A Level Mathematics B (MEI)

H640/01 MEI Pure Mathematics and Mechanics

Pure

Question Set 1

1. **In this question you must show detailed reasoning.**

Show that $\int_4^9 (2x+\sqrt{x})\mathrm{d}x = \frac{233}{3}$. [3]

$\int_4^9 2x+\sqrt{x}\ dx$

$\int_4^9 2x+x^{1/2}\ dx$

$\left[\frac{2}{2}x^2+\frac{2}{3}x^{3/2}\right]_4^9$

$\left[9^2+\frac{2}{3}\times 9^{3/2}\right]-\left[4^2-\frac{2}{3}\times 4^{3/2}\right]$

$99-\frac{64}{3}=\frac{233}{3}$

2. Show that the line which passes through the points $(2, -4)$ and $(-1, 5)$ does not intersect the line $3x+y=10$. [3]

$\frac{5--4}{-1-2}=-3$

$y=mx+c$

$-4=(-3\times 2)+c$

$2=c$

$y=-3x+2$

$y=10-3x$

$-3x+2=10-3x$

$0=8$

this is inconsistent therefore the lines do not meet

3. The function $f(x)$ is given by $f(x) = (1 - ax)^{-3}$, where a is a non-zero constant. In the binomial expansion of $f(x)$, the coefficients of x and x^2 are equal.

a) Find the value of a. [3]

$$f(x) = (1 - ax)^{-3}$$

$$= 1 + (-ax)(-3) + \frac{(-3-1)(-3)}{2!}(-ax)^2$$

$$= 1 + 3ax + 6a^2x^2$$

$$3ax = 6a^2x^2$$

$$3a = 6a^2$$

$$1 = 2a$$

$$\tfrac{1}{2} = a$$

b) Using this value for a,

(i) state the set of values of x for which the binomial expansion is valid, [1]

$$-1 < \tfrac{1}{2}a < 1$$

$$-2 < a < 2 \qquad |a| < 2$$

c) (ii) write down the quadratic function which approximates $f(x)$ when x is small. [1]

$$\left(3 \times \frac{1}{2}\right)x + 6 \times \left(\frac{1}{2}\right)^2 (x^2) + 1$$

$$\frac{3}{2}x^2 + \frac{3}{2}x + 1$$

4. Prove that $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{1}{\sin\theta} = \cot\theta$. [4]

$$\frac{\sin\theta}{1-\cos\theta} - \frac{1}{\sin\theta}$$

$$\frac{(\sin\theta)(\sin\theta) - 1(1-\cos\theta)}{(1-\cos\theta)(\sin\theta)}$$

$$\frac{\sin^2\theta - 1 + \cos\theta}{(1-\cos\theta)(\sin\theta)}$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\frac{1-\cos^2\theta - 1 + \cos\theta}{(1-\cos\theta)(\sin\theta)} = \frac{\cos\theta\cancel{(-\cos\theta+1)}}{\cancel{(1-\cos\theta)}(\sin\theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

b) Hence find the exact roots of the equation $\frac{\sin\theta}{1-\cos\theta} - \frac{1}{\sin\theta} = 3\tan\theta$ in the interval $0 \leqslant \theta \leqslant \pi$. [3]

$\cot\theta = 3\tan\theta$

$\frac{1}{\tan\theta} = 3\tan\theta$

$1 = 3\tan^2\theta$

$\frac{1}{3} = \tan\theta$

$\pm\sqrt{\frac{1}{3}} = \tan\theta$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

5. An arithmetic series has first term 9300 and 10th term 3900.

a) Show that the 20th term of the series is negative. [3]

$a_n = a_1 + (n-1)d$

$3900 = 9300 + (10-1)d$

$-600 = d$

$a_{20} = 9300 + (-600 \times 19)$

$a_{20} = -2100$

a_{20} is negative

b) The sum of the first n terms is denoted by S. Find the greatest value of S as n varies. [4]

$S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{n}{2}((2\times 9300) + (n-1)(-600))$

$= \frac{n}{2}(18600 - 600n + 600)$

$S_n = 9600n - 300n^2$. $\frac{d}{dn} = 9600 - 600n$.

max value of S_n is when n is 16

$S_n = (9600 \times 16) - (300 \times 16^2)$

$S_n = 76800$

6.

a) Express $7\cos x - 2\sin x$ in the form $R\cos(x+\alpha)$ where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact value of R and the value of α correct to 3 significant figures. **[4]**

$7\cos x - 2\sin x = R\cos(x+a)$

$= R\cos x\cos a - R\sin x\sin a$

$7 = R\cos\alpha$

$-2 = -R\sin\alpha$

$2 = R\sin\alpha$

$\tan\alpha = 2/7$

$\alpha = 0.278$

$R = \sqrt{7^2+2^2}$

$R = \sqrt{53}$

$= \sqrt{53}\cos(x+0.278)$

b) Give details of a sequence of two transformations which maps the curve $y = \sec x$ onto the curve $y = \dfrac{1}{7\cos x - 2\sin x}$. **[3]**

$y = \dfrac{1}{\sqrt{53}(x+0.278)}$

translate 0.278 to the left

Stretch scale factor $\frac{1}{\sqrt{53}}$ parallel to the y axis

7. Fig. 7 shows a circle with centre O and radius r cm. The chord AB is such that angle AOB $= x$ radians. The area of the shaded segment formed by AB is 5% of the area of the circle.

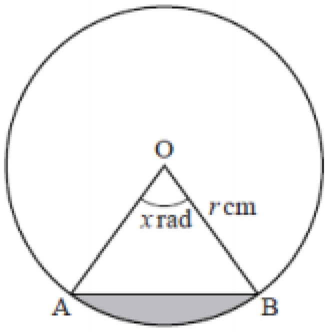


Fig. 7

a) Show that $x - \sin x - \frac{1}{10}\pi = 0$. [4]

The Newton-Raphson method is to be used to find x.

Sector Area $= \frac{1}{2}\theta r^2 = \frac{1}{2}x r^2$

Triangle OAB area $= \frac{1}{2}ab\sin C = \frac{1}{2}r^2\sin x$

Shaded Area $= \frac{1}{2}x r^2 - \frac{1}{2}\sin x r^2 = \frac{1}{2}r^2(x - \sin x)$

Shaded Area $= 0.05 \times \pi r^2$ so

$\frac{1}{2}r^2(x - \sin x) = 0.05\pi r^2$

$x - \sin x = \frac{1}{10}\pi$ so $x - \sin x - \frac{1}{10}\pi = 0$

b) Write down the iterative formula to be used for the equation in part (a). [1]

$f(x_n) = x_n - \sin x_n - \frac{1}{10}\pi$ so $f'(x_n) = 1 - \cos x_n$

$$x_{n+1} = x_n - \frac{x_n - \sin x - \frac{1}{10}\pi}{1 - \cos x_n}$$

c) Use three iterations of the Newton-Raphson method with $x_0 = 1.2$ to find the value of x to a suitable degree of accuracy. [3]

$x_0 = 1.2$, $x_1 = 1.27245...$, $x_2 = 1.26895...$,

$x_3 = 1.26894...$ so $x = 1.269$ to 3dp

8. A model for the motion of a small object falling through a thick fluid can be expressed using the differential equation

$$\frac{dv}{dt} = 9.8 - kv,$$

where $v \, \mathrm{m\,s^{-1}}$ is the velocity after t s and k is a positive constant.

a) Given that $v = 0$ when $t = 0$, solve the differential equation to find v in terms of t and k. [7]

$$\frac{dv}{dt} = 9.8 - kv$$

$$\frac{1}{9.8 - kv}\frac{dv}{dt} = 1$$

$$\int \frac{1}{9.8 - kv}\, dv = \int 1 \, dt$$

$$-\frac{1}{k}\ln|9.8 - kv| = t + C$$

$$-\tfrac{1}{k}\ln(9.8) = C$$

$$-\frac{1}{k}\ln|9.8 - kv| = t - \tfrac{1}{k}\ln(9.8)$$

$$\ln|9.8 - kv| = \ln(9.8) - tk$$

$$9.8kv = e^{\ln(9.8) - tk}$$

$$= e^{\ln(9.8)} \div e^{tk}$$

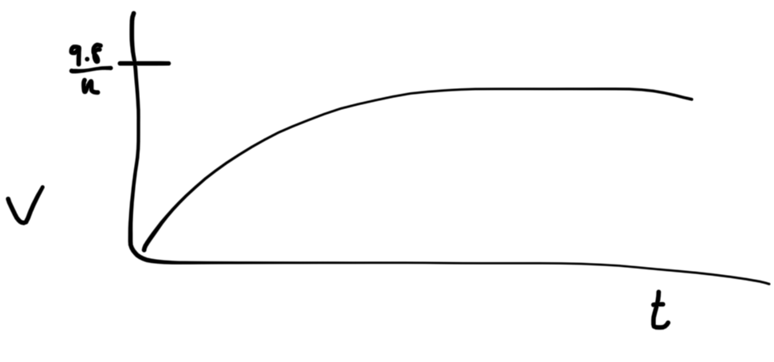
$$= \frac{9.8}{e^{tk}}$$

$$-kv = \frac{9.8}{e^{tk}} - 9.8$$

$$kv = 9.8 - \frac{9.8}{e^{tk}}$$

$$v = \frac{9.8}{k}\left(1 - \frac{1}{e^{tk}}\right)$$

b) Sketch the graph of v against t. [2]



c) Find the value of k. [2]

$$\frac{9.8}{k} = 7$$

$$k = \frac{9.8}{7}$$

$$k = 1.4$$

d) Find the value of t for which $v = 3.5$. [1]

$$3.5 = \frac{9.8}{1.4}\left(1 - \frac{1}{e^{t \times 1.4}}\right)$$

$$\frac{1}{2} = \left(1 - \frac{1}{e^{t \times 1.4}}\right)$$

$$e^{t \times 1.4} = 2$$

$$t \times 1.4 = \ln 2$$

$$t = \ln 2 \div 1.4$$

$$t = 0.495$$

$$t = 0.5$$

Total Marks for Question Set 1: 52