Oxford Cambridge and RSA

A Level Mathematics A

H240/03 Pure Mathematics and Mechanics

Question Set 1

1. A circle with centre C has equation $x^2 + y^2 + 8x - 2y - 7 = 0$.

Find

(a) the coordinates of C, [2]

$x^2 + y^2 + 8x - 2y - 7 = 0$

$(x^2 + 8x) + (y^2 - 2y) - 7 = 0$

$(x^2 + 8x + 16 - 16) + (y^2 - 2y + 1 - 1) - 7 = 0$

$(x + 4)^2 + (y - 1)^2 - 16 - 1 - 7 = 0$

$(x + 4)^2 + (y - 1)^2 = 24 = (2\sqrt{6})^2$

$C(-4, 1)$

(b) the radius of the circle. [1]

$2\sqrt{6}$

2 Solve the equation $|2x - 1| = |x + 3|$. [3]

$|2x - 1| = |x + 3|$

$2x - 1 = x + 3$

$x = 4$

$2x - 1 = -(x + 3)$

$2x - 1 = -x - 3$

$3x = -2$

$x = -\frac{2}{3}$

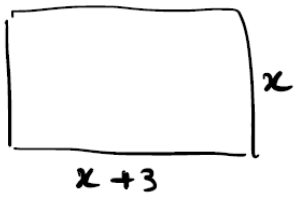
3 **In this question you must show detailed reasoning.**

A gardener is planning the design for a rectangular flower bed. The requirements are:

- the length of the flower bed is to be 3 m longer than the width,
- the length of the flower bed must be at least 14.5 m,
- the area of the flower bed must be less than 180 m^2.

The width of the flower bed is x m.

By writing down and solving appropriate inequalities in x, determine the set of possible values for the width of the flower bed. **[6]**



$x+3 \geq 14.5$

$x \geq 11.5$

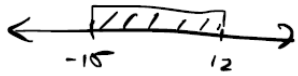
$(x+3)x < 180$

$x^2+3x-180<0$

$15 = 15$

$-12 = -12$

$(x+15)(x-12)<0$



$-15<x<12$

$11.5 \leq x < 12$

4

In this question you must show detailed reasoning.

The functions f and g are defined for all real values of x by

$$f(x) = x^3 \quad \text{and} \quad g(x) = x^2 + 2.$$

(a) Write down expressions for

(i) $fg(x)$, **[1]**

$f(x) = x^3 \qquad g(x) = x^2+2$

$f(g(x)) = (x^2+2)^3$

$= 8 + 12x^2 + 6x^4 + x^6$

(ii) $gf(x)$. **[1]**

$g(f(x)) = (x^3)^2+2 = x^6+2$

(b) Hence find the values of x for which $\quad \mathrm{fg}(x) - \mathrm{gf}(x) = 24$. **[6]**

$$f(g(x)) - g(f(x)) = (8 + 12x^2 + 6x^4 + x^6) - (x^6 + 2)$$
$$= 8 + 12x^2 + 6x^4 + x^6 - x^6 - 2$$
$$24 = 6 + 12x^2 + 6x^4$$

$$6x^4 + 12x^2 - 18 = 0 \qquad x^2 = -3 \text{ or } 1$$
$$6(x^4 + 2x^2 - 3) = 0 \qquad \times$$
$$6(x^2 + 3)(x^2 - 1) = 0$$

$$x = \pm 1$$

5 **(a)** Use the trapezium rule, with two strips of equal width, to show that

$$\int_0^4 \frac{1}{2 + \sqrt{x}}\,dx \approx \frac{11}{4} - \sqrt{2}. \qquad \textbf{[5]}$$

$$x = 0 \sim 2 \Rightarrow 2 \qquad y = \frac{1}{2}, \; y = \frac{1}{2+\sqrt{2}}$$
$$x = 2 \sim 4 \Rightarrow 2 \qquad y = \frac{1}{2+\sqrt{2}}, \; y = \frac{1}{4}$$

$$\int_0^4 \frac{1}{2+\sqrt{x}}\,dx = \frac{1}{2} \times 2 \times \left(\frac{1}{2} + \frac{1}{2+\sqrt{2}} + \frac{1}{2+\sqrt{2}} + \frac{1}{4}\right)$$
$$= \frac{3}{4} + \frac{2}{2+\sqrt{2}} = \frac{3}{4} + \frac{2(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}$$
$$= \frac{3}{4} + \frac{4 - 2\sqrt{2}}{4-2} = \frac{3}{4} + 2 - \sqrt{2}$$
$$= \frac{11}{4} - \sqrt{2}$$

(b) Use the substitution $x = u^2$ to find the exact value of

$$\int_0^4 \frac{1}{2+\sqrt{x}}\,dx.$$

[6]

$$\int_0^4 \frac{1}{2+\sqrt{x}}\,dx$$

$x = u^2$

$u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ $\quad dx = 2\sqrt{x}\,du$

$= 2u\,du$

$$= \int_0^2 \frac{1}{2+u} \times 2u\,du$$

$u = \sqrt{4} = 2 \quad u = \sqrt{0} = 0$

$$= \int_0^2 \frac{2u}{2+u}\,du = 2\int_0^2 \frac{u}{2+u}\,du$$

$v = 2+u$

$\frac{dv}{du} = 1 \quad du = dv$

$$= 2\int_2^4 \frac{v-2}{v}\,dv = 2\int_2^4 1\,dv - 4\int_2^4 \frac{1}{v}\,dv$$

$v = 2+2 = 4 \quad v = 2+0 = 2$

$$= 2[v]_2^4 - 4[\ln v]_2^4$$

$$= 2(4-2) - 4(\ln 4 - \ln 2)$$

$$= 4 - 4\ln\left(\frac{2^2}{2}\right) = 4 - 4\ln 2 = 2(2-2\ln 2)$$

(c) Using your answers to parts **(a)** and **(b**, show that

$$\ln 2 \approx k + \frac{\sqrt{2}}{4},$$

where k is a rational number to be determined. [2]

$$\frac{11}{4} - \sqrt{2} = 4 - 4\ln 2$$

$$4\ln 2 = \sqrt{2} + \frac{5}{4}$$

$$\ln 2 = \frac{5}{16} + \frac{\sqrt{2}}{4}$$

6 It is given that the angle θ satisfies the equation $\sin\left(2\theta + \frac{1}{4}\pi\right) = 3\cos\left(2\theta + \frac{1}{4}\pi\right)$.

(a) Show that $\tan 2\theta = \frac{1}{2}$. [3]

$$\sin\left(2\theta + \tfrac{1}{4}\pi\right) = 3\cos\left(2\theta + \tfrac{1}{4}\pi\right)$$

$$\sin 2\theta \cos\tfrac{1}{4}\pi + \cos 2\theta \sin\tfrac{1}{4}\pi = 3\cos 2\theta\cos\tfrac{1}{4}\pi - 3\sin 2\theta \sin\tfrac{1}{4}\pi$$

$$\sin 2\theta\cos\tfrac{1}{4}\pi + 3\sin 2\theta\sin\tfrac{1}{4}\pi = -\cos 2\theta\sin\tfrac{1}{4}\pi + 3\cos 2\theta\cos\tfrac{1}{4}\pi$$

$$\sin 2\theta\left(\cos\tfrac{1}{4}\pi + 3\sin\tfrac{1}{4}\pi\right) = \cos 2\theta\left(3\cos\tfrac{1}{4}\pi - \sin\tfrac{1}{4}\pi\right)$$

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{3\cos\frac{1}{4}\pi - \sin\frac{1}{4}\pi}{\cos\frac{1}{4}\pi + 3\sin\frac{1}{4}\pi} = \frac{1}{2}$$

$$\tan 2\theta = \frac{1}{2}$$

(b) Hence find, in surd form, the exact value of $\tan\theta$, given that θ is an obtuse angle. [5]

$$\tan 2\theta = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \frac{1}{2}$$

$$2\tan\theta = 2(1 - \tan^2\theta)$$

$$\tan^2\theta + 2\tan\theta - 2 = 0$$

$$\tan\theta = \frac{-2 \pm \sqrt{-2 - 4(1)(-2)}}{2\times 1} = \frac{-2 \pm \sqrt{6}}{2}$$

$$\tan\theta = \frac{-2 - \sqrt{6}}{2}$$

7 The gradient of the curve $y = f(x)$ is given by the differential equation

$$(2x-1)^3\frac{dy}{dx} + 4y^2 = 0$$

and the curve passes through the point $(1, 1)$. By solving this differential equation show that

$$f(x) = \frac{ax^2 - ax + 1}{bx^2 - bx + 1},$$

where a and b are integers to be determined. [9]

$$(2x-1)^3\frac{dy}{dx} + 4y^2 = 0$$

$$(2x-1)^3 dy + 4y^2 dx = 0$$

$$-\frac{1}{(2x-1)^3}dx = \frac{1}{4y^2}dy$$

$$-\int \frac{1}{(2x-1)^3}dx = \frac{1}{4}\int \frac{1}{y^2}dy$$

$$-\int \frac{1}{u^3}\times\frac{du}{2} = \frac{1}{4}\int y^{-2}dy$$

$$-\frac{1}{2}\times -\frac{1}{2}u^{-2} = -\frac{1}{4}y^{-1} + c$$

$$\frac{1}{4u^2} = -\frac{1}{4y} + c$$

$$\frac{1}{4(2x-1)^2} = -\frac{1}{4y} + c$$

$$u = 2x-1 \quad \frac{du}{dx} = 2 \quad dx = \frac{du}{2}$$

$(1,1)$ $$\frac{1}{4(2(1)-1)^2} = -\frac{1}{4(1)} + c$$

$$c = \frac{1}{2}$$

$$\frac{1}{4(2x-1)^2} = -\frac{1}{4y} + \frac{1}{2}$$

$$8y = -8(2x-1)^2 + 16y(2x-1)^2$$

$$(16(2x-1)^2 - 8)y = 8(2x-1)^2$$

$$y = \frac{\cancel{8}(4x^2-4x+1)}{\cancel{16}^{\,2}(4x^2-4x+1) - \cancel{8}_{\,1}}$$

$$f(x) = \frac{4x^2-4x+1}{8x^2-8x+1}$$

Total Marks for Question Set 1: 50 Marks