AQA Realising potential

2021 ASSESSMENT MATERIALS

AS
MATHS

Model Solutions

Differentiation and Integration (Topics G,H)

Total number of marks: 40

1 Given that $\frac{dy}{dx} = \frac{1}{6x^2}$ find y.

Circle your answer.

[1 mark]

$\frac{-1}{3x^3} + c$ $\qquad \frac{1}{2x^3} + c$ $\qquad \frac{-1}{6x} + c$ $\qquad \frac{-1}{3x} + c$

$$y = \int \frac{1}{6x^2}\, dx = \frac{1}{6}\int x^{-2}\, dx = \frac{1}{6}x - 1 \times x^{-1} + C = -\frac{1}{6x} + C$$

3 It is given that

$$y = 3x^4 + \frac{2}{x} - \frac{x}{4} + 1$$

Find an expression for $\frac{d^2y}{dx^2}$

$$-\frac{2}{x^2} = -2x^{-2} = +4x^{-3}$$

[3 marks]

$$\frac{dy}{dx} = 12x^3 - \frac{2}{x^2} - \frac{1}{4} \quad \text{and} \quad \frac{d^2y}{dx^2} = 36x^2 + \frac{4}{x^3}$$

5 Differentiate from first principles

$$y = 4x^2 + x$$

[4 marks]

Definition of derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f(x) = 4x^2 + x$$
$$f(x+h) = 4(x+h)^2 + x + h = 4(x^2 + 2hx + h^2) + x + h = 4x^2 + 8hx + 4h^2 + x + h$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{4x^2} + 8hx + 4h^2 + \cancel{x} + h - \cancel{4x^2} - \cancel{x}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{8hx + 4h^2 + h}{h} = \lim_{h \to 0} 8x + 4h + 1$$

Then take the limit as $h \to 0$ $\Rightarrow f'(x) = 8x + 1$

5 $f'(x) = \left(2x - \frac{3}{x}\right)^2$ and $f(3) = 2$

Find $f(x)$.

[4 marks]

$$f(x) = \int f'(x)\, dx = \int \left(2x - \frac{3}{x}\right)^2 dx = \int 4x^2 - 12 + 9x^{-2}\, dx$$

$$f(x) = \frac{4x^3}{3} - 12x - \frac{9}{x} + C$$

Then using $f(3) = 2$ we can find C

$$\Rightarrow 2 = \frac{4 \times (3)^3}{3} - 12(3) - \frac{9}{3} + C \Rightarrow 36 - 36 - 3 + C = 2$$

$$\Rightarrow C = 5 \Rightarrow f(x) = \frac{4x^3}{3} - 12x - \frac{9}{x} + 5$$

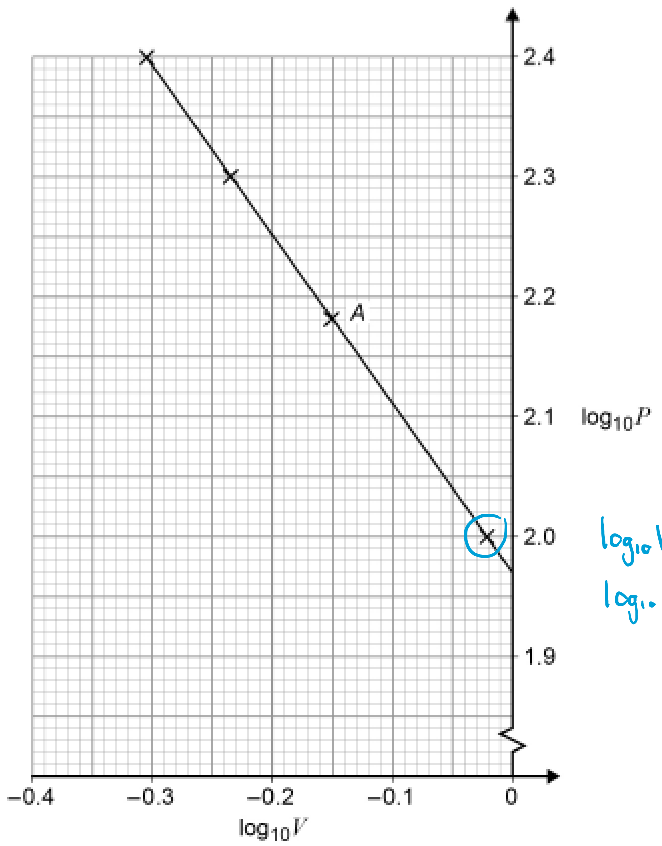
8 Maxine measures the pressure, P kilopascals, and the volume, V litres, in a fixed quantity of gas.

Maxine believes that the pressure and volume are connected by the equation

$$P = cV^d$$

where c and d are constants.

Using four experimental results, Maxine plots $\log_{10}P$ against $\log_{10}V$, as shown in the graph below.



$\log_{10}V = -0.03$

$\log_{10}P = 2.0$

8 (a) Find the value of P and the value of V for the data point labelled A on the graph.

[2 marks]

Data point A : $\log_{10}V = -0.15$ and $\log_{10}P = 2.18$

$\Rightarrow V = 10^{-0.15} = 0.7079\ldots = \underline{0.708}$ and $P = 10^{2.18} = 151.356\ldots = \underline{151}$ (3 s.f)

8 (b) Calculate the value of each of the constants c and d.

[4 marks]

$P = cV^d$, then from 8a $\Rightarrow 151 = c(0.708)^d \Rightarrow c = \dfrac{151}{0.708^d}$

Then we use another data point which would then let us solve the pair of equations simultaneously.

Take $\log_{10}V = -0.03 \Rightarrow V = 0.933$ and $\log_{10}P = 2.0 \Rightarrow P = 100 \Rightarrow C = \dfrac{100}{0.933^d}$

$\Rightarrow \dfrac{151}{0.708^d} = \dfrac{100}{0.933^d} \Rightarrow \dfrac{151}{100} = \left(\dfrac{0.708}{0.933}\right)^d \Rightarrow d\ln\left(\dfrac{0.708}{0.933}\right) = \ln\left(\dfrac{151}{100}\right)$

$\Rightarrow d = \underline{-1.51} \Rightarrow c = \underline{89.6}$

9 (a) (i) Find

$$\int (4x - x^3)\,dx$$

[2 marks]

$\int 4x - x^3 \; dx = \frac{4x^2}{2} - \frac{x^4}{4} + c = \underline{2x^2 - \frac{x^4}{4} + c}$

9 (a) (ii) Evaluate

$$\int_{-2}^{2} (4x - x^3)\,dx$$

[1 mark]

evaluate expression at 2 and −2 and subtract!

From part a: $\int_{-2}^{2} 4x - x^3 \; dx = \left[2x^2 - \frac{x^4}{4}\right]_{-2}^{2} = (4) - (4) = \underline{0}$

9 (b) Using a sketch, explain why the integral in part **(a)(ii)** does **not** give the area enclosed between the curve $y = 4x - x^3$ and the x-axis.

[2 marks]

We have a graph which has area above the x-axis (0 to 2) and it has equal area below the x-axis (0 to −2) and these cancel each other out to give 0.

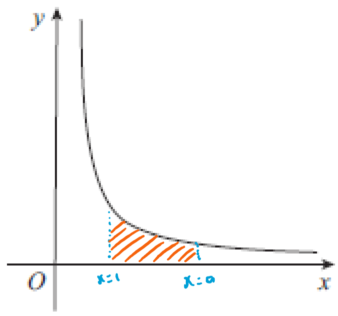
9 (c) Find the area enclosed between the curve $y = 4x - x^3$ and the x-axis.

[2 marks]

$\int_0^2 4x - x^3 \; dx = 4$ using previous working and $\int_{-2}^{0} 4x - x^3 \; dx = 4$

$\Rightarrow$ Total Area $= 4 + 4 = \underline{8 \text{ units}}$

6 A curve has equation $y = \frac{2}{x\sqrt{x}}$



The region enclosed between the curve, the x-axis and the lines $x = 1$ and $x = a$ has area 3 units.

Given that $a > 1$, find the value of a.

Fully justify your answer.

[5 marks]

$y = 2x^{-1}x^{-1/2} = 2x^{-3/2}$

Then $\int_1^a y \, dx = \int_1^a 2x^{-3/2} \, dx = \left[-4x^{-1/2}\right]_1^a = \left(-\frac{4}{\sqrt{a}}\right) - \left(-\frac{4}{\sqrt{1}}\right) = 3$

$\Rightarrow -\frac{4}{\sqrt{a}} + 4 = 3 \Rightarrow \frac{-4}{\sqrt{a}} = -1$

$\Rightarrow \sqrt{a} = 4$

$\Rightarrow \underline{a = 16}$

8 A curve has equation

$$y = x^3 + px^2 + qx - 45$$

The curve passes through point R (2, 3)

The gradient of the curve at R is 8

8 (a) Find the value of p and the value of q.

[5 marks]

$\frac{dy}{dx} = 3x^2 + 2px + q$ and at $x = 2$, $\frac{dy}{dx} = 8$.

$\Rightarrow 3(2)^2 + 2p(2) + q = 8$

$\Rightarrow 4p + q = -4$

$\Rightarrow q = -4 - 4p$ and then $R(2,3)$ being a point $\Rightarrow$

$y = x^3 + px^2 + qx - 45$

$3 = (2)^3 + p(2)^2 + 2q - 45$

$\Rightarrow \frac{40 - 4p}{2} = q \Rightarrow q = 20 - 2p$

$\Rightarrow -4 - 4p = 20 - 2p \Rightarrow 24 = -2p \Rightarrow p = -12$

$\Rightarrow q = -4 - 4(-12) = 44 \Rightarrow \underline{p = -12}, \underline{q = 44}$

8 (b) Calculate the area enclosed between the normal to the curve at R and the coordinate axes.

[5 marks]

The gradient at $R = 8$ implies that the gradient of the normal will be $m = -1/8$.

$\Rightarrow y - y_0 = m(x - x_0)$ where $m = -\frac{1}{8}$ with $x_0 = 2$ and $y_0 = 3$

$y = -\frac{1}{8}(x - 2) + 3$

$y = -\frac{1}{8}x + \frac{1}{4} + 3$

$\Rightarrow y = -\frac{1}{8}x + \frac{13}{4}$

$\frac{1}{8}x = \frac{13}{4} \Rightarrow x = \frac{13}{4} \times 8 = 26$

Then for $x = 0 \Rightarrow y = \frac{13}{4}$ and $y = 0 \Rightarrow x = 26$

Then Area $= \left(\frac{13}{4} \times 26\right) \times \frac{1}{2} = \underline{42.25 \text{ units}}$