Oxford Cambridge and RSA

A Level Mathematics B (MEI)

H640/03 MEI Pure Mathematics and Comprehension

Question Set 6

1 Show that the equation of the line in Fig. C2 is $ry + hx = hr$, as given in line 24. **[2]**

2 **(a)** **(i)** Show that the cross-sectional area in Fig. C3.2 is $\pi x(2r - x)$. **[2]**

(ii) Hence show that the cross-sectional area is $\frac{\pi r^2}{h^2}(h^2 - y^2)$, as given in line 37. **[2]**

(b) Verify that the formula $\frac{\pi r^2}{h^2}(h^2 - y^2)$ for the cross-sectional area is also valid for

(i) Fig. C3.1, **[1]**

(ii) Fig. C3.3. **[1]**

3 **(a)** Express $\lim_{\delta y \to 0} \sum_{0}^{h}(h^2 - y^2)\delta y$ as an integral. **[1]**

(b) Hence show that $V = \frac{2}{3}\pi r^2 h$, as given in line 41. **[3]**

4 A typical tube of toothpaste measures 5.4 cm across the straight edge at the top and is 12 cm high. It contains 75 ml of toothpaste so it needs to have an internal volume of $75\,\text{cm}^3$.

Comment on the accuracy of the formula $V = \frac{2}{3}\pi r^2 h$, as given in line 41, for the volume in this case. **[3]**

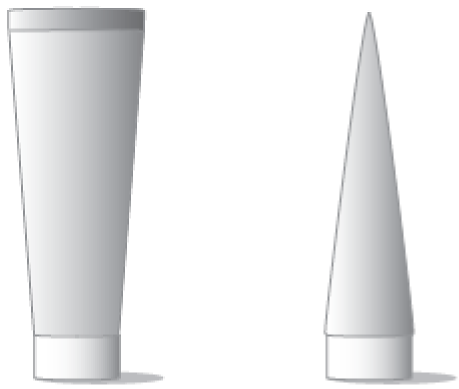
Total Marks for Question Set 6: 15

Resource Materials

Question Set No: 6

Modelling a tube

Products such as toothpaste and hand cream are often sold in tubes which have a circular cross-section at the end which has the opening for the product to be dispensed. The other end of the tube is closed and is a straight line. The front view and side view of such a tube are shown in Fig. C1. The circular end will be defined to be the bottom end of the tube and the straight line end will be defined to be the top end.



Front view Side view

Fig. C1

There is no simple formula for the volume of a tube of this shape, but a good approximation can be derived using mathematical modelling.

The cross-section at the bottom of the tube is a circle; the cross-section at the top is a straight line. Observation of tubes suggests that they are made by starting with a cylinder and closing one end by bringing the sides together in a straight line. This means that the tube will have a volume smaller than the cylinder that was used when making it. If the base radius of the tube is r, the height is h and the volume is V then

$V < \pi r^2 h$.

Modelling assumptions

The following table lists the modelling assumptions which will be made, together with some comments justifying each of them.

Modelling assumption	Comments
The perimeter of the cross-section of the tube is constant all the way up.	This follows from starting with a cylinder to make the tube.
The nozzle at the bottom of the tube and the cap will be ignored.	Experience suggests that the nozzle and cap are not filled with the product when the tube is first opened so their volumes are not relevant.
The front width of the tube increases at a constant rate from the bottom end to the top end.	Observation suggests that this is a good approximation.
The side width of the tube decreases at a constant rate from the bottom end to the top end.	This situation is shown in Fig. C2; observation suggests that this is a close approximation for tubes of typical sizes.

Modelling the cross-section

Taking the y-axis as the axis of symmetry of the tube and looking at the tube from the side, as shown in Fig. C2, means that the side width of the tube is $2x$ at height y.

When $y = 0$, $x = r$ and when $y = h$, $x = 0$.

Assuming that the relationship between x and y is linear means that the side width decreases at a constant rate as y increases; this leads to $ry + hx = hr$.

The cross-section at the bottom of the tube is a circle, as shown in Fig. C3.1; at the top of the tube, the cross-section is a line, as shown in Fig. C3.3.

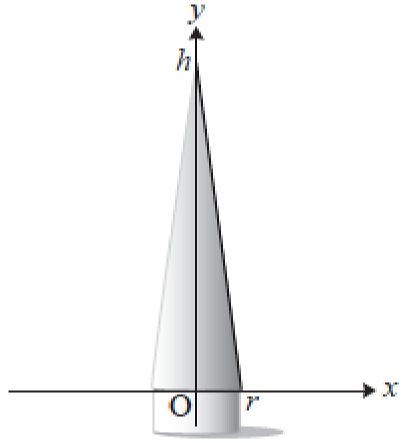


Fig. C2

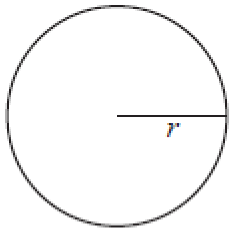


Fig. C3.1

Fig. C3.2

Fig. C3.3

The exact 'oval' shape of the cross-section at intermediate points is not easy to determine, so a simple approximation for the shape is used.

When the width of the tube is $2x$, the cross-section will be modelled as a rectangle with semicircular ends, as shown in Fig. C3.2. The radius of the semicircular ends is x. To ensure that the total perimeter of the cross-section is a constant, the length, l, of the rectangular part of the cross-section is given by $l = \pi(r - x)$. It can be shown that this ensures that the front width of the tube increases at a constant rate as y increases, as required by the modelling assumptions.

Calculating the volume

Finding the area of the cross-section shown in Fig. C3.2 and using $ry + hx = hr$ gives the cross-sectional area in terms of y as $\frac{\pi r^2}{h^2}(h^2 - y^2)$.

Imagine slicing the tube into thin horizontal slices, with cross-section as shown in Fig. C3.2 and thickness δy. The volume of the tube is given by $\sum_0^h \frac{\pi r^2}{h^2}(h^2 - y^2)\delta y$; since r and h are constants for the tube, this can be written as $\frac{\pi r^2}{h^2}\sum_0^h (h^2 - y^2)\delta y$.

Taking the limit as $\delta y \to 0$ and evaluating the resulting integral gives $V = \frac{2}{3}\pi r^2 h$. This is less than the volume of the cylinder, $\pi r^2 h$, as expected.