Oxford Cambridge and RSA

A Level Mathematics B (MEI)

H640/03 MEI Pure Mathematics and Comprehension

Question Set 2

1 Line 8 states that $\frac{a+b}{2} \geqslant \sqrt{ab}$ for $a,\ b \geqslant 0$. Explain why the result cannot be extended to apply in each of the following cases.

(a) One of the numbers a and b is positive and the other is negative **[1]**

(b) Both numbers a and b are negative **[1]**

2 Lines 5 and 6 outline the stages in a proof that $\frac{a+b}{2} \geqslant \sqrt{ab}$. Starting from $(a-b)^2 \geqslant 0$, give a detailed proof of the inequality of arithmetic and geometric means. **[3]**

3 Consider a geometric sequence in which all the terms are positive real numbers. Show that, for any three consecutive terms of this sequence, the middle one is the geometric mean of the other two. **[3]**

4 (a) In Fig. C1.3, angle CBD = θ. Show that angle CDA is also θ, as given in line 23. **[2]**

(b) Prove that $h = \sqrt{ab}$, as given in line 24. **[2]**

5 It is given in lines 31–32 that the square has the smallest perimeter of all rectangles with the same area. Using this fact, prove by contradiction that among rectangles of a given perimeter, $4L$, the square with side L has the largest area. **[3]**

Total Marks for Question Set 2: 15

Resource Materials

Question Set No: 2

Arithmetic and Geometric Means

Arithmetic and geometric mean of two numbers

For two real numbers a and b, the arithmetic mean of the numbers is defined to be $\frac{a+b}{2}$. For two non-negative real numbers a and b, the geometric mean of the two numbers is defined to be $\sqrt{ab}$.

Squares of real numbers cannot be negative, so we know that $(a-b)^2 \geqslant 0$. It follows that $a^2+b^2 \geqslant 2ab$ and so $(a+b)^2 \geqslant 4ab$. Hence the arithmetic mean of two real non-negative numbers is greater than, or equal to, their geometric mean.

$$\frac{a+b}{2} \geqslant \sqrt{ab} \text{ for } a, b \geqslant 0$$

This result is known as the inequality of the arithmetic and geometric means. If the two numbers a and b are equal then the arithmetic mean equals the geometric mean.

The three real numbers a, $\frac{a+b}{2}$, b form an arithmetic sequence. The three non-negative real numbers a, $\sqrt{ab}$, b form a geometric sequence.

Constructing the arithmetic and geometric mean of two numbers

Lengths representing the arithmetic and geometric mean of two positive numbers can be constructed with a straight edge and compasses.

Fig. C1.1 shows a straight line ACB with AC of length a and CB of length b.

Fig. C1.1

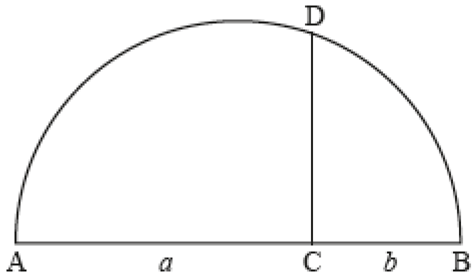

Fig. C1.2

The line AB is first bisected, to locate its midpoint. A semicircle with AB as diameter is then drawn, and a line at C perpendicular to the diameter is constructed. Fig. C1.2 shows this semicircle, with the perpendicular line through C meeting the semicircle at D.

The radius of the semicircle is the arithmetic mean of a and b, and the length of CD is the geometric mean of a and b.

To prove that the length of CD is the geometric mean of a and b, consider triangles ACD and BCD, as shown in Fig. C1.3. Letting angle CBD $= \theta$, it follows that angle CDA is also θ. Finding expressions for $\tan\theta$ in each of triangles ACD and BCD leads to $h = \sqrt{ab}$, where h is the length of CD.

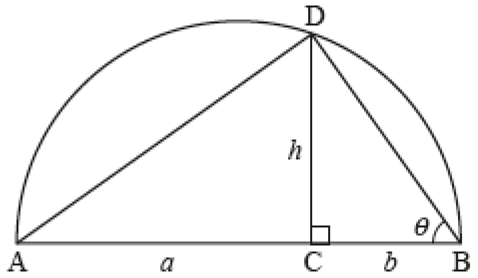

Fig. C1.3

The relationship between a, b and h in Fig. C1.3 means that a square with side CD has the same area as a rectangle with sides equal to AC and CB. Fig. C2 shows the square and a rectangle ACFG with CF equal in length to CB. This diagram illustrates how a straight edge and compasses can be used to construct a square with area equal to that of a given rectangle. This method appears in Euclid's books on Geometry (the 'Elements') which were published around 2300 years ago.

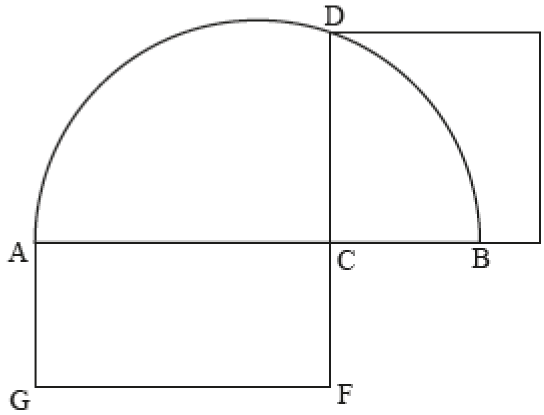

Fig. C2

Areas of rectangles

The inequality of arithmetic and geometric means implies that the square has the smallest perimeter of all rectangles with the same area.

Consider a rectangle of given area A that has sides of lengths x and y, so that $xy = A$. The perimeter of this rectangle is $2(x + y)$. From the inequality of arithmetic and geometric means, we know that $\frac{x+y}{2} \geqslant \sqrt{xy}$ so that $2(x+y) \geqslant 4\sqrt{xy}$. But the right-hand side of this last inequality has the fixed value $4\sqrt{A}$ whatever x and y are. For a square of area A, each side has length $\sqrt{A}$ and so $4\sqrt{A}$ is the perimeter of this square. Therefore, the perimeter of any rectangle of area A is not less than this, so the square has the smallest perimeter of all rectangles with given area.