A Level Mathematics A

H240/01 Pure Mathematics

Question Set 4

1 The function f is defined by $f(x) = (x-3)^2 - 17$ for $x \geqslant k$, where k is a constant.

(a) Given that $f^{-1}(x)$ exists, state the least possible value of k. **[1]**

1. $f(x) = (x-3)^2 - 17$

a) $f(x) = y$

$y = (x-3)^2 - 17$

$y + 17 = (x-3)^2$

$\sqrt{y+17} = x - 3$

$x = \sqrt{y+17} + 3$

$y = \sqrt{x+17} + 3$

$f^{-1}(x) = \underbrace{\sqrt{x+17}}_{\text{can't be negative}} + 3$

$x + 17 \geq 0$

$x \geq -17$

-17

(b) Evaluate ff(5). **[2]**

b) $f(f(x)) = [((x-3)^2 - 17) - 3]^2 - 17$

$f(f(5)) = [((5-3)^2 - 17) - 3]^2 - 17 = 239$

(c) Solve the equation $f(x) = x$. **[3]**

c) $f(x) = x$

$(x-3)^2 - 17 = x$

$x^2 - 6x + 9 - 17 = x$

$x^2 - 7x - 8 = 0$

$(x-8)(x+1) = 0$

$x = 8$ or -1

$x = 8, -1$

(d) Explain why your solution to part **(c)** is also the solution to the equation $f(x) = f^{-1}(x)$. **[1]**

d) when $x = 8$, $\sqrt{x+17} + 3$ gives same value of 8 when $x = 8$ has been substituted, equalling x like previously

2 Sam starts a job with an annual salary of £16000. It is promised that the salary will go up by the same amount every year. In the second year Sam is paid £17200.

(a) Find Sam's salary in the tenth year. [2]

2.
a) $u_1 = a = £16\,000$ $u_2 = £17200$

$u_n = a + (n-1)d$ $d = 17200 - 16\,000 = 1200$

$u_{10} = 16000 + (10-1) \times 1200 = £20\,800$

(b) Find the number of complete years needed for Sam's **total** salary to first exceed £500000. [4]

b) $S_n = \frac{1}{2}n[2a + (n-1)d]$

$\frac{1}{2}n[2 \times 16000 + (n-1) \times 1200] > 500\,000$

$n[32000 + 1200n - 1200] > 1\,000\,000$

$n[30800 + 1200n] > 1000000$

$12\cancel{0}\cancel{0}n^2 + 308\cancel{0}\cancel{0}n - 10000\cancel{0}\cancel{0} > 0$

$3n^2 + 77n - 2500 > 0$

$n > 18.8$, $n < -44.4$ ×

$n = 19$ years

(c) Comment on how realistic this model may be in the long term. [1]

c) not realistic as the salary always increase without limit according to model but there will be the maximum salary which would stop it from increasing further

3 Let $f(x) = 2x^3 + 3x$. Use differentiation from first principles to show that $f'(x) = 6x^2 + 3$. [6]

3. $f(x) = 2x^3 + 3x$ $f(x+h) = 2(x+h)^3 + 3(x+h)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^3 + 3(x+h) - (2x^3 + 3x)}{h}$$

$$= \lim_{h \to 0} \frac{2(h^3 + 3h^2x + 3hx^2 + x^3) + 3(x+h) - (2x^3 + 3x)}{h}$$

$$= \lim_{h \to 0} \frac{2h^3 + 6h^2x + 6hx^2 + \cancel{2x^3} + \cancel{3x} + 3h - \cancel{2x^3} - \cancel{3x}}{h}$$

$$= \lim_{h\to 0} \frac{2h^3 + 6h^2x + (6x^2+3)h}{h}$$

$$= \lim_{h\to 0} 2h^2 + 6hx + 6x^2 + 3$$

as $h \to 0$, $2h^2+6hx+6x^2+3 \to 6x^2+3$ so $f'(x) = 6x^2+3$

4 A cylindrical tank is initially full of water. There is a small hole at the base of the tank out of which the water leaks.

The height of water in the tank is x m at time t seconds. The rate of change of the height of water may be modelled by the assumption that it is proportional to the square root of the height of water.

When $t = 100$, $x = 0.64$ and, at this instant, the height is decreasing at a rate of $0.0032\,\mathrm{ms}^{-1}$.

(a) Show that $\frac{dx}{dt} = -0.004\sqrt{x}$. **[2]**

4. $t = 100$ $\quad x = 0.64$ $\quad \frac{dx}{dt} = -0.0032$

a) $\frac{dx}{dt} = k\sqrt{x}$ $\quad -0.0032 = k\sqrt{0.64}$ $\quad k = -0.004$

$$\frac{dx}{dt} = -0.004\sqrt{x}$$

(b) Find an expression for x in terms of t. **[4]**

b) $\frac{dx}{dt} = -0.004\sqrt{x}$ $\quad \frac{1}{\sqrt{x}}\,dx = -0.004\,dt$

$\int \frac{1}{\sqrt{x}}\,dx = \int -0.004\,dt$ $\quad \int x^{-\frac{1}{2}}\,dx = -0.004\int 1\,dt$

$2x^{\frac{1}{2}} = -0.004t + c$

$2 \times 0.64^{\frac{1}{2}} = -0.004 \times 100 + c$ $\quad c = 2$

$2\sqrt{x} = -0.004t + 2$

$\sqrt{x} = -0.002t + 1$

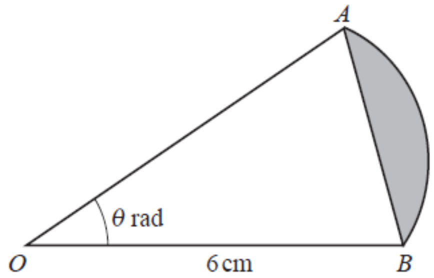
(c) Hence determine at what time, according to this model, the tank will be empty. **[2]**

c) $x = 0$ when empty $\quad \sqrt{0} = -0.002t + 1$

$0 = -0.002t + 1$

$0.002t = 1$

$t = 500$ $\quad$ after 500s



The diagram shows a sector AOB of a circle with centre O and radius 6 cm.
The angle AOB is θ radians.
The area of the segment bounded by the chord AB and the arc AB is $7.2\,\text{cm}^2$.

(a) Show that $\theta = 0.4 + \sin\theta$. **[3]**

5. a) sector area: $A = \frac{1}{2}r^2\theta$ $\quad A = \frac{1}{2}(6)^2\theta = 18\theta$

triangle area: $A = \frac{1}{2}ab\sin C$ $\quad A = \frac{1}{2}(6)^2\sin\theta = 18\sin\theta$

$18\theta - 18\sin\theta = 7.2$

$18\theta = 7.2 + 18\sin\theta$

$\theta = 0.4 + \sin\theta$

(b) Let $F(\theta) = 0.4 + \sin\theta$.

By considering the value of $F'(\theta)$ where $\theta = 1.2$, explain why using an iterative method based on the equation in part **(a)** will converge to the root, assuming that 1.2 is sufficiently close to the root. **[2]**

b) $F(\theta) = 0.4 + \sin\theta$

$F'(\theta) = \cos\theta$ where $\theta = 1.2$ $\quad F'(\theta) = 0.362$

$0.362 < 1$ for all values of x converging to root

(c) Use the iterative formula $\theta_{n+1} = 0.4 + \sin\theta_n$ with a starting value of 1.2 to find the value of θ correct to 4 significant figures.
You should show the result of each iteration. **[3]**

c) $\theta_{n+1} = 0.4 + \sin\theta_n$

$\theta_1 = 1.2$

$\theta_2 = 0.4 + \sin 1.2 = 1.3320$

$\theta_3 = 0.4 + \sin 1.3320 = 1.3716$

$\theta_4 = 0.4 + \sin 1.3716 = 1.3802$

$\theta_5 = 0.4 + \sin 1.3802 = 1.3819$

$\theta_6 = 0.4 + \sin 1.3819 = 1.3822$

$\theta_7 = 0.4 + \sin 1.3822 = 1.3823$

$\theta_8 = 0.4 + \sin 1.3823 = 1.3823$

1.382

(d) Use a change of sign method to show that the value of θ found in part **(c)** is correct to 4 significant figures. **[3]**

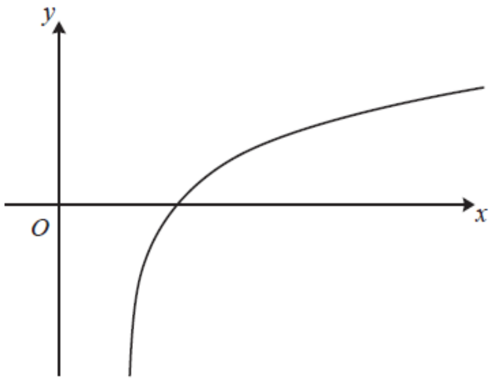
d) $F(\theta) = 0.4 + \sin\theta$

$F(1.3823) = 0.4 + \sin 1.3823 - 1.3823$

$F(3.564) = 0.4 + \sin 3.564 = -9.957 \times 10^{-3}$

change of sign so value of θ found correct

6



The diagram shows part of the curve $y = \ln(x-4)$.

(a) Use integration by parts to show that $\int \ln(x-4)\,dx = (x-4)\ln|x-4| - x + c$. **[5]**

6. $y = \ln(x-4)$

a) $\int \ln(x-4)\,dx$

$u = \ln(x-4) \quad u' = \frac{1}{x-4}$

$v = x \quad v' = 1$

$= \int \ln(x-4) \times (1)\,dx$

$= (\ln(x-4))(x) - \int (x)\left(\frac{1}{x-4}\right) dx$

$= x\ln(x-4) - \int \frac{x}{x-4}\,dx$

$w = x-4$

$\frac{dw}{dx} = 1 \quad dx = dw$

$= x\ln(x-4) - \int \frac{w+4}{w}\,dw$

$= x\ln(x-4) - \left(\int 1\,d + \int \frac{4}{w}\,d\right)$

$= x\ln(x-4) - w - 4\ln w$

$= x\ln(x-4) - (x-4) - 4\ln(x-4)$

$= (x-4)\ln(x-4) - x + 4 = (x-4)\ln|x-4| - x + 4$

(b) State the equation of the vertical asymptote to the curve $y = \ln(x-4)$. **[1]**

b) $x = 4$ as curve moved to the right by $\binom{4}{0}$ so original asymptote of $x = 0$ shifts to the right by $\binom{4}{0}$

(c) Find the total area enclosed by the curve $y = \ln(x-4)$, the x-axis and the lines $x = 4.5$ and $x = 7$. Give your answer in the form $a\ln 3 + b\ln 2 + c$ where a, b and c are constants to be found. **[4]**

c) $\int_{4.5}^{7} \ln(x-4)\,dx$

$= \left[(x-4)\ln|x-4| - x + 4\right]_{4.5}^{7}$

$= \left[(7-4)\ln|7-4| - 7 + 4\right] - \left[(4.5-4)\ln|4.5-4| - 4.5 + 4\right]$

$= (3\ln 3 - 3) - (0.5\ln 0.5 - 0.5)$

$= 3\ln 3 - 3 - 0.5\ln 2^{-1} + 0.5$

$= 3\ln 3 + 0.5\ln 2 - 2.5$

Total Marks for Question Set 4: 49 Marks