

**A Level Mathematics B (MEI)**

**H640/03** MEI Pure Mathematics and Comprehension

**Question Set 4**

- 1 (a) Show that if  $a = 1$  and  $b > 1$  then  $a^b < b^a$ . [2]

$$\begin{aligned} \text{if } a &= 1 \\ a^b &= 1 \text{ and } b^a = b \\ b &> 1 \therefore b^a < a^b \end{aligned}$$

- (b) Find integer values of  $a$  and  $b$  with  $b > a > 1$  and  $a^b$  not greater than  $b^a$  (a counter example to the conjecture given in lines 7-8). [1]
- $$b = 4$$
- $$a = 3$$

- 2 In this question you must show detailed reasoning.

Show that  $\int_e^\pi \frac{1}{x} dx = \ln \pi - 1$  as given in line 37. [2]

$$\begin{aligned} &= \left[ \ln x \right]_e^\pi \\ &= \ln \pi - \ln e \\ &= \ln \pi - 1 \end{aligned}$$

- 3 Show that  $e^x$  is an increasing function for all values of  $x$ , as stated in line 39. [2]

$$\frac{dy}{dx} = e^x \frac{dy}{dx} > 0 \text{ for all values of } x \text{ and } e^2 > e \quad \therefore \text{no turning points}$$

∴ always increasing

- 4 (a) Show that the only stationary point on the curve  $y = \frac{\ln x}{x}$  occurs where  $x = e$ , as given in line 45. [3]

$$\ln x \rightarrow \frac{1}{x} \quad \frac{1 - \ln x}{x^2} = 0 \quad \ln x = 1$$

$$x = e$$

- (b) Show that the stationary point is a maximum. [3]

$$y = \frac{\ln x}{x} \quad \left| \begin{array}{l} u = 1 - \ln x \\ \frac{du}{dx} = -1/x \\ v = x^2 \\ \frac{dv}{dx} = 2x \end{array} \right.$$

$$\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$$

$$\frac{-x^2}{x} - 2x(1 - \ln x) = \frac{-x - 2x + 2x \ln x}{x^2} = \frac{-3x + 2x \ln x}{x^2}$$

$$= \frac{-3 + 2 \ln x}{x}$$

when  $x = e$

$$\frac{-3 - 2 \ln e}{e^2} = -0.049787 \dots$$

$$\rightarrow -0.0498 < 0 \therefore \text{a maximum}$$

- (c) It follows from part (b) that, for any positive number  $a$  with  $a \neq e$ ,

$$\frac{\ln e}{e} > \frac{\ln a}{a}$$

Use this fact to show that  $e^a > a^e$ . [2]

$$\begin{aligned} a \ln e &> e \ln a \\ \ln e^a &> \ln a^e \quad e^a > a^e \end{aligned}$$

Total Marks for Question Set 4: 15

# Resource Materials

Question Set No: 4

## Which is bigger?

Which is bigger:  $\pi^e$  or  $e^\pi$ ? Using a calculator confirms that  $e^\pi$  is the larger, but how can this be proved without the use of a calculator?

### Simpler examples

It is often helpful in mathematics to consider simpler examples. It is easy to work out that  $3^4 > 4^3$ . In the expression  $3^4$ , 3 is the base and 4 is the exponent. Working with integers greater than 1, it is easy to find many examples where  $a^b > b^a$  if  $a < b$ . That is, using the smaller base and the larger exponent gives the larger result. This might lead us to conjecture that  $a^b > b^a$  if  $a < b$  and both  $a$  and  $b$  are integers greater than 1. However, it is also possible to find counter examples to this conjecture. 5

Exponents can also be rational numbers, and in general  $x^{\frac{p}{q}}$  denotes  $(\sqrt[q]{x})^p$  where  $p$  and  $q$  are integers and  $q$  is positive. So, any rational power of a positive number,  $x$ , can be defined. However, both  $e$  and  $\pi$  are irrational numbers. Considering the original question about  $\pi^e$  and  $e^\pi$  raises the issue of what is meant by an irrational power of a number. 10

### Extending the definition of power to irrational numbers

What, for example, is meant by  $2^\pi$ ? 15

An irrational number corresponds to a non-recurring infinite decimal. Rounding the decimal gives a rational approximation to the irrational number. For example, the following sequence gives increasingly accurate approximations to  $\pi$ .

3, 3.1, 3.14, 3.142, 3.1416, 3.14159, ...

Using a spreadsheet gives a sequence of approximations to  $2^\pi$ , as shown in Fig. C1. The limit of this sequence of approximations is the value of  $2^\pi$ . This limit cannot be evaluated with a spreadsheet but it is, in principle, possible to find the value to any required degree of accuracy. 20

	A	B	
1	$k$	$2^k$	
2	3	8	
3	3.1	8.574188	
4	3.14	8.815241	
5	3.142	8.82747	
6	3.1416	8.825023	
7	3.14159	8.824962	

Fig. C1

$2^x$  and  $x^2$  are increasing functions of  $x$  for  $x > 0$  and this allows us to deduce that  $\pi^2 > 2^\pi$ , as follows.

We know that  $\pi$  is between 3 and 3.142

25

$$\pi < 3.142 \Rightarrow 2^\pi < 2^{3.142} = 8.82747$$

$$\pi > 3 \Rightarrow \pi^2 > 3^2 = 9$$

$$\text{So } \pi^2 > 9 > 8.82747 > 2^\pi$$

$$\text{Hence } \pi^2 > 2^\pi$$

Which is bigger:  $\pi^e$  or  $e^\pi$ ?

30

An indirect method, using calculus, enables us to prove that  $e^\pi$  is larger than  $\pi^e$ . Fig. C2 shows the curve  $y = \frac{1}{x}$  in the first quadrant together with the rectangle with vertices at the points  $(e, 0)$ ,  $(e, \frac{1}{e})$ ,  $(\pi, \frac{1}{e})$  and  $(\pi, 0)$ . We use the fact that the area under the curve between  $e$  and  $\pi$  is less than the area of this rectangle.

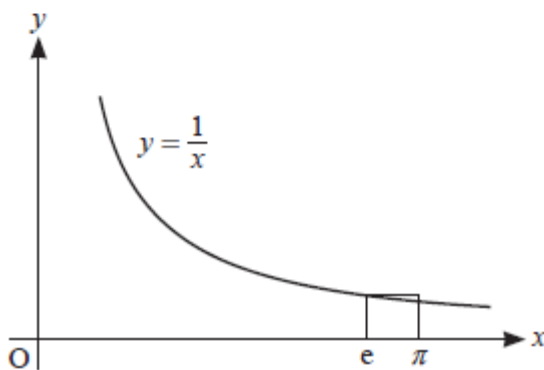


Fig. C2

The area of the rectangle is  $\frac{1}{e}(\pi - e)$

35

$$\int_e^\pi \frac{1}{x} dx < \frac{1}{e}(\pi - e)$$

$$\ln \pi - 1 < \frac{\pi}{e} - 1$$

$$\ln \pi < \frac{\pi}{e}$$

$e^x$  is an increasing function for all values of  $x$

$$\text{hence } \pi < e^{\frac{\pi}{e}}$$

40

Assuming that the usual rules of indices apply to irrational powers of irrational numbers, raising both sides of the inequality to the power  $e$  gives the desired result.

Using a similar method, it can be shown that  $e^a > a^e$  for any positive number  $a \neq e$ .

An alternative method for showing that  $e^a > a^e$  for any positive number  $a$  is to show that the only stationary point on the curve  $y = \frac{\ln x}{x}$  (a maximum) occurs where  $x = e$ .

45

---

# OCR

Oxford Cambridge and RSA

## **Copyright Information**

OCR is committed to seeking permission to reproduce all third-party content that it uses in its assessment materials. OCR has attempted to identify and contact all copyright holders whose work is used in this paper. To avoid the issue of disclosure of answer-related information to candidates, all copyright acknowledgements are reproduced in the OCR Copyright Acknowledgements Booklet. This is produced for each series of examinations and is freely available to download from our public website ([www.ocr.org.uk](http://www.ocr.org.uk)) after the live examination series.

If OCR has unwittingly failed to correctly acknowledge or clear any third-party content in this assessment material, OCR will be happy to correct its mistake at the earliest possible opportunity.

For queries or further information please contact The OCR Copyright Team, The Triangle Building, Shaftesbury Road, Cambridge CB2 8EA.

OCR is part of the Cambridge Assessment Group; Cambridge Assessment is the brand name of University of Cambridge Local Examinations Syndicate (UCLES), which is itself a department of the University of Cambridge