Oxford Cambridge and RSA

A Level Mathematics A

H240/02 Pure Mathematics and Statistics

Question Set 5

1 **(a)** Differentiate the following with respect to x.

(i) $(2x+3)^7$ **[2]**

$14(2x+3)^6$

(ii) $x^3 \ln x$ **[3]**

$3x^2 \ln x + x^2$

(b) Find $\int \cos 5x \, dx$. **[2]**

$\frac{1}{5} \sin 5x + c$

(c) Find the equation of the curve through (1, 3) for which $\frac{dy}{dx} = 6x - 5$. **[2]**

$y = 3x^2 - 5x + 5$

$y = 3x^2 - 5x + c$

$3 = 3 - 5 + c$

$c = 5$

2 Simplify fully $\frac{2x^3 + x^2 - 7x - 6}{x^2 - x - 2}$. **[4]**

$\frac{(x-2)(x+1)(2x+3)}{(x-2)(x+1)} = 2x+3$

3 In this question you should assume that $-1 < x < 1$.

(a) For the binomial expansion of $(1-x)^{-2}$

(i) find and simplify the first four terms, **[2]**

$1 + 2x + \frac{x^2 \times -2 \times -3}{2!} + \frac{-1x^3 \times -2 \times -3 \times -4}{3!} = 1 + 2x + 3x^2 + 4x^3$

(ii) write down the term in x^n. **[1]**

$\frac{(-1)^n \times n \times (n-1) \ldots \times 2}{n!} x^n$

(b) Write down the sum to infinity of the series $1+x+x^2+x^3+\dots$. **[1]**

$$\frac{1}{1-x}$$

(c) Hence or otherwise find and simplify an expression for $2+3x+4x^2+5x^3+\dots$ in the form $\frac{a-x}{(b-x)^2}$ where a and b are constants to be determined. **[3]**

$(a-x)(1-x)^{-2}$

$= a-x(1+2x+3x^2\dots)$

$a+2ax-x \quad a=2$

$a=2 \quad 2ax-x=3x \therefore a=2$

$$\frac{2-x}{(1-x)^{-2}}$$

4 **In this question you must show detailed reasoning.**

Solve the equation $3\sin^4\phi+\sin^2\phi=4$, for $0 \leqslant \phi < 2\pi$, where ϕ is measured in radians. **[5]**

Let $x=\sin^2\phi$

$(3x+4)(x-1)$

not a solution as $\sin^2\phi > 0$

$\sin^2\phi=1$

$\sin\phi=\pm1$

$\phi=\frac{\pi}{2}, \frac{3\pi}{2}$

5 **(a)** Determine the set of values of n for which $\frac{n^2-1}{2}$ and $\frac{n^2+1}{2}$ are positive integers. **[3]**

$n \geq 3$ and n is odd

A 'Pythagorean triple' is a set of three positive integers a, b and c such that $a^2+b^2=c^2$.

(b) Prove that, for the set of values of n found in part **(a)**, the numbers n, $\frac{n^2-1}{2}$ and $\frac{n^2+1}{2}$ form a Pythagorean triple. **[2]**

$n^2+\left(\frac{n^2-1}{2}\right)^2=\left(\frac{n^2+1}{2}\right)^2 \quad n^2+\frac{n^4-2n^2+1}{4}=\frac{n^4+2n^2+1}{4} \quad n^2=\frac{4n^2}{4} \quad n^2=n^2 \therefore$ true

6 Prove that $\sqrt{2}\cos(2\theta+45°) \equiv \cos^2\theta - 2\sin\theta\cos\theta - \sin^2\theta$, where θ is measured in degrees. **[3]**

$\sqrt{2}(\cos 2\theta \cos 45 - \sin 2\theta \sin 45)$

$= \cos 2\theta - \sin 2\theta$

$\cos^2\theta - \sin^2\theta \qquad 2\cos\theta\sin\theta$

$= \cos^2\theta - 2\cos\theta\sin\theta - \sin^2\theta$

7 A and B are fixed points in the x-y plane. The position vectors of A and B are **a** and **b** respectively.

State, with reference to points A and B, the geometrical significance of

(a) the quantity $|\mathbf{a}-\mathbf{b}|$, **[1]**

$\overrightarrow{BA}$

(b) the vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$. **[1]**

midpoint of OA and OB

The circle P is the set of points with position vector **p** in the x-y plane which satisfy

$\left|\mathbf{p}-\frac{1}{2}(\mathbf{a}+\mathbf{b})\right| = \frac{1}{2}|\mathbf{a}-\mathbf{b}|$.

(c) State, in terms of **a** and **b**,

(i) the position vector of the centre of P, **[1]**

$\frac{1}{2}(a+b)$

Using C

(ii) the radius of P. **[1]**

$0.5|a-b|$

It is now given that $\mathbf{a} = \begin{pmatrix}2\\-1\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}4\\5\end{pmatrix}$ and $\mathbf{p} = \begin{pmatrix}x\\y\end{pmatrix}$.

(d) Find a cartesian equation of P. **[4]**

$r^2 = (-2)^2 + (-6)^2$
$= 40$

$(x-3)^2 + (y-2)^2 = 40$

8 The rate of change of a certain population P at time t is modelled by the equation $\frac{dP}{dt} = (100 - P)$.

Initially $P = 2$[redacted].

(a) Determine an expression for P in terms of t. **[7]**

$$\frac{dP}{100-P} = dt$$

$$-\ln|100-P| = t + C$$

$$\ln|100-P| = -t + C$$

$$100 - P = e^{-t+C} \qquad e^{C} = A$$

$$100 - P = Ae^{-t}$$

$$P = 100 - Ae^{-t}$$

$$2 = 100 - Ce^{0\times -1}$$

$$C = 98$$

$$\underline{P = 100 - 98e^{-t}}$$

(b) Describe how the population changes over time. **[2]**

It goes up but to a max of 100

Total Marks for Question Set 5: 50 Marks