A Level Mathematics B (MEI)

H640/03 MEI Pure Mathematics and Comprehension

Question Set 3

1 Find the value of $\sum_{r=1}^{5} 2^r(r-1)$. **[2]**

$r=1$ $r=2$ $r=3$ $r=4$ $r=5$

$\rightarrow (2(1-1)) + (2^2(2-1)) + (2^3(3-1)) + (2^4(4-1)) + (2^5(5-1))$

$= 0 + 4 + (8 \times 2) + (16 \times 3) + (32 \times 4)$

$= 0 + 4 + 16 + 48 + 128$

$= 196$

2 The graph of $y = |1-x| - 2$ is shown in Fig. 2.

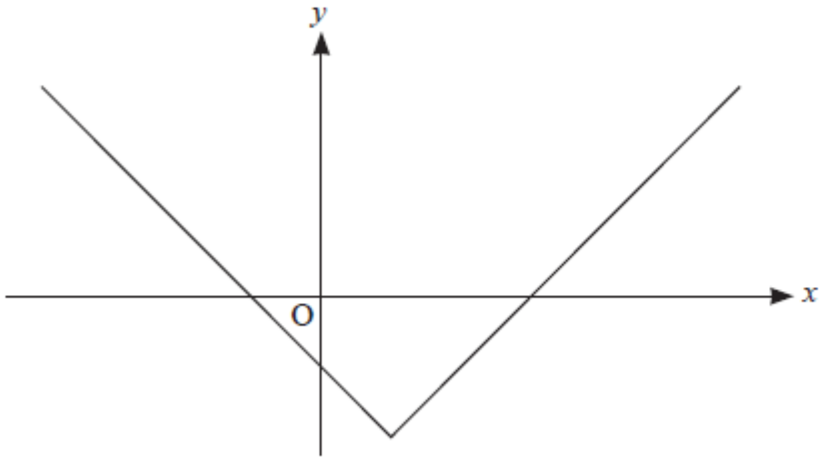


Fig. 2

Determine the set of values of x for which $|1-x| > 2$. **[4]**

$1 - x > 2$ OR $-1 + x > 2$

$-1 > 1$ $\quad x > 3$

3 A particular phone battery will last 10 hours when it is first used. Every time it is recharged, it will only last 98% of its previous time.

Find the maximum total length of use for the battery. **[3]**

$\sum_{r=1}^{\infty} 10 \times 0.98^{(r-1)} = \frac{a}{1-r}$ ($a = 10$, $r = 0.98$) $= \frac{10}{0.02} = 500$ hours

4 Fig. 4 shows the regular octagon ABCDEFGH.

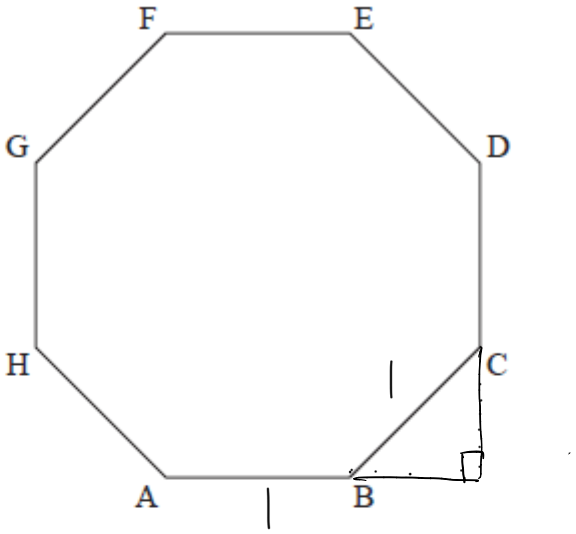


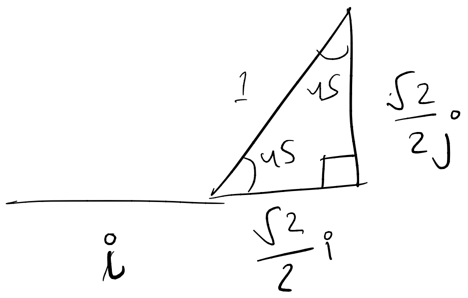
Fig. 4

$\overrightarrow{AB} = \mathbf{i}$, $\overrightarrow{CD} = \mathbf{j}$, where $\mathbf{i}$ is a unit vector parallel to the x-axis and $\mathbf{j}$ is a unit vector parallel to the y-axis.

Find an exact expression for $\overrightarrow{BC}$ in terms of $\mathbf{i}$ and $\mathbf{j}$. **[3]**

$$\overrightarrow{BC} = \frac{\sqrt{2}}{2}\,i + \frac{\sqrt{2}}{2}\,j$$

$$\overrightarrow{BC} = \frac{\sqrt{2}}{2}(i + j)$$



$$\frac{1}{\sin 90} \times \sin 45 = \frac{\sqrt{2}}{2}$$

5 Fig. 5 shows part of the curve $y = \operatorname{cosec} x$ together with the x- and y-axes.

$y = \frac{1}{\sin x}$

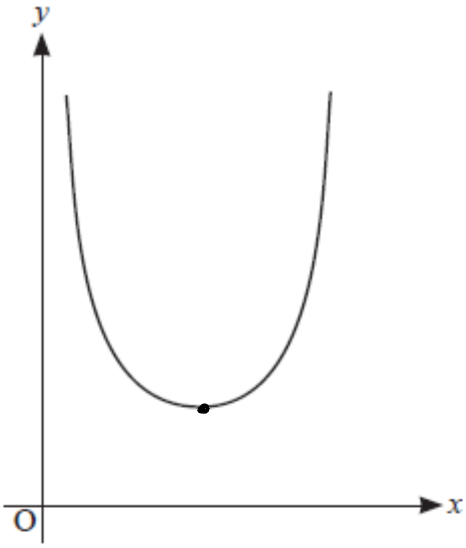


Fig. 5

(a) For the section of the curve which is shown in Fig. 5, write down

(i) the equations of the two vertical asymptotes, $x=0,\ x=\pi \rightarrow \sin\pi=0$ **[2]**

$\frac{1}{0}$ not possible.

(ii) the coordinates of the minimum point. $\left(\frac{\pi}{2}, 1\right)$ **[1]**

(b) Show that the equation $x = \operatorname{cosec} x$ has a root which lies between $x = 1$ and $x = 2$. **[2]**

$x = \operatorname{cosec} x$

$f(x) = x - \operatorname{cosec} x$

$f(1) = -0.1883951058$

$f(2) = 0.9002498297$

there is a sign change Therefore there is a root

(c) Use the iteration $x_{n+1} = \operatorname{cosec}(x_n)$, with $x_0 = 1$, to find

(i) the values of x_1 and x_2, correct to 5 decimal places, **[1]**

$x_1 = \operatorname{cosec} 1 = 1.188395106 = 1.18840$ (5dp)

$x_2 = \operatorname{cosec}(1.188...) = 1.07785184 = 1.07785$ (5dp)

(ii) this root of the equation, correct to 3 decimal places. 1.114 **[1]**

root = 1.14 (method, on your calculator do cosec (ANS) till answer doesn't change)

(d) There is another root of $x = \operatorname{cosec} x$ which lies between $x = 2$ and $x = 3$.

Determine whether the iteration $x_{n+1} = \operatorname{cosec}(x_n)$ with $x_0 = 2.5$ converges to this root. **[1]**

Yes (1.114)

(e) Sketch the staircase or cobweb diagram for the iteration, starting with $x_0 = 2.5$, on the diagram in the Resource Material. **[3]**

6 **(a)** **(i)** Write down the derivative of e^{kx}, where k is a constant. Ke^{kx} **[1]**

(ii) A business has been running since 2009. They sell maths revision resources online.

Give a reason why an exponential growth model might be suitable for the annual profits for the business. **[1]**

Popularity of the business will grow, so profits will also increase at a faster rate every year.

Fig. 6 shows the relationship between the annual profits of the business in thousands of pounds (y) and the time in years after 2009 (x). The graph of $\ln y$ plotted against x is approximately a straight line.

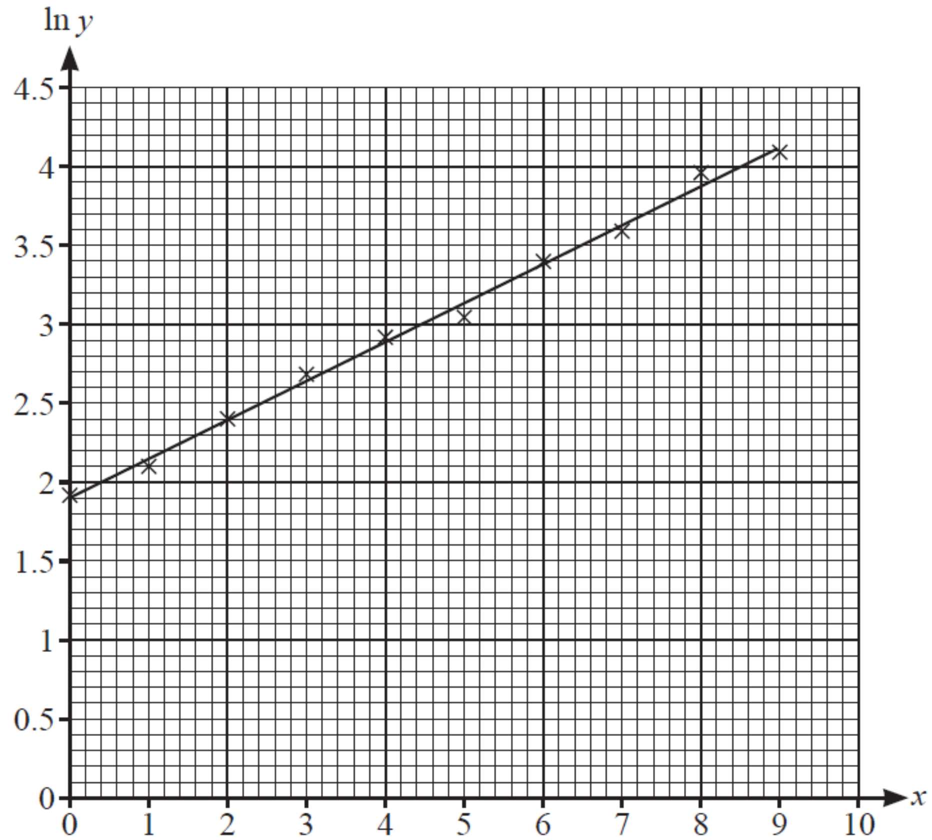


Fig. 6

(b) Show that the straight line is consistent with a model of the form $y = Ae^{kx}$, where A and k are constants. **[2]**

$\ln y = \ln Ae^{kx} \rightarrow \ln A + \cancel{\ln e}^{kx} = \underline{\ln A + kx} = \ln y$

(c) Estimate the values of A and k. [4]

$A = \ln 1.9$
$= 0.641853886 2$
$= 0.642$ (3sf)

$k = \dfrac{\ln 4.1 - \ln 1.9}{9} = 0.0854892319 5$
$= 0.0855$ (3sf)

(d) Use the model to predict the profit in the year 2020. [3]

$x = 11,\ y = 1.64$

$y = 0.642 e^{0.0855 \times 11}$

$y = 1.64433003$

$= £1644.33$

(e) How reliable do you expect the prediction in part (d) to be? Justify your answer. [1]

not as reliable as $x = 11$ is an extrapolation so the model is not as reliable.

7 (a) Express $\dfrac{1}{x}+\dfrac{1}{A-x}$ as a single fraction. [1]

$\dfrac{A-x+x}{x(A-x)} = \dfrac{A}{x(A-x)}$

(b) **In this question you must show detailed reasoning.**

Find the number of fish in the lake when $t = 10$, as predicted by the model. [8]

$\dfrac{dx}{dt} = \dfrac{x(400-x)}{400}$

$\dfrac{400}{x(400-x)}\,dx = 1\,dt \rightarrow \displaystyle\int \frac{400}{x(400-x)}\,dx = t + C$

$= \displaystyle\int \frac{1}{x} + \frac{1}{400-x}\,dx = t + c$

$= \ln x - \ln|400-x| = t + c$

find c:

$\ln x - \ln|400-x| = t + c$

$\ln 100 - \ln|400-100| = 0 + c$

$\ln 100 - \ln 300 = c$

$\ln \frac{1}{3} = c$

$\therefore \quad \ln x - \ln|400 - x| = t + \ln 1/3$

when $t = 10$

$$\ln x - \ln|400 - x| = 10 + \ln 1/3$$

$$\ln \frac{x}{400 - x} = 10 + \ln 1/3$$

$$\frac{x}{400 - x} = e^{10} + 1/3$$

$$x = 22026.79913\,(400 - x)$$

$$\underline{\underline{x = 400 \text{ fish}}}$$

8 (a) The curve $y = \dfrac{1}{(1+x^2)^2}$ is shown in Fig. 8.

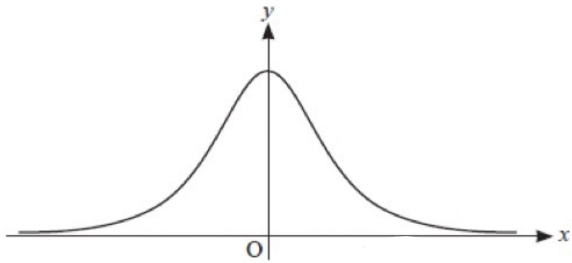


Fig. 8

(i) Show that $\dfrac{d^2y}{dx^2} = \dfrac{20x^2-4}{(1+x^2)^4}$.

$$y = (1+x^2)^{-2}$$

$$\frac{dy}{dx} = -2(2x)(1+x^2)^{-3}$$

$$= -4x(1+x^2)^{-3} = -\frac{4x}{(1+x^2)^3}$$

$$u = x \qquad v = (1+x^2)^3$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 6x(1+x)^2$$

$$\rightarrow -4\left(\frac{(1+x^2)^3 - 6x^2(1+x^2)^2}{(1+x^2)^6}\right)$$

$$= -4\left(\frac{-5x^2+1}{(1+x^2)^4}\right) = \frac{-20x^2+4}{(1+x^2)^4}$$

(ii) **In this question you must show detailed reasoning.**

Find the set of values of x for which the curve is concave downwards. [3]

$$\frac{d^2y}{dx^2} < 0 \longrightarrow 20x^2 - 4 < 0$$

$$20x^2 < 4$$

$$x^2 < \frac{1}{5}$$

$$x < \pm\frac{1}{\sqrt{5}}$$

$$0 < x < \frac{1}{\sqrt{5}}$$

(b) Use the substitution $x = \tan\theta$ to find the exact value of $\int_{-1}^{1} \frac{1}{(1+x^2)^2}\,dx$. [8]

$$\int_{-1}^{1} \frac{1}{(1+x^2)^2}\,dx$$

$x = \tan\theta$

$\frac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta\,d\theta$

$$\int_{-1}^{1} \frac{1}{(1+\tan\theta^2)^2}\sec^2\theta\,d\theta$$

$$\int_{-1}^{1} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int_{-1}^{1} \frac{1}{\sec^2\theta}\,d\theta = \int_{-1}^{1} \cos^2\theta\,d\theta$$

$$\rightarrow \int_{-1}^{1} \frac{\cos 2\theta + 1}{2}\,d\theta = \frac{1}{2}\int_{-1}^{1} \cos 2\theta + 1\,d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right) - \frac{1}{2}\left(-\frac{1}{2} - \frac{\pi}{4}\right)$$

$$= \frac{1}{4} + \frac{\pi}{8} + \frac{1}{4} + \frac{\pi}{8} = \frac{2}{4} + \frac{2\pi}{8} = \frac{2}{4} + \frac{\pi}{4}$$

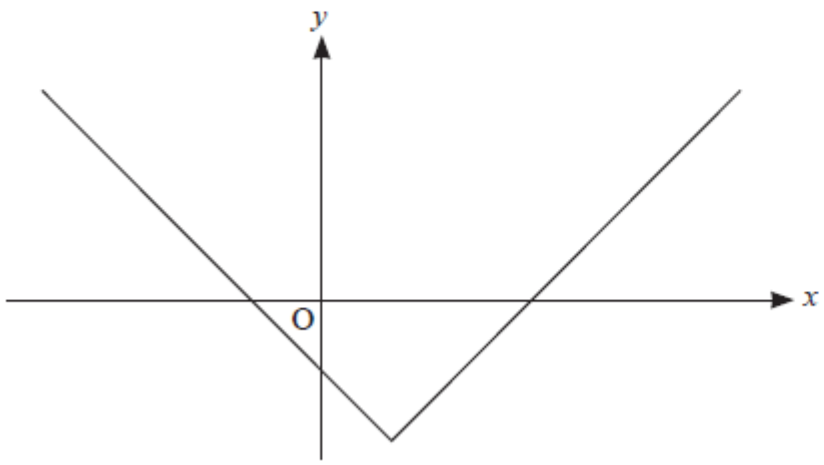
$$= \underline{\underline{\frac{2+\pi}{4}}} = \text{exact value}$$

Total Marks for Question Set 3: 60

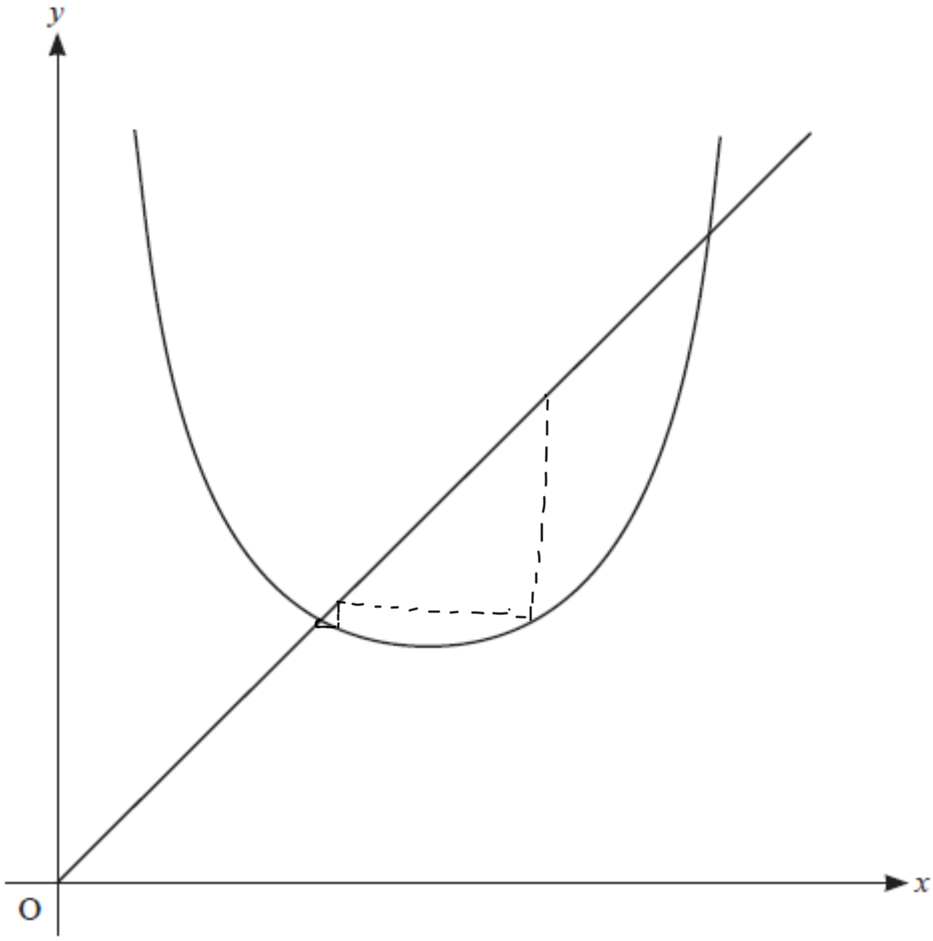
Resource Materials

Question Set No: 3

Q2



Q5(e)



Q6(c)