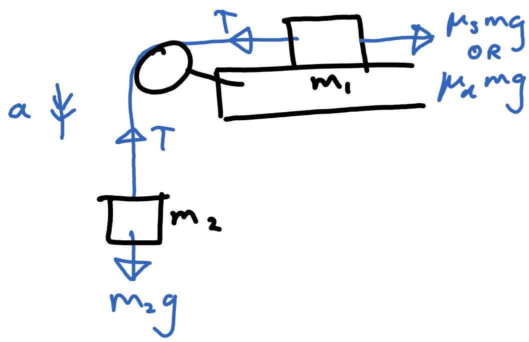
(1) in (2): $m_2 g - m_1 a = m_2 a$

$$a = \frac{m_2 g}{m_1 + m_2}$$

In (1): $T = \dfrac{m_1 m_2 g}{m_1 + m_2}$

Now and for the rest of this question, consider friction acting on the table but not on the pulley. Friction force F_{fr} is proportional to the mass's weight; $F_{fr} = \mu_s mg$ or $F_{fr} = \mu_d mg$ depending whether the mass is at rest (μ_s = static friction coefficient) or in motion (μ_d = dynamic friction coefficient). Both coefficients are known.

b) Derive expressions for the acceleration of the masses and for the tension of the string. What condition needs to be satisfied for m_1 to accelerate? [3]



$1: [F = ma] \leftarrow$

$T - \mu_d mg = m_1 a$

$2: [F = ma] \downarrow$

$m_2 g - T = m_2 a$

$(1)+(2): \quad m_2 g - \mu_d m_1 g = (m_2 + m_1) a$

$$a = \frac{g(m_2 - m_1 \mu_d)}{m_2 + m_1}$$

$$\text{In } (2): \quad T = m_2 g - m_2 g \frac{(m_2 - m_1 \mu_d)}{m_2 + m_1}$$

$$= m_2 g \left(1 + \frac{m_1 \mu_d - m_2}{m_1 + m_2}\right)$$

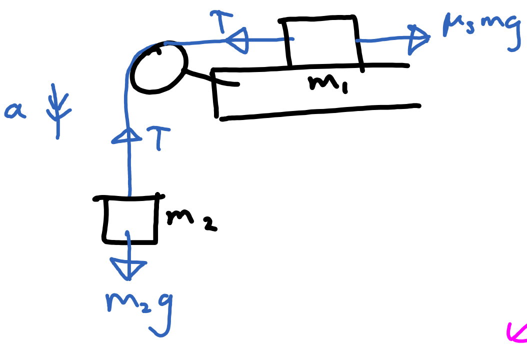
For acceleration, $T > \mu_s m_1 g$ and $T < m_2 g$

$\Rightarrow \mu_s m_1 g < T < m_2 g$

$\therefore \mu_s m_1 < m_2$

c) The table on which the mass m_1 is resting is now rotating about the vertical axis going through the middle of the table with angular speed ω. Assuming that the object with mass m_1 can be treated as point like, derive expressions for the minimal, $r_{\min}$, and maximal, $r_{\max}$, distance between m_1 and the axis of rotation such that for $r_{\min} < r < r_{\max}$, m_1 will not be moving radially.

[5]



$\uparrow \; T = m_2 g$

$\leftarrow \; [F = ma]$

$T \pm \mu_s m_2 g = m_1 \omega^2 r$

$r = \dfrac{m_2 g \pm \mu_s m_2 g}{m_1 \omega^2}$

Depending on r, friction can act radially inwards or outwards

$\therefore \; r_{\min} = \dfrac{m_2 g - \mu_s m_2 g}{m_1 \omega^2}$ $\qquad r_{\max} = \dfrac{m_2 g + \mu_s m_2 g}{m_1 \omega^2}$

[LAST PAGE]