A Level Mathematics B (MEI)

H640/03 MEI Pure Mathematics and Comprehension

Question Set 5

1 The function $f(x)$ is defined for all real x by

$f(x) = 3x - 2$.

(a) Find an expression for $f^{-1}(x)$. [2]

$y = 3x - 2$

$\frac{y+2}{3} \quad x$

$\frac{x+2}{3} = y$

(c) Find the set of values of x for which $f(x) > f^{-1}(x)$. [2]

$3x - 2 > \frac{x+2}{3}$

$9x - 6 > x + 2$

$8x > 8$

$x > 1$

2 (a) Find the transformation which maps the curve $y = x^2$ to the curve $y = x^2 + 8x - 7$. [4]

$y = (x+4)^2 - 23$

translation to the left +4

and translation down by 23

$\therefore$ translation $\binom{-4}{-23}$

(b) Write down the coordinates of the turning point of $y = x^2 + 8x - 7$. [1]

$(x+4)^2 - 23 \qquad (-4, -23)$

3 (a) Express $\frac{1}{(x+2)(x+3)}$ in partial fractions. [3]

$\frac{A}{x+2} + \frac{B}{x+3} \rightarrow A(x+3) + B(x+2) = 1$

let $x = -3$

$B(-3+2) = -B = 1$

$B = -1$

let $x = -2$

$A(-2+3) = 1$

$A = 1$

$\frac{1}{x+2} - \frac{1}{x+3}$

(b) Find $\int \frac{1}{(x+2)(x+3)} dx$ in the form $\ln(f(x)) + c$, where c is the constant of integration and $f(x)$ is a function to be determined. [3]

$\int \frac{1}{x+2} - \frac{1}{x+3} dx = \ln x+2 - \ln x+3 + C$

$= \ln \frac{x+2}{x+3} + C$

4 **In this question you must show detailed reasoning.**

Show that $\frac{1}{\sqrt{10}+\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{12}} + \frac{1}{\sqrt{12}+\sqrt{13}} = \frac{3}{\sqrt{10}+\sqrt{13}}$ [3]

$\frac{3}{\sqrt{10}+\sqrt{13}} \times \frac{\sqrt{13}-\sqrt{10}}{\sqrt{13}-\sqrt{10}} = \sqrt{13}-\sqrt{10}$

$\frac{1}{\sqrt{10}+\sqrt{11}} \times \frac{\sqrt{11}-\sqrt{10}}{\sqrt{11}-\sqrt{10}} = \sqrt{11}-\sqrt{10}$

$\therefore \sqrt{11}-\sqrt{10} - \sqrt{11}+\sqrt{12} - \sqrt{12}+\sqrt{13} = \sqrt{13}-\sqrt{10}$ $\therefore$ true as LHS=RHS

5 A student's attempt to prove by contradiction that there is no largest prime number is shown below.

> If there is a largest prime, list all the primes.
> Multiply all the primes and add 1.
> The new number is not divisible by any of the primes in the list and so it must be a new prime.

The proof is incorrect and incomplete.
Write a correct version of the proof. [3]

Let the list of all primes be $P_1, P_2 \ldots P_n$ where P_n is the largest prime
then do $P_1 \times P_2 \ldots \times P_n + 1$ and let this $= N$
N has no factors on the finite list of Primes so there must be more primes not on the list
and $\therefore$ there can be no largest prime

6 A circle has centre C (10, 4). The x-axis is a tangent to the circle, as shown in Fig. 6.

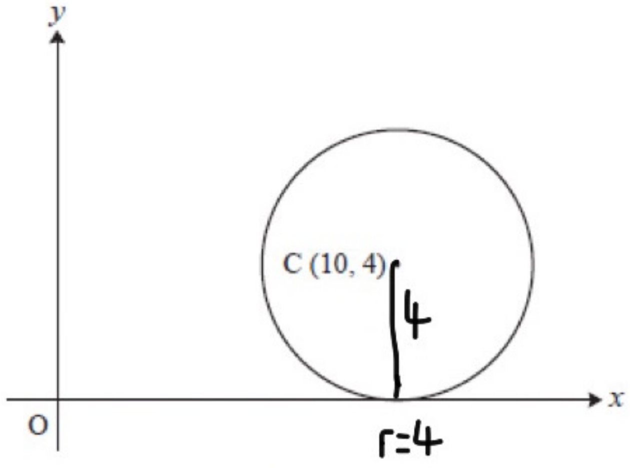


Fig. 6

(a) Find the equation of the circle. $(x-10)^2+(y-4)^2=16$ [2]

(b) Show that the line $y = x$ is not a tangent to the circle. [4]

let $y = x$
$(x-10)^2 + (y-4)^2 = 16$
$x^2 - 20x + 100 + x^2 - 8x + 16 = 16$
$2x^2 - 28x + 100 = 0$

$b^2 - 4ac < 0$
$28^2 - 4(2)(100) < 0$
$-16 < 0$
$\therefore$ no solutions
not a tangent

(c) Write down the position vector of the midpoint of OC. [1]

$\frac{1}{2}\begin{pmatrix}10\\4\end{pmatrix} = \frac{5}{2}$

7 **In this question you must show detailed reasoning.**

(a) Express $\ln 3 \times \underset{\overset{\uparrow}{2\ln 3}}{\ln 9} \times \underset{3\ln 3}{\ln 27}$ in terms of $\ln 3$. $\therefore 6(\ln 3)^3$ [2]

(b) Hence show that $\ln 3 \times \ln 9 \times \ln 27 > 6$. [2]

$\ln 3 = 1.0986$ $\quad \ln 3 > 1$

$\therefore 6(\ln 3)^3 > 6$

8 **In this question you must show detailed reasoning.**

A is the point (1, 0), B is the point (1, 1) and D is the point where the tangent to the curve $y = x^3$ at B crosses the x-axis, as shown in Fig. 8. The tangent meets the y-axis at E.

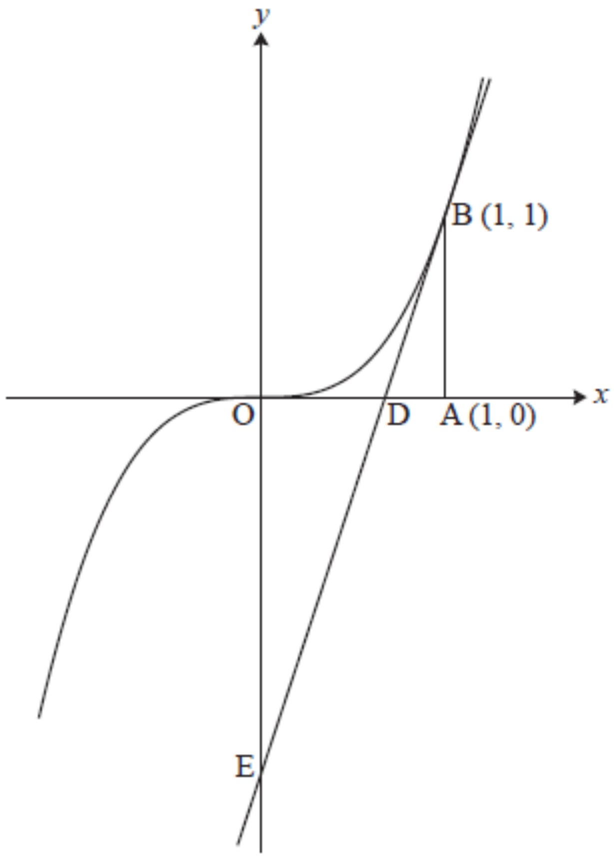


Fig. 8

(a) Find the area of triangle ODE. **[6]**

$\frac{dy}{dx} = 3x^2$ at $x = 1$

$\frac{dy}{dx} = 3(1^2) = 3$, $y = 3x + c$

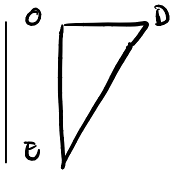
$1 = 3 + c$

$c = -2$

$y = 3x - 2$

$D = \left(\frac{2}{3}, 0\right)$

$E = (0, -2)$

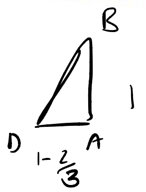


$\frac{1}{2} \times \left(\frac{2}{3}\right) \times 2$

$= \frac{2}{3}$

(b) Find the area of the region bounded by the curve $y = x^3$, the tangent at B and the y-axis. **[4]**

$\int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}$ $\longrightarrow$



$= \frac{1}{2} \times 1 \times \frac{1}{3}$

$= \frac{1}{6}$

$\frac{1}{4} - \frac{1}{6} + 2/3 = \frac{3}{4}$

9 **In this question you must show detailed reasoning.**

The curve $xy + y^2 = 8$ is shown in Fig. 9.

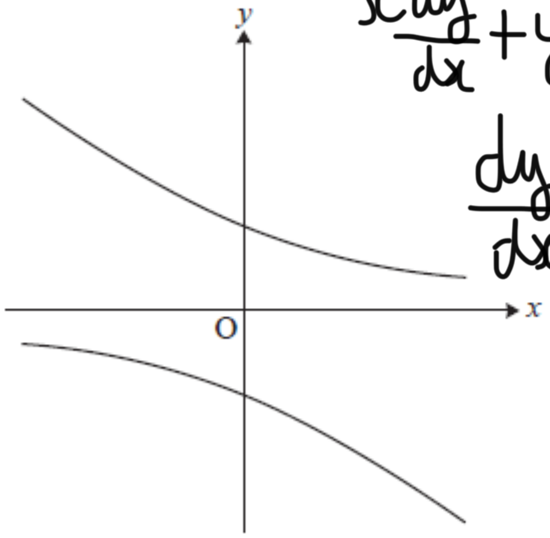


Fig. 9

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y}{2y+x} = -\frac{1}{2}$$

as 2 is gradient of the normal

$$2y = 2y + x$$

$x = 0 \quad y = \pm 2\sqrt{2}$

$(0, 2\sqrt{2})$

$(0, -2\sqrt{2})$

Find the coordinates of the points on the curve at which the normal has gradient 2. **[6]**

$$xy + y^2 = 8$$

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$x + 2y\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x+2y} = \frac{-1}{2}$$

$$-2y = -x - 2y$$

$$0 = -x$$

$$\underline{x = 0}$$

$$0y + y^2 = 8$$

$$y^2 = 8$$

$$y = \sqrt[2]{8} = \pm 2\sqrt{2}$$

gradient normal = 2

$\therefore$ gradient of curve = $-\frac{1}{2}$

$\therefore (0, 2\sqrt{2})$

and $(0, -2\sqrt{2})$

10 Show that $f(x) = \frac{e^x}{1+e^x}$ is an increasing function for all values of x. [4]

$e^x \to e^x$

$1+e^x \to e^x$

$\frac{e^x+e^{2x}-e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$ ∴ positive for all x ∴ no turning points

when $x=0$ $\frac{dy}{dx} = \frac{1}{4}$ when $x=1$ $\frac{dy}{dx} > \frac{1}{4}$ ∴ increasing always

11 By using the substitution $u = 1+\sqrt{x}$, find $\int \frac{x}{1+\sqrt{x}}\,dx$. [6]

$\frac{du}{dx} = \frac{x^{-\frac{1}{2}}}{2}$

$dx = 2du\,x^{\frac{1}{2}}$

$dx = 2du(u-1)$

$\therefore 2\int \frac{(u+1)^2}{u} \times (u-1)\,du$

Total Marks for Question Set 5: 60

$= \frac{(u^2+2u+1)(u-1)}{u} = u^2+2u+1-u-2-\frac{1}{u}$

$= 2\int u^2+u-1+\frac{1}{u}\,du$

$= \frac{2}{3}u^3+u^2-2u+2\ln u$

$= \frac{2}{3}(1+\sqrt{x})^3+(1+\sqrt{x})^2-2(1+\sqrt{x})+2\ln(1+\sqrt{x})$

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