A Level Mathematics B (MEI)

H640/01 MEI Pure Mathematics and Mechanics

Question Set 5

1 Show that $(x-2)$ is a factor of $3x^3 - 8x^2 + 3x + 2$. [3]

if $(x-2)$ is a factor $f(2) = 0$

$(3 \times 2^3) - (8 \times 2^2) + (3 \times 2) + 2$

$24 - 32 + 6 + 2 = 0$

$\therefore$ $(x-2)$ is a factor

2 By considering a change of sign, show that the equation $e^x - 5x^3 = 0$ has a root between 0 and 1. [2]

When $x = 0$, $e^0 - 5(0^3) = 1 > 0$

When $x = 1$, $e^1 - 5(1^3) = e - 5 < 0$.

So there is a change of sign between $x = 0$ and $x = 1$ so there is a root.

3 **In this question you must show detailed reasoning.**

Solve the equation $\sec^2\theta + 2\tan\theta = 4$ for $0° \leqslant \theta < 360°$. [4]

$\sec^2\theta = \tan^2\theta + 1$

$\therefore$

$\tan^2\theta + 1 + 2\tan\theta - 4 = 0$

$\tan^2\theta + 2\tan\theta - 3 = 0$

$\tan\theta = x$

$x^2 + 2x - 3 = 0$

$x = 1$

$x = -3$

$\tan\theta = 1$

$\tan\theta = -3$

$\theta = 45, 225, 108.4, 288.4$

4 Aleela and Baraka are saving to buy a car. Aleela saves £50 in the first month. She increases the amount she saves by £20 each month.

(a) Calculate how much she saves in two years. [2]

$a_1 = 50 \quad d = 20$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_n = \frac{24}{2}((2 \times 50) + (23 \times 20))$

$S_n = 6720$

(b) Explain why the amounts Baraka saves each month form a geometric sequence. [1]

Because the amount that she is saving is increasing by 1.12 each month. Therefore it can be modelled as a geometric sequence.

(c) Determine whether Baraka saves more in two years than Aleela. [3]

$S_n = \frac{a_1(1-r^n)}{1-r}$

$S_n = \frac{50(1 - 1.12^{24})}{1 - 1.12}$

$S_n = 5707$

$\therefore$ Aleela saves more as $6720 > 5907$

5 **(a)** Show that $8\sin^2 x\cos^2 x$ can be written as $1 - \cos 4x$. [3]

$8\sin^2 x \cos^2 x$

$8\sin x(1 - \sin^2 x)$

$\cos 2x = 1 - 2\sin^2 x$

$\frac{-\cos 2x + 1}{2} = \sin^2 x$

$8\left(\frac{1-\cos 2x}{2}\right)\left(\frac{2}{2} - \left(\frac{1-\cos 2x}{2}\right)\right)$

$8\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1}{2} + \frac{\cos 2x}{2}\right)$

$2(1 - \cos 2x)(1 + \cos 2x)$

$2(1 - \cos^2 2x)$

$\cos 4x = 2\cos^2 2x - 1$

$2\left(\frac{2}{2} - \frac{\cos 4x + 1}{2}\right)$

$1(1 - \cos 4x)$

OR

$8\sin^2 x\cos^2 x$

$= 2(2\sin x\cos x)^2$

$= 2\sin^2 2x$

$= 1 - \cos 4x$

(b) Hence find $\int \sin^2 x \cos^2 x \, dx$. [3]

$$\int \sin^2 x \cos^2 x \, dx$$

$$\frac{1}{8}\int 1 - \cos 4x \, dx$$

$$\frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]$$

$$\frac{x}{8} - \frac{\sin 4x}{32} + C$$

6 Fig. 6 shows the graph of $y = (k - x)\ln x$ where k is a constant ($k > 1$).

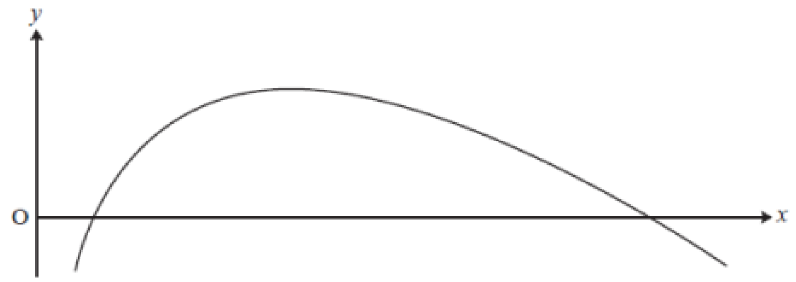

Fig. 6

Find, in terms of k, the area of the finite region between the curve and the x-axis. [8]

$$y = (k - x)\ln x$$

$$u = \ln x \qquad \int \rightarrow v = kx - \frac{x^2}{2}$$

$$\frac{dv}{dx} = k - x$$

$$\therefore (k-x)(\ln x) - \int \left(kx - \frac{x^2}{2}\right)\left(\frac{1}{x}\right)$$

$$k\ln x - x\ln x - kx - \frac{x^2}{4} + C$$

$$kx\ln x - \frac{x^2}{2}\ln(x)$$

$$x\ln x\left(k - \frac{x}{2}\right) - x\left(k - \frac{x}{4}\right) + C$$

$$0 = (k - x)\ln x$$

$$x = 1$$

$$x = k$$

$$\left[x\ln x\left(k - \frac{x}{2}\right) - x\left(k - \frac{x}{4}\right)\right]_1^k$$

$$\left[k^2\ln k - \frac{k^2}{2}\ln k - k^2 + \frac{k^2}{4}\right] - \left[-k + 1/4\right]$$

$$\frac{1}{2}k^2\ln k - \frac{3}{4}k^2 + k - \frac{1}{4}$$

7 Fig. 7 shows the circle $(x-1)^2+(y+1)^2=25$, the line $4y=3x-32$ and the tangent to the circle at the point A (5, 2). D is the point of intersection of the line $4y=3x-32$ and the tangent at A.

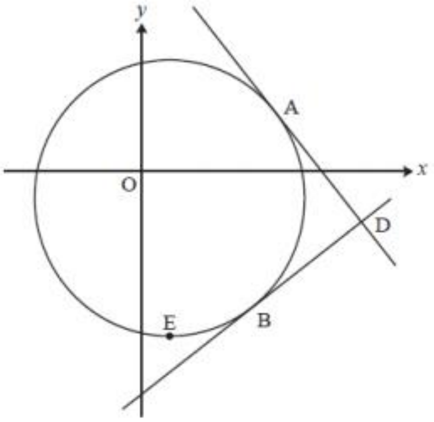

Fig. 7

(a) Write down the coordinates of C, the centre of the circle. [1]

$(1, -1)$

(b) (i) Show that the line $4y = 3x - 32$ is a tangent to the circle. [4]

$4y = 3x - 32$

$y = 3/4x - 8$

$(x-1)^2 + ((\frac{3}{4}x - 8) + 1)^2$

$(x^2 - 2x + 1) + (3/4x - 7)^2$

$x^2 - 2x + 1 + \frac{9x^2}{16} - \frac{21x}{2} + 49 = 25$

$\frac{25}{16}x^2 - \frac{25}{2}x + 25 = 0$

because the discriminant is zero they only touch at one point therefore it is a tangent. ← $b^2 - 4ac = \left(\frac{-25}{2}\right)^2 - \left(4 \times \frac{25}{16} \times 25\right) = 0$

(ii) Find the coordinates of B, the point where the line $4y = 3x - 32$ touches the circle. [1]

$\frac{25}{16}x^2 - \frac{25}{2}x + 25 = 0$ $\therefore (4, -5)$

$x = 4$

$\therefore y = 9^2 + (y+1)^2 = 25$

$(y+1)^2 = 16$

$y+1 = \pm\sqrt{16}$

$y+1 = -4$

$y = -5$

(c) Prove that ADBC is a square. [3]

$\overrightarrow{CA}$ $(5, 2) - (1, -1) = (4, 3)$

$|\overrightarrow{CA}| = 5$

$\overrightarrow{CB}$ $(4, -5) - (1, -1) = (3, -4)$

$|\overrightarrow{CB}| = 5$

gradient of $\vec{AD} = -4/3$

gradient of $\vec{BD} = 3/4$

$\therefore$ ADBC is a square.

8 The function f(x) is defined by $f(x) = \sqrt[3]{27-8x^3}$. Jenny uses her scientific calculator to create a table of values for f(x) and f′(x).

x	f(x)	f′(x)
0	3	0
0.25	2.9954	−0.056
0.5	2.9625	−0.228
0.75	2.8694	−0.547
1	2.6684	−1.124
1.25	2.2490	−1.977
1.5	0	ERROR

(a) Use calculus to find an expression for f′(x) and hence explain why the calculator gives an error for f′(1.5). [3]

$f(x) = (27-8x^3)^{1/3}$

$f'(x) = \frac{1}{3} \times -24x^2 \times (27-8x^3)^{-2/3}$

$= \frac{-8x^2}{(27-8x^3)^{2/3}}$

$f'(1.5) = -\frac{8 \times 1.5^2}{0}$ and dividing by 0 gives the error

(b) Find the first three terms of the binomial expansion of f(x). [3]

$27^{1/3}\left(1 - \frac{8}{27}x^3\right)^{1/3}.$

$3\left(1 - \frac{8}{27}x^3\right)^{1/3}$

$1 + \left(\frac{1}{3} \times \left(\frac{-8}{27}x^3\right)\right) + \frac{(1/3)(1/3-1)}{2!}\left(\frac{-8}{27}x^3\right)^2$

$3\left(1 - \frac{8}{81}x^3 - \frac{64}{6561}x^6\right) = 3 - \frac{8}{27}x^3 - \frac{64}{2187}x^6.$

(c) Jenny integrates the first three terms of the binomial expansion of f(x) to estimate the value of $\int_0^1 \sqrt[3]{27-8x^3}\,dx$. Explain why Jenny's method is valid in this case. (You do not need to evaluate Jenny's approximation.) [2]

The binomial expansion is valid for $\left|-\frac{8}{27}x^3\right| < 1$ so $|x| < 1.5$. As the limits of the integral are within $|x| < 1.5$, this is valid.

(d) Use the trapezium rule with 4 strips to obtain an estimate for $\int_0^1 \sqrt[3]{27-8x^3}\,dx$. **[3]**

$\frac{0.25}{2}\left(3+2.6684+2(2.9954+2.9625+2.8694)\right)$

$=2.9153$

The calculator gives 2.921 174 38 for $\int_0^1 \sqrt[3]{27-8x^3}\,dx$. The graph of $y = f(x)$ is shown in Fig. 8

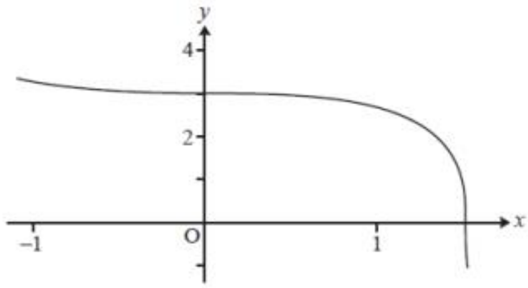

Fig. 8

(e) Explain why the trapezium rule gives an underestimate. **[1]**

Because the curve concaves downwards.

Total Marks for Question Set 5: 55