

**A Level Mathematics A**  
**H240/01 Pure Mathematics**

**Question Set 4**

1 The function  $f$  is defined by  $f(x) = (x-3)^2 - 17$  for  $x \geq k$ , where  $k$  is a constant.

(a) Given that  $f^{-1}(x)$  exists, state the least possible value of  $k$ .

[1]

1.  $f(x) = (x-3)^2 - 17$

a)  $f(x) = y$       $y = (x-3)^2 - 17$   
 $y + 17 = (x-3)^2$   
 $\sqrt{y+17} = x-3$   
 $x = \sqrt{y+17} + 3$

$$y = \sqrt{x+17} + 3$$
$$f^{-1}(x) = \sqrt{x+17} + 3$$

can't be negative

$$x+17 \geq 0$$
$$x \geq -17$$

-17

(b) Evaluate  $ff(5)$ .

[2]

b)  $f(f(x)) = \left[ \left( (x-3)^2 - 17 \right) - 3 \right]^2 - 17$

$$f(f(5)) = \left[ \left( (5-3)^2 - 17 \right) - 3 \right]^2 - 17 = 239$$

(c) Solve the equation  $f(x) = x$ .

[3]

c)  $f(x) = x$       $(x-3)^2 - 17 = x$   
 $x^2 - 6x + 9 - 17 = x$   
 $x^2 - 7x - 8 = 0$   
 $(x-8)(x+1) = 0$   
 $x = 8 \text{ or } -1$

$x = 8, -1$

(d) Explain why your solution to part (c) is also the solution to the equation  $f(x) = f^{-1}(x)$ .

[1]

d) when  $x=8$ ,  $\sqrt{x+17} + 3$  gives same value of 8 when  $x=8$  has been substituted, equalling  $x$  like previously

- 2 Sam starts a job with an annual salary of £16000. It is promised that the salary will go up by the same amount every year. In the second year Sam is paid £17200.

(a) Find Sam's salary in the tenth year. [2]

$$2. \quad u_1 = a = \pounds 16000 \quad u_2 = \pounds 17200$$

$$a) \quad u_n = a + (n-1)d \quad d = 17200 - 16000 = 1200$$

$$u_{10} = 16000 + (10-1) \times 1200 = \pounds 20800$$

(b) Find the number of complete years needed for Sam's total salary to first exceed £500000. [4]

$$b) \quad S_n = \frac{1}{2}n [2a + (n-1)d]$$

$$\frac{1}{2}n [2 \times 16000 + (n-1) \times 1200] > 500000$$

$$n [32000 + 1200n - 1200] > 1000000$$

$$n [30800 + 1200n] > 1000000$$

$$1200n^2 + 30800n - 1000000 > 0$$

$$3n^2 + 77n - 2500 > 0$$

$$n > 18.8 \quad , \quad n < -44.4$$

$$n = 19 \text{ years}$$

(c) Comment on how realistic this model may be in the long term. [1]

c) not realistic as the salary always increase without limit according to model but there will be the maximum salary which would stop it from increasing further

- 3 Let  $f(x) = 2x^3 + 3x$ . Use differentiation from first principles to show that  $f'(x) = 6x^2 + 3$ . [6]

$$3. \quad f(x) = 2x^3 + 3x \quad f(x+h) = 2(x+h)^3 + 3(x+h)$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2(x+h)^3 + 3(x+h) - (2x^3 + 3x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2(h^3 + 3h^2x + 3hx^2 + x^3) + 3(x+h) - (2x^3 + 3x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2h^3 + 6h^2x + 6hx^2 + 2x^3 + 3x + 3h - 2x^3 - 3x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2h^3 + 6h^2x + (6x^2 + 3)h}{h}$$

$$= \lim_{h \rightarrow 0} 2h^2 + 6hx + 6x^2 + 3$$

as  $h \rightarrow 0$ ,  $2h^2 + 6hx + 6x^2 + 3 \rightarrow 6x^2 + 3$  so  $f'(x) = 6x^2 + 3$

- 4 A cylindrical tank is initially full of water. There is a small hole at the base of the tank out of which the water leaks.

The height of water in the tank is  $x$  m at time  $t$  seconds. The rate of change of the height of water may be modelled by the assumption that it is proportional to the square root of the height of water.

When  $t = 100$ ,  $x = 0.64$  and, at this instant, the height is decreasing at a rate of  $0.0032 \text{ ms}^{-1}$ .

- (a) Show that  $\frac{dx}{dt} = -0.004\sqrt{x}$ . [2]

4.  $t = 100$        $x = 0.64$        $\frac{dx}{dt} = -0.0032$

a)  $\frac{dx}{dt} = k\sqrt{x}$        $-0.0032 = k\sqrt{0.64}$        $k = -0.004$

$$\frac{dx}{dt} = -0.004\sqrt{x}$$

- (b) Find an expression for  $x$  in terms of  $t$ . [4]

b)  $\frac{dx}{dt} = -0.004\sqrt{x}$        $\frac{1}{\sqrt{x}} dx = -0.004 dt$

$$\int \frac{1}{\sqrt{x}} dx = \int -0.004 dt$$

$$\int x^{-\frac{1}{2}} dx = -0.004 \int 1 dt$$

$$2x^{\frac{1}{2}} = -0.004t + C$$

$$2 \times 0.64^{\frac{1}{2}} = -0.004 \times 100 + C$$

$$C = 2$$

$$2\sqrt{x} = -0.004t + 2$$

$$\sqrt{x} = -0.002t + 1$$

- (c) Hence determine at what time, according to this model, the tank will be empty. [2]

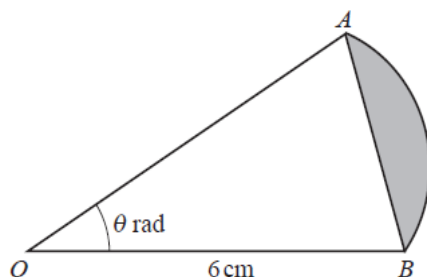
c)  $x = 0$  when empty       $\sqrt{0} = -0.002t + 1$

$$0 = -0.002t + 1$$

$$0.002t = 1$$

$$t = 500$$

after 500s



The diagram shows a sector  $AOB$  of a circle with centre  $O$  and radius  $6 \text{ cm}$ .  
The angle  $AOB$  is  $\theta$  radians.  
The area of the segment bounded by the chord  $AB$  and the arc  $AB$  is  $7.2 \text{ cm}^2$ .

(a) Show that  $\theta = 0.4 + \sin \theta$ .

[3]

5. a) sector area:  $A = \frac{1}{2} r^2 \theta$   $A = \frac{1}{2} (6)^2 \theta = 18\theta$   
 triangle area:  $A = \frac{1}{2} ab \sin C$   $A = \frac{1}{2} (6)^2 \sin \theta = 18 \sin \theta$   
 $18\theta - 18 \sin \theta = 7.2$   
 $18\theta = 7.2 + 18 \sin \theta$   
 $\theta = 0.4 + \sin \theta$

(b) Let  $F(\theta) = 0.4 + \sin \theta$ .

By considering the value of  $F'(\theta)$  where  $\theta = 1.2$ , explain why using an iterative method based on the equation in part (a) will converge to the root, assuming that  $1.2$  is sufficiently close to the root. [2]

b)  $F(\theta) = 0.4 + \sin \theta$   
 $F'(\theta) = \cos \theta$  where  $\theta = 1.2$   $F'(1.2) = 0.362$   
 $0.362 < 1$  for all values of  $x$  converging to root

(c) Use the iterative formula  $\theta_{n+1} = 0.4 + \sin \theta_n$  with a starting value of  $1.2$  to find the value of  $\theta$  correct to 4 significant figures.

You should show the result of each iteration.

[3]

c)  $\theta_{n+1} = 0.4 + \sin \theta_n$   
 $\theta_1 = 1.2$   
 $\theta_2 = 0.4 + \sin 1.2 = 1.3320$   
 $\theta_3 = 0.4 + \sin 1.3320 = 1.3716$   
 $\theta_4 = 0.4 + \sin 1.3716 = 1.3802$   
 $\theta_5 = 0.4 + \sin 1.3802 = 1.3819$   
 $\theta_6 = 0.4 + \sin 1.3819 = 1.3822$   
 $\theta_7 = 0.4 + \sin 1.3822 = 1.3823$   
 $\theta_8 = 0.4 + \sin 1.3823 = 1.3823$

1.382

- (d) Use a change of sign method to show that the value of  $\theta$  found in part (c) is correct to 4 significant figures. [3]

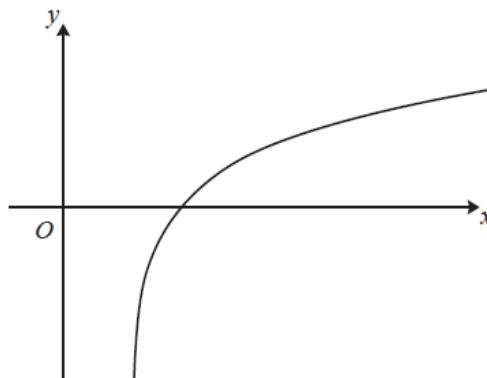
d)  $F(\theta) = 0.4 + \sin \theta$

$$F(1.3823) = 0.4 + \sin 1.3823 = 1.3823$$

$$F(3.564) = 0.4 + \sin 3.564 = -9.957 \times 10^{-3}$$

change of sign so  
value of  $\theta$  found correct

6



The diagram shows part of the curve  $y = \ln(x-4)$ .

- (a) Use integration by parts to show that  $\int \ln(x-4) dx = (x-4)\ln|x-4| - x + c$ . [5]

b.  $y = \ln(x-4)$

a)  $\int \ln(x-4) dx$

$$u = \ln(x-4) \quad u' = \frac{1}{x-4}$$

$$v = x \quad v' = 1$$

$$= \int \ln(x-4) \times (1) dx$$

$$= (\ln(x-4))(x) - \int (x) \left(\frac{1}{x-4}\right) dx$$

$$= x \ln(x-4) - \int \frac{x}{x-4} dx$$

$$w = x-4$$

$$\frac{dw}{dx} = 1 \quad dx = dw$$

$$= x \ln(x-4) - \int \frac{w+4}{w} dw$$

$$= x \ln(x-4) - \left( \int 1 dw + \int \frac{4}{w} dw \right)$$

$$= x \ln(x-4) - w - 4 \ln w$$

$$= x \ln(x-4) - (x-4) - 4 \ln(x-4)$$

$$= (x-4) \ln(x-4) - x + 4 = (x-4) \ln|x-4| - x + 4$$

(b) State the equation of the vertical asymptote to the curve  $y = \ln(x-4)$ .

[1]

b)  $x = 4$  as curve moved to the right by (4)  
so original asymptote of  $x = 0$  shifts  
to the right by (4)

(c) Find the total area enclosed by the curve  $y = \ln(x-4)$ , the  $x$ -axis and the lines  $x = 4.5$  and  $x = 7$ . Give your answer in the form  $a \ln 3 + b \ln 2 + c$  where  $a$ ,  $b$  and  $c$  are constants to be found.

[4]

c)

$$\begin{aligned} & \int_{4.5}^7 \ln(x-4) dx \\ &= [(x-4) \ln|x-4| - x + 4]_{4.5}^7 \\ &= [(7-4) \ln|7-4| - 7 + 4] - [(4.5-4) \ln|4.5-4| - 4.5 + 4] \\ &= (3 \ln 3 - 3) - (0.5 \ln 0.5 - 0.5) \\ &= 3 \ln 3 - 3 - 0.5 \ln 2^{-1} + 0.5 \\ &= 3 \ln 3 + 0.5 \ln 2 - 2.5 \end{aligned}$$

**Total Marks for Question Set 4: 49 Marks**

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