A Level Mathematics B (MEI)

H640/03 MEI Pure Mathematics and Comprehension

Question Set 4

1 (a) Show that if $a = 1$ and $b > 1$ then $a^b < b^a$. [2]

If $a=1$

$a^b = 1$ and $b^a = b$

$b > 1$ $\therefore$ $b^a = a^b$

(b) Find integer values of a and b with $b > a > 1$ and a^b not greater than b^a (a counter example to the conjecture given in lines 7–8). [1]

$b = 4$

$a = 3$

2 **In this question you must show detailed reasoning.**

Show that $\int_e^{\pi} \frac{1}{x} dx = \ln \pi - 1$ as given in line 37. [2]

$= [\ln x]_e^{\pi} = \ln \pi - \ln e$

$= \ln \pi - 1$

3 Show that e^x is an increasing function for all values of x, as stated in line 39. [2]

$\frac{dy}{dx} = e^x$ $\frac{dy}{dx} > 0$ for all values of x and $e^2 > e$ $\therefore$ no turning points $\therefore$ always increasing

4 (a) Show that the only stationary point on the curve $y = \frac{\ln x}{x}$ occurs where $x = e$, as given in line 45. [3]

$\ln x \to \frac{1}{x}$ $\frac{1 - \ln x}{x^2} = 0$ $\ln x = 1$ $x = e$

(b) Show that the stationary point is a maximum. [3]

$y = \frac{\ln x}{x}$

$\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$

$u = 1 - \ln x$

$\frac{du}{dx} = -1/x$

$v = x^2$

$\frac{dv}{dx} = 2x$

$\frac{\frac{-x^2}{x} - 2x(1 - \ln x)}{(x^2)^2} = \frac{-x - 2x + 2x\ln x}{x^4} = \frac{-3x + 2x\ln x}{x^4}$

$= \frac{-3 + 2\ln x}{x^3}$

when $x = e$

$\frac{-3 - 2\ln e}{e^8} = -0.049787...$

$\to -0.0498 < 0$ $\therefore$ a maximum

(c) It follows from part (b) that, for any positive number a with $a \neq e$,

$\frac{\ln e}{e} > \frac{\ln a}{a}$.

Use this fact to show that $e^a > a^e$. [2]

$a\ln e > e\ln a$

$\ln e^a > \ln a^e$ $e^a > a^e$

Total Marks for Question Set 4: 15

Resource Materials

Question Set No: 4

Which is bigger?

Which is bigger: π^e or e^π? Using a calculator confirms that e^π is the larger, but how can this be proved without the use of a calculator?

Simpler examples

It is often helpful in mathematics to consider simpler examples. It is easy to work out that $3^4 > 4^3$. In the expression 3^4, 3 is the base and 4 is the exponent. Working with integers greater than 1, it is easy to find many examples where $a^b > b^a$ if $a < b$. That is, using the smaller base and the larger exponent gives the larger result. This might lead us to conjecture that $a^b > b^a$ if $a < b$ and both a and b are integers greater than 1. However, it is also possible to find counter examples to this conjecture.

Exponents can also be rational numbers, and in general $x^{\frac{p}{q}}$ denotes $(\sqrt[q]{x})^p$ where p and q are integers and q is positive. So, any rational power of a positive number, x, can be defined. However, both e and π are irrational numbers. Considering the original question about π^e and e^π raises the issue of what is meant by an irrational power of a number.

Extending the definition of power to irrational numbers

What, for example, is meant by 2^π?

An irrational number corresponds to a non-recurring infinite decimal. Rounding the decimal gives a rational approximation to the irrational number. For example, the following sequence gives increasingly accurate approximations to π.

3, 3.1, 3.14, 3.142, 3.1416, 3.14159, …

Using a spreadsheet gives a sequence of approximations to 2^π, as shown in Fig. C1. The limit of this sequence of approximations is the value of 2^π. This limit cannot be evaluated with a spreadsheet but it is, in principle, possible to find the value to any required degree of accuracy.

	A	B
1	k	2^k
2	3	8
3	3.1	8.574188
4	3.14	8.815241
5	3.142	8.82747
6	3.1416	8.825023
7	3.14159	8.824962

Fig. C1

2^x and x^2 are increasing functions of x for $x > 0$ and this allows us to deduce that $\pi^2 > 2^\pi$, as follows.

We know that π is between 3 and 3.142

$\pi < 3.142 \Rightarrow 2^{\pi} < 2^{3.142} = 8.82747$

$\pi > 3 \Rightarrow \pi^2 > 3^2 = 9$

So $\pi^2 > 9 > 8.82747 > 2^{\pi}$

Hence $\pi^2 > 2^{\pi}$

Which is bigger: π^{e} or e^{π}?

An indirect method, using calculus, enables us to prove that e^{π} is larger than π^{e}. Fig. C2 shows the curve $y = \frac{1}{x}$ in the first quadrant together with the rectangle with vertices at the points (e, 0), $\left(\mathrm{e}, \frac{1}{\mathrm{e}}\right)$, $\left(\pi, \frac{1}{\mathrm{e}}\right)$ and $(\pi, 0)$. We use the fact that the area under the curve between e and π is less than the area of this rectangle.

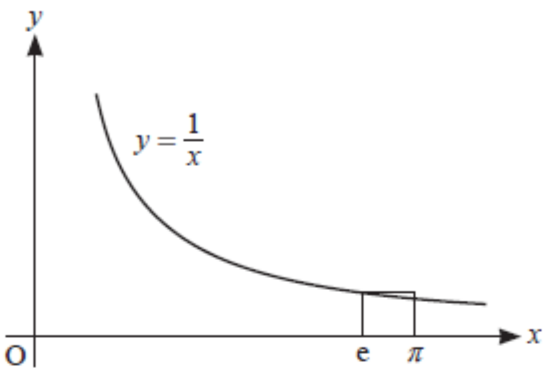

Fig. C2

The area of the rectangle is $\frac{1}{\mathrm{e}}(\pi - \mathrm{e})$

$\int_{\mathrm{e}}^{\pi} \frac{1}{x}\,\mathrm{d}x < \frac{1}{\mathrm{e}}(\pi - \mathrm{e})$

$\ln \pi - 1 < \frac{\pi}{\mathrm{e}} - 1$

$\ln \pi < \frac{\pi}{\mathrm{e}}$

e^x is an increasing function for all values of x

hence $\pi < \mathrm{e}^{\frac{\pi}{\mathrm{e}}}$

Assuming that the usual rules of indices apply to irrational powers of irrational numbers, raising both sides of the inequality to the power e gives the desired result.

Using a similar method, it can be shown that $\mathrm{e}^a > a^{\mathrm{e}}$ for any positive number $a \neq \mathrm{e}$.

An alternative method for showing that $\mathrm{e}^a > a^{\mathrm{e}}$ for any positive number a is to show that the only stationary point on the curve $y = \frac{\ln x}{x}$ (a maximum) occurs where $x = \mathrm{e}$.