Pearson Edexcel

Model Solutions

Additional Assessment Materials

Summer 2021

Pearson Edexcel GCE in Mathematics

9MA0 (Public release version)

Resource Set 1: Topic 8

Integration

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Additional Assessment Materials, Summer 2021

General guidance to Additional Assessment Materials for use in 2021

Context

- Additional Assessment Materials are being produced for GCSE, AS and A levels (with the exception of Art and Design).
- The Additional Assessment Materials presented in this booklet are an optional part of the range of evidence teachers may use when deciding on a candidate's grade.
- 2021 Additional Assessment Materials have been drawn from previous examination materials, namely past papers.
- Additional Assessment Materials have come from past papers both published (those materials available publicly) and unpublished (those currently under padlock to our centres) presented in a different format to allow teachers to adapt them for use with candidate.

Purpose

- The purpose of this resource to provide qualification-specific sets/groups of questions covering the knowledge, skills and understanding relevant to this Pearson qualification.
- This document should be used in conjunction with the mapping guidance which will map content and/or skills covered within each set of questions.
- These materials are only intended to support the summer 2021 series.

1. A curve C has equation $y = \mathrm{f}(x)$

Given that

- $\mathrm{f}'(x) = 6x^2 + ax - 23$ where a is a constant
- the y intercept of C is -12
- $(x + 4)$ is a factor of $\mathrm{f}(x)$

find, in simplest form, $\mathrm{f}(x)$

(6)

We know that $\int f'(x)\,dx = f(x) + C = y + C$

$y = \int 6x^2 + ax - 23 \; dx = 2x^3 + \dfrac{ax^2}{2} - 23x + C$.

We are told that the y-intercept is $-12 \Rightarrow y = 2x^3 + \dfrac{ax^2}{2} - 23x - 12$

$x+4$ factor $\Rightarrow f(-4) = 0$, so we can solve for a.

$f(-4) = 2(-4)^3 + \dfrac{a(-4)^2}{2} - 23(-4) - 12 \Rightarrow 8a = 48 \Rightarrow a = 6$

$\Rightarrow y = f(x) = 2x^3 + 3x^2 - 23x - 12$

$f(x) = (x+4)(2x^2 - 5x - 3)$

$f(x) = (x+4)(x-3)(2x+1)$

	$2x^2$	$-5x$	-3
x	$2x^3$	$-5x^2$	$-3x$
$+4$	$8x^2$	$-20x$	-12

(Total for Question 1 is 6 marks)

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2. (*a*) Given that

$$\frac{x^2 + 8x - 3}{x + 2} \equiv Ax + B + \frac{C}{x + 2} \qquad x \in ¡ \quad x \neq -2$$

find the values of the constants A, B and C

(3)

$x^2 + 8x - 3 = Ax(x+2) + B(x+2) + C$

$x = -2 \Rightarrow -15 = C$

$x = 0 \Rightarrow -3 = 2B - 15 \Rightarrow 2B = 12 \Rightarrow B = 6$

$x = 1 \Rightarrow 6 = 3A + 6\cdot 3 - 15 \Rightarrow 3A = 3 \Rightarrow A = 1 \Rightarrow A = 1,\ B = 6$ and $C = -15$ $\therefore x + 6 - \dfrac{15}{x+2}$

(*b*) Hence, using algebraic integration, find the exact value of

$$\int_0^6 \frac{x^2+8x-3}{x+2}\,dx$$

giving your answer in the form $a + b \ln 2$ where a and b are integers to be found.

(4)

$$\int_0^6 \frac{x^2+8x-3}{x+2}\,dx = \int_0^6 x+6-\frac{15}{x+2}\,dx$$

$$= \int_0^6 x + \int_0^6 6 - 15\int_0^6 \frac{1}{x+2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^6 + \left[6x\right]_0^6 - 15 \times 2\ln 2$$

$$= 54 - 30\ln(2)$$

$= a + b\ln(2)$ where $\underline{a = 54}$ and $\underline{b = -30}$

$$= \int_2^8 \frac{1}{u}\,du$$

$u = x+2$

new limits

$u = 6+2 = 8$

$u = 0+2 = 2$

$= \ln(8) - \ln(2)$

$= \ln(2^3) - \ln(2)$

$= 3\ln(2) - \ln(2)$

$= 2\ln(2)$

(Total for Question 2 is 7 marks)

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3. Show that $\displaystyle\int_0^2 2x\sqrt{x+2}\,dx = \frac{32}{15}(2+\sqrt{2})$.

(7)

$\displaystyle\int_0^2 2x\sqrt{x+2}\,dx = 2\int_0^2 x\sqrt{x+2}\,dx$ then for integration by substitution, let $u = x+2$

$u = 0+2 = 2$

$u = 2+2 = 4$

$$= 2\int_2^4 (u-2)\sqrt{u}\,du = 2\int_2^4 u^{3/2} - \int 2u^{1/2}\,du = 2\left[\frac{2}{5}u^{5/2}\right]_2^4 - 2\left[\frac{4}{3}u^{3/2}\right]_2^4$$

$$= \left(2 \times \frac{2}{5} \times 4^{5/2} - 2 \times \frac{2}{5} \times 2^{5/2}\right) - \left(2 \times \frac{4}{3} \times 4^{3/2} - 2 \times \frac{4}{3} \times 2^{3/2}\right)$$

$$= \left(\frac{128}{5} - \frac{4}{5}\cdot 2^{5/2}\right) - \left(\frac{64}{3} - \frac{8}{3}\cdot 2^{3/2}\right)$$

$$= \frac{64}{15} - \frac{4}{5}\cdot 2^{5/2} + \frac{8}{3}\cdot 2^{3/2}$$

$$= \left(\frac{64}{15} + \frac{32\cdot\sqrt{2}}{15}\right)$$

$= \underline{\dfrac{32}{15}\left(2+\sqrt{2}\right)}$ as required.

(Total for Question 3 is 7 marks)

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4. Given that $k \in \mathbb{Z}^+$,

(a) show that $\displaystyle\int_k^{3k} \frac{2}{(3x-k)}\,dx$ is independent of k, Recall $\int \frac{1}{x}\,dx = x' \cdot \ln(x)$

(4)

$$\int_K^{3k} \frac{2}{3x-K}\,dx = \frac{2}{3}\ln(3(3K)-k) - \frac{2}{3}\ln(3k-K) = \frac{2}{3}\left(\ln(8K) - \ln(2K)\right)$$

$$= \underline{\underline{\frac{2}{3}\ln(4)}}$$ which is independent of k.

(b) show that $\displaystyle\int_k^{2k} \frac{2}{(2x-k)^2}\,dx$ is inversely proportional to k.

(3)

$u = 2x-K = 3k$ and $du = 2\,dx$

$u = 2k-K = K$ $\quad dx = \frac{du}{2}$

$$= 2\int_K^{2K} \frac{1}{(2x-K)^2} = 2\int_K^{3k} \frac{1}{2}\frac{1}{u^2}\,du$$

$$= \int_K^{3K} u^{-2}\,du = \left[-\frac{1}{u}\right]_K^{3K} = -\frac{1}{3K} + \frac{1}{K} = \underline{\underline{\frac{2}{3K}}}$$ which is $\underline{\underline{\text{clearly proportional}}}$ to K.

(Total for Question 4 is 7 marks)

5.

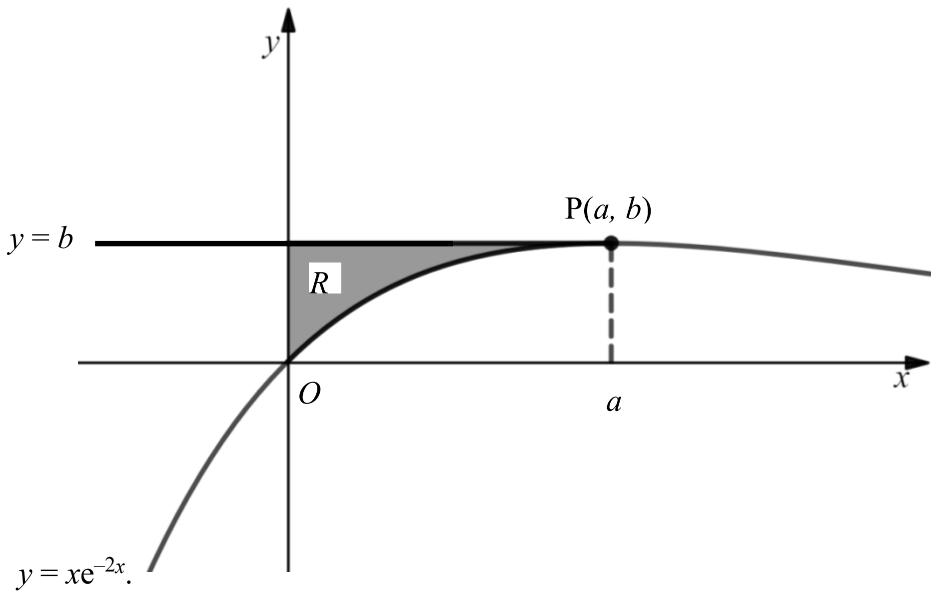

Figure 3

In this question you must show all stages of your working.
Solutions relying on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = xe^{-2x}.$$

The point $P(a, b)$ is the turning point of the curve.

$x \to 1$
$e^{-2x} \to -2e^{-2x}$

(a) Find the value of a and the exact value of b. **(4)**

Turning Point $\Rightarrow \frac{dy}{dx} = 0 =$ $y = xe^{-2x} \Rightarrow \frac{dy}{dx} = e^{-2x} - 2xe^{-2x} = 0$ — Since minimum point

$\Rightarrow e^{-2x} = 2xe^{-2x}$

$\Rightarrow 1 = 2x \Rightarrow x = 1/2$

$\Rightarrow y = \frac{1}{2}e^{-2x 1/2} = \underline{\frac{1}{2e}}$

$\Rightarrow$ we have $a = \frac{1}{2}$ nd $b = \frac{1}{2e}$

The finite region R, shown shaded in Figure 3, is bounded by the curve, the line with equation $y = b$ and the y-axis.

(b) Find the exact area of R. **(5)**

We want to find the Shade area R, and it will be found using $A = \int x \, dy$.
Our limits will be 0 and $y = \underline{-1/2e}$. It is hard to find an expression for x explicitly,
So we will find the Square a ($0 \to a$ length and $a \to b$ width) and then subtract the white Area.

Square Area $= L \times W = (\frac{1}{2} - 0) \times (\quad - 0) = \frac{1}{4e}$.

Then white area $= \int_0^{1/2} xe^{-2x} \, dx =$ let $u = -2x$ then $du = -2 \, dx$ and $dx = \frac{du}{-2}$ $x = -\frac{u}{2}$

$a =$

$= \int_0^{-1} -\frac{u}{2} e^u \frac{du}{-2} = + \frac{1}{4} \int_{-1}^{0} ue^u \, du$.

We can then integrate by parts:

$f(x) = e^u \longrightarrow f'(x) = e^u$

$g(x) = u \longrightarrow g'(x) = 1$

$\Rightarrow \int_{-1}^{0} ue^u \, du = ue^u - \int e^u \, du = ue^u - e^u$

$\Rightarrow \left[ue^u - e^u\right]_{-1}^{0} = -1 - \left(-\frac{1}{e} - \frac{1}{e}\right) = -1 + \frac{2}{e}$

$\Rightarrow + \frac{1}{4}\left(\frac{2}{e} - 1\right) = \text{white area} = \underline{-\frac{1}{4} + \frac{2}{4e}}$

Then $R = \text{Square Area} - \text{white Area} = \frac{1}{4e} - \left(\frac{1}{4} + \frac{2}{4e}\right) = \frac{1}{4e} + \frac{1}{4} - \frac{2}{4e} = \frac{1}{4} - \frac{1}{4e}$

$\Rightarrow$ Exact Area of is $\underline{\frac{1}{4} - \frac{1}{4e}}$

(Total for Question 5 is 9 marks)

6. (*a*) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3\,dx}{(x-1)(3+2\sqrt{x-1})} = \int_p^q \frac{6\,du}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

$x = u^2 + 1$

$u = \sqrt{x-1}$

$du = \frac{1}{2\sqrt{x-1}}\,dx$

$dx = 2\sqrt{x-1}\,du$

$dx = 2u\,du$

limits: $u = \sqrt{10-1} = 3$

$u = \sqrt{5-1} = 2$

$\Rightarrow \int_2^3 \frac{3}{u^2(3+2u)}\,2u\,du$

$\Rightarrow \int_2^3 \frac{6}{u(3+2u)}\,du$ as required.

(*b*) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3\,dx}{(x-1)(3+2\sqrt{x-1})} = \ln a$$

where a is a rational constant to be found.

(6)

$\int_5^{10} \frac{3\ dx}{x-1\ \ 3+2\sqrt{x-1}} = \int_2^3 \frac{6}{u(3+2u)}\,du$, but we can now make use of partial fractions; $\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u} \Rightarrow 6 = (3+2u)A + Bu \Rightarrow u = -\frac{3}{2} \Rightarrow 6 = -\frac{3}{2}B \Rightarrow B = -4$

Then let $u=0 \Rightarrow 6 = 3A \Rightarrow A = 2$

$\Rightarrow \int_2^3 \frac{6}{u(3+2u)}\,du = \int_2^3 \frac{2}{u}\,du - \int_2^3 \frac{4}{3+2u}\,du = \left[2\ln u\right]_2^3 - 2\ln\left[3+2u\right]_2^3$

$= 2\ln(3) - 2\ln(2) - 2\ln(9) + 2\ln(7)$

$= \ln(3^2) - \ln(2^2) - \ln(9^2) + \ln(7^2)$

$= \ln\left(\frac{9\times 49}{4\times 81}\right) = \ln\left(\frac{49}{36}\right)$

$\Rightarrow$ we have $\ln(a)$ with $a = \frac{49}{36}$.

(Total for Question 6 is 10 marks)

7. A large spherical balloon is deflating.

At time t seconds the balloon has radius r cm and volume V cm^3

The volume of the balloon is modelled as decreasing at a constant rate.

(*a*) Using this model, show that

$$\frac{dr}{dt} = -\frac{k}{r^2}$$

where k is a positive constant.

(3)

We use the Chain Rule; we know the volume will change with respect to time, i.e. $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$. We also know that Volume of a sphere is $V = \frac{4}{3}\pi r^3$ $\Rightarrow \frac{dV}{dr} = 4\pi r^2$ and we know that $\frac{dV}{dt}$ is Constant, let it be denoted by k.

negative since deflating/volume reducing.

$\Rightarrow -k = 4\pi r^2 \times \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{-k}{4\pi r^2}$ and then we can omit the 4π as this is/can be part of the constant value k, hence $\underline{\frac{dr}{dt} = \frac{-k}{r^2}}$ as required.

Given that

- the initial radius of the balloon is 40 cm
- after 5 seconds the radius of the balloon is 20 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(*b*) solve the differential equation to find a complete equation linking r and t.

(5)

We have that at $t = 0$, $r = 40$cm
$t = 5$, $r = 20$cm

Then $\frac{dr}{dt} = -\frac{k}{r^2}$ then we can rearrange and integrate both sides:

$\int r^2 \, dr = \int -k \, dt \Rightarrow \frac{r^3}{3} = -kt + C$

$\Rightarrow t = 0 \Rightarrow \frac{40^3}{3} = C = \frac{64{,}000}{3}$

Then we can find k: $\frac{20^3}{3} = -5k + \frac{64{,}000}{3} \Rightarrow k = \frac{11{,}200}{3}$

$\Rightarrow r^3 = 3\left(-\frac{11{,}200}{3}t + \frac{64000}{3}\right) \Rightarrow r^3 = \underline{64{,}000 - 11{,}200t}$

(*c*) Find the limitation on the values of t for which the equation in part (*b*) is valid.

(2)

We can't have negative time values ;

$\Rightarrow 64{,}000 - 11{,}200t \geq 0$

$\Rightarrow 11{,}200t \leq 64{,}000$

$\Rightarrow t \leq \frac{40}{7} = \underline{5.71 \text{ seconds}}$

(Total for Question 7 is 10 marks)

8.

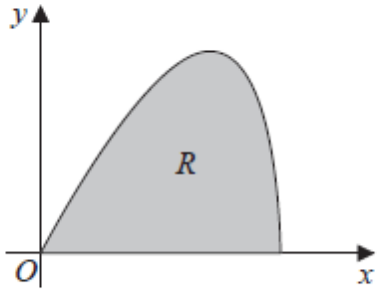

Figure 3

The curve shown in Figure 3 has parametric equations

$$x = 6 \sin t \qquad y = 5 \sin 2t \qquad 0 \le t \le \frac{\pi}{2}$$

The region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

(*a*) (i) Show that the area of R is given by $\int_0^{\frac{\pi}{2}} 60 \sin t \cos^2 t \, dt$

(3)

We know that the area of R will be $\int y \, dx$ and $x = 6\sin t$, $y = 5\sin 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \Rightarrow \frac{dy}{dt} = 10\cos(2t)$ and $\frac{dx}{dt} = 6\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{6\cos t} \Rightarrow dx = 6\cos t \, dt$

Then we want $\int y \, dx \Rightarrow \int 5\sin 2t \times 6\cos t \, dt = \int 30\sin 2t \cos t \, dt = \int 30 \times 2\sin t \cos t \cos t \, dt$
$= \int 60\sin t \cos^2 t \, dt$.

And then noting $0 \le t \le \frac{\pi}{2}$, we conclude that the area of R is given by $\int_0^{\pi/2} 60\sin t \cos^2 t \, dt$ as required.

(ii)Hence show, by algebraic integration, that the area of R is exactly 20

(3)

let $u = \cos t$ then $du = -\sin t \, dt \Rightarrow \int_0^{\pi/2} 60\sin t\cos^2 t \, dt = \int_1^0 60\sin t \, u^2 \frac{du}{-\sin t}$

$dt = \frac{du}{-\sin t}$

(take negative out then swap limits and x-1 again. $\Rightarrow +60$

$\Rightarrow 60\int_0^1 u^2 \, du = 60\left[\frac{u^3}{3}\right]_0^1 = 60\left(\frac{1}{3}\right) = 20$ as required

(Total for Question 8 is 6 marks)

9. Given that A is constant and

$$\int_1^4 (3\sqrt{x} + A) \ \ \mathrm{d}x = 2A^2,$$

show that there are exactly two possible values for A.

$\int_1^4 3\sqrt{x} + A \ dx = \left[2x^{3/2} + Ax\right]_1^4 = (16 + 4A) - (2 + A) = 14 + 3A$

$\Rightarrow \quad 14 + 3A = 2A^2 \Rightarrow \quad 2A^2 - 3A - 14 = 0$ then use quardratic formula :

+ case   - case

$A = \dfrac{3 \pm \sqrt{(-3)^2 - 4(2)(-14)}}{2(2)} \Rightarrow A = \underline{\dfrac{7}{2}}$ and $A = \underline{-2}$

(Total for Question 9 is 5 marks)