Pearson Edexcel

Additional Assessment Materials

Summer 2021

Pearson Edexcel GCE in Mathematics

9MA0 (Public release version)

Resource Set 1: Topic 7

Differentiation (Test 1)

Model Solutions

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Additional Assessment Materials, Summer 2021

General guidance to Additional Assessment Materials for use in 2021

Context

- Additional Assessment Materials are being produced for GCSE, AS and A levels (with the exception of Art and Design).
- The Additional Assessment Materials presented in this booklet are an optional part of the range of evidence teachers may use when deciding on a candidate's grade.
- 2021 Additional Assessment Materials have been drawn from previous examination materials, namely past papers.
- Additional Assessment Materials have come from past papers both published (those materials available publicly) and unpublished (those currently under padlock to our centres) presented in a different format to allow teachers to adapt them for use with candidate.

Purpose

- The purpose of this resource to provide qualification-specific sets/groups of questions covering the knowledge, skills and understanding relevant to this Pearson qualification.
- This document should be used in conjunction with the mapping guidance which will map content and/or skills covered within each set of questions.
- These materials are only intended to support the summer 2021 series.

1. The curve C has equation

$$y = 3x^4 - 8x^3 - 3$$

(a) Find (i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) Verify that C has a stationary point when $x = 2$

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)

(Total for Question 1 is 7 marks)

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a) i) $\dfrac{dy}{dx} = 12x^3 - 24x^2$

$\dfrac{d^2y}{dx^2} = 36x^2 - 48x$

b) $\left.\dfrac{dy}{dx}\right|_{x=2} = 12(2)^3 - 24 \cdot 2^2 = 0$

so stationary point as $\dfrac{dy}{dx} = 0$

c) $$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 36 \cdot (2)^2 - 48 \cdot 2 = 48$$

$$48 > 0$$

so the stationary point is a local minimum

2. Given that θ is measured in radians, prove, from first principles, that

$$\frac{\mathrm{d}}{\mathrm{d}\theta} = (\cos\,\theta) = -\sin\,\theta.$$

You may assume the formula for cos ($A \pm B$) and that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$.

(5)

(Total for Question 2 is 5 marks)

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad f(\theta) = \cos\theta$$

$$\lim_{h \to 0} \frac{\cos(\theta + h) - \cos\theta}{h}$$

$$= \lim_{h \to 0} \frac{\cos\theta \cosh - \sin\theta \sinh - \cos\theta}{h}$$

$$= \lim_{h \to 0} \frac{\cos\theta \cosh - \cos\theta}{h} - \frac{\sin\theta \sinh}{h}$$

$$= \lim_{h \to 0} \frac{\cos\theta(\cosh - 1)}{h} - \sin\theta \cdot \frac{\sinh}{h}$$

$$= \cos\theta \cdot 0 - \sin\theta \cdot 1$$

$$= -\sin\theta$$

3. A curve C has equation

$$y = x^2 - 2x - 24\sqrt{x}, \quad x > 0.$$

(a) Find (i) $\dfrac{dy}{dx}$,

(ii) $\dfrac{d^2y}{dx^2}$.

(3)

(b) Verify that C has a stationary point when $x = 4$.

(2)

(c) Determine the nature of this stationary point, giving a reason for your answer.

(2)

(Total for Question 3 is 7 marks)

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a) $y = x^2 - 2x - 24x^{\frac{1}{2}}$

$\dfrac{dy}{dx} = 2x - 2 - 12x^{-\frac{1}{2}}$

$\dfrac{d^2y}{dx^2} = 2 + 6x^{-\frac{3}{2}}$

b) $\left.\frac{dy}{dx}\right|_{x=4} = 2 \cdot 4 - 2 - 12 \cdot (4)^{-\frac{1}{2}} = 0$

so stationary point as $\frac{dy}{dx} = 0$

c) $\left.\frac{d^2y}{dx^2}\right|_{x=4} = 2 + 6 \cdot (4)^{-\frac{3}{2}}$

$= \frac{11}{4}$

so the stationary point is local minium as $\frac{d^2y}{dx^2} > 0$

4. Given that

$$y = \frac{3\sin\theta}{2\sin\theta + 2\cos\theta}, \qquad -\frac{\pi}{4} < \theta < \frac{3\pi}{4},$$

show that

$$\frac{dy}{d\theta} = \frac{A}{1 + \sin 2\theta}, \qquad -\frac{\pi}{4} < \theta < \frac{3\pi}{4},$$

where A is a rational constant to be found.

(5)

(Total for Question 4 is 5 marks)

$$y = \frac{3\sin\theta}{2\sin\theta + 2\cos\theta}$$ using the quotient rule

$f = 3\sin\theta \qquad g = 2\sin\theta + 2\cos\theta$

$f' = 3\cos\theta \qquad g' = 2\cos\theta - 2\sin\theta$

$$\frac{dy}{d\theta} = \frac{3\cos\theta(2\sin\theta + 2\cos\theta) - 3\sin\theta(2\cos\theta - 2\sin\theta)}{(2\sin\theta + 2\cos\theta)^2}$$

$$\frac{dy}{d\theta} = \frac{6\sin\theta\cos\theta + 6\cos^2\theta + 6\sin^2\theta - 6\sin\theta\cos\theta}{4\sin^2\theta + 4\cos^2\theta + 8\sin\theta\cos\theta}$$

$$\frac{dy}{d\theta} = \frac{6(\sin^2\theta + \cos^2\theta)}{4(\sin^2\theta + \cos^2\theta) + 8\sin\theta\cos\theta}$$

$$= \frac{6}{4 + 4\sin 2\theta}$$

$$= \frac{6}{4} \cdot \frac{1}{1 + \sin 2\theta}$$

so $A = \frac{6}{4} = \frac{3}{2}$

5.

$$y = \frac{5x^2 + 10x}{(x+1)^2} \qquad x \neq -1$$

(*a*) Show that $\dfrac{dy}{dx} = \dfrac{A}{(x+1)^n}$ where A and n are constants to be found.

(4)

(*b*) Hence deduce the range of values for x for which $\dfrac{dy}{dx} < 0$

(1)

(Total for Question 5 is 5 marks)

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a) $y = \dfrac{5x^2 + 10x}{(x+1)^2}$ Quotient rule

$f = 5x^2 + 10x \qquad g = (x+1)^2$

$f' = 10x + 10 \qquad g' = 2(x+1)$

$$= \frac{(10x+10)(x+1)^2 - 2(x+1)(5x^2+10x)}{\left((x+1)^2\right)^2}$$

$$= \frac{10(x+1)^3 - 2(x+1)(5x^2+10x)}{(x+1)^4}$$

$$= \frac{10(x+1)^2 - 2(5x^2+10x)}{(x+1)^3}$$

$$= \frac{10(x^2+2x+1) - 10x^2 - 20x}{(x+1)^3}$$

$$= \frac{10x^2 + 20x + 10 - 10x^2 - 20x}{(x+1)^3}$$

$$\frac{dy}{dx} = \frac{10}{(x+1)^3}$$

b) $\frac{10}{(x+1)^3} < 0 \Rightarrow (x+1) < 0$

$\Rightarrow x < -1$

6.

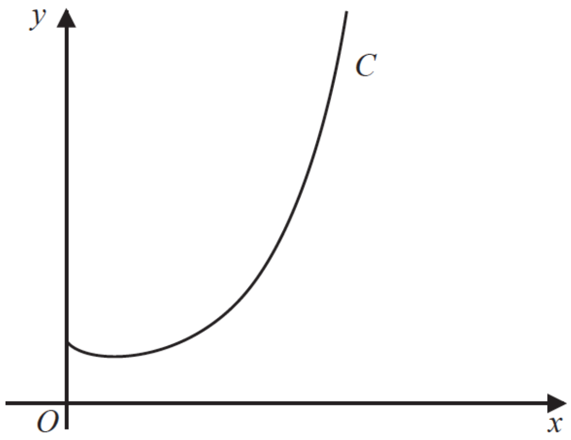

Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x$, $x > 0$.

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{1-x_n}$$

Using this formula with $x_1 = 1.5$

(c) find x_4 to 3 decimal places,

(2)

(d) describe the long-term behaviour of x_n

(2)

(Total for Question 6 is 11 marks)

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a) $\ln y = \ln x^x$

$\Rightarrow \ln y = x \ln x$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

$$\Rightarrow \frac{dy}{dx} = y(1+\ln x)$$ since $y = x^x$

$$\Rightarrow \frac{dy}{dx} = x^x(1+\ln x)$$

$\frac{dy}{dx} = 0$ at the turning point

$$0 = x^x(1+\ln x)$$

$$\Rightarrow \underbrace{x^x = 0}_{\text{Impossible}} \quad \text{or} \quad \ln x = -1$$

$$\ln x = -1 \Rightarrow e^{-1} = x$$

b) $y = x^x$ $P(\alpha, 2)$

$2 = \alpha^\alpha$

$\alpha^\alpha - 2$

when $x = 1.5$

$\alpha^\alpha - 2 = -0.162...$

when $x = 1.6$

$\alpha^\alpha - 2 = 0.121...$

so there is a change of sign so α is between 1.5 and 1.6

c) $x_{n+1} = 2x_n^{1-x_n}$

$x_1 = 1.5$

$x_2 = 1.6330$

$x_3 = 1.4663$

$x_4 = 1.6731$

d) x_n diverges and oscallates between 1 and 2

7.

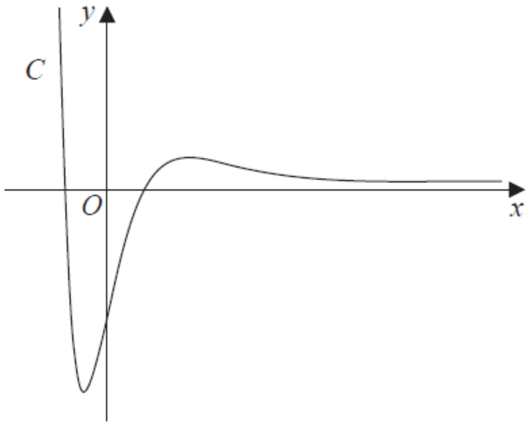

Figure 2

Figure 2 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 4(x^2 - 2)\mathrm{e}^{-2x} \qquad x \in \mathbb{R}$$

(*a*) Show that $\mathrm{f}'(x) = 8(2 + x - x^2)\mathrm{e}^{-2x}$

(3)

(*b*) Hence find, in simplest form, the exact coordinates of the stationary points of C.

(3)

(Total for Question 7 is 6 marks)

Using the product rule

$f = 4e^{-2x}$ $\qquad g = x^2 - 2$

$f' = -8e^{-2x}$ $\qquad g' = 2x$

$f'(x) = 8xe^{-2x} - 8e^{-2x}(x^2 - 2)$

$$f'(x) = 8xe^{-2x} - 8x^2e^{-2x} + 16e^{-2x}$$

$$= 8e^{-2x}(2 + x - x^2)$$

b) $f'(x) = 0 = 8e^{-2x}(2+x-x^2)$

so $\underline{8e^{-2x} = 0}$, $\underline{2+x-x^2 = 0}$

No solutions

$(-x+2)(x+1) = 0$

$x = 2, -1$

$(2, y_1)$ $(-1, y_2)$

$y = 4e^{-2x}(x^2-2)$

$y_1 = 4e^{-4}(4-2)$

$= 4e^{-4} \cdot 2$

$y_1 = 8e^{-4}$

$y_2 = 4e^{2}(1-2)$

$y_2 = -4e^{2}$

so

$(2, 8e^{-4})$ $(-1, -4e^{2})$

8.

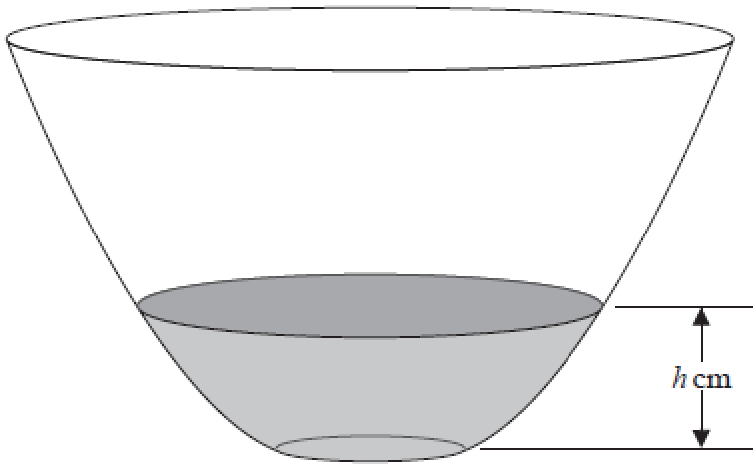

Figure 4

Figure 4 shows a bowl with a circular cross-section. Initially the bowl is empty. Water begins to flow into the bowl.

At time t seconds after the water begins to flow into the bowl, the height of the water in the bowl is h cm.

The volume of water, V cm^3, in the bowl is modelled as

$$V = 4\pi h(h + 6), \quad 0 \leq h \leq 25.$$

The water flows into the bowl at a constant rate of 80π cm3 s^{-1}.

(a) Show that, according to the model, it takes 36 seconds to fill the bowl with water from empty to a height of 24 cm.

(1)

(b) Find, according to the model, the rate of change of the height of the water, in cm s^{-1}, when $t = 8$.

(8)

(Total for Question 8 is 9 marks)

a) $V = 4\pi h(h+6)$

$$\frac{dV}{dt} = 80\pi$$

so $V = 80t\pi$

$$80t\pi = 4\pi h(h+6)$$

$t = 36$

$$2880\pi = 2880\pi$$

so when $t = 36$ $h = 24$

b) Want to find $\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$

$\underline{\frac{dh}{dV}}$: $V = 4\pi h(h+6)$

$$V = 4\pi h^2 + 24\pi h$$

$$\frac{d}{dV}\left(\qquad \right)\frac{d}{dV}$$

$$1 = 8\pi h \frac{dh}{dV} + 24\pi \frac{dh}{dV}$$

$$1 = \frac{dh}{dV}(8\pi h + 24\pi)$$

$$\Rightarrow \frac{dh}{dV} = \frac{1}{8\pi h + 24\pi}$$

so $$\frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{1}{8\pi h + 24\pi} \cdot 80\pi = \frac{80\pi}{8\pi h + 24\pi}$$

$$\frac{dh}{dt} = \frac{80}{8h + 24}$$

$$80t\pi = 4\pi h(h+6)$$

$$20t = h^2 + 6h$$

<u>when $t = 8$</u>

$$160 = h^2 + 6h$$

$$h^2 + 6h - 160 = 0$$

$$h = 10, -16$$

$h = -16$ is not valid

$$\frac{dh}{dt} = \frac{80}{8h+24} \quad \text{when} \quad h = 10$$

$$\frac{dh}{dt} = \frac{80}{80+24} = \frac{10}{13}$$

9.

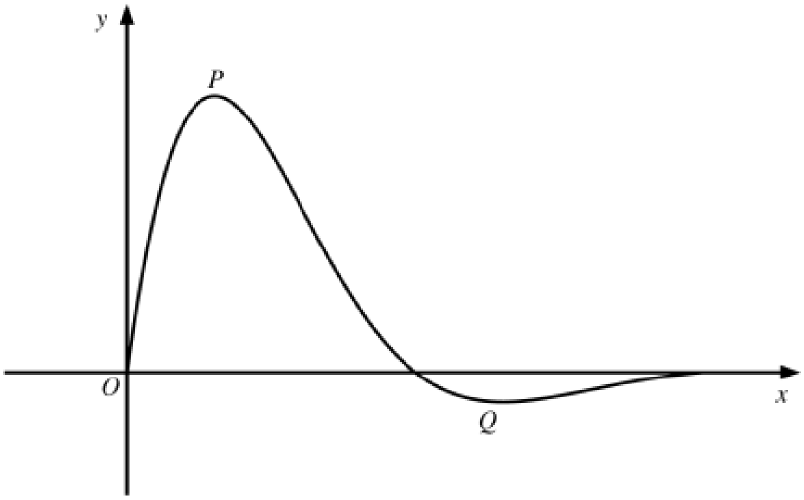

Figure 5

Figure 5 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{4\sin 2x}{\mathrm{e}^{(\sqrt{2})x-1}}, \quad 0 \leq x \leq \pi$$

The curve has a maximum turning point at P and a minimum turning point at Q, as shown in Figure 5.

(a) Show that the x-coordinates of point P and point Q are solutions of the equation

$$\tan 2x = \sqrt{2}$$

(4)

(Total for Question 9 is 4 marks)

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a) $f(x) = \dfrac{4\sin 2x}{e^{x\sqrt{2}-1}}$ Quotient rule

$f = 4\sin 2x \qquad g = e^{x\sqrt{2}-1}$

$f' = 8\cos 2x \qquad g' = \sqrt{2}e^{x\sqrt{2}-1}$

so $f'(x) = \dfrac{8\cos 2x\left(e^{x\sqrt{2}-1}\right) - 4\sqrt{2}\sin 2x e^{x\sqrt{2}-1}}{\left(e^{x\sqrt{2}-1}\right)^2}$

$$f'(x) = \frac{8\cos 2x - 4\sqrt{2}\sin 2x}{e^{x\sqrt{2}-1}}$$

$$f'(x) = 0 = 8\cos 2x - 4\sqrt{2}\sin 2x$$

$$0 = 8\cos 2x - 4\sqrt{2}\sin 2x$$

$$4\sqrt{2}\sin 2x = 8\cos 2x$$

$$0 \doteq$$

$\Rightarrow$ $\dfrac{4\sqrt{2}\sin 2x}{\cos 2x} = 8$

$\Rightarrow$ $\dfrac{\sin 2x}{\cos 2x} = \dfrac{2}{\sqrt{2}}$

$\Rightarrow$ $\tan 2x = \sqrt{2}$

10. A scientist is studying a population of mice on an island. The number of mice, N, in the population, t months after the start of the study, is modelled by the equation

$$N = \frac{900}{3 + 7\mathrm{e}^{-0.25t}}, \quad t \in \mathbb{R}, \quad t \geq 0.$$

(a) Find the number of mice in the population at the start of the study.

(1)

(b) Show that the rate of growth $\frac{\mathrm{d}N}{\mathrm{d}t}$ is given by $\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(300 - N)}{1200}$.

(4)

The rate of growth is a maximum after T months.

(c) Find, according to the model, the value of T.

(4)

According to the model, the maximum number of mice on the island is P.

(d) State the value of P.

(1)

(Total for Question 10 is 7 marks)

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a) $N = \dfrac{900}{3+7e^{-0.25t}}$ $\quad t = 0$

$\Rightarrow N = \dfrac{900}{3+7}$

$N = 90$

b) $N = \dfrac{900}{3+7e^{-0.25t}}$ $\underline{\text{Quotient rule}}$

$f = 900 \quad g = 3+7e^{-0.25t}$

$f' = 0 \quad g' = -\frac{7}{4}e^{-0.25t}$

$$\frac{0 - -1575e^{-0.25t}}{\left(3+7e^{-0.25t}\right)^2}$$

$$= \frac{1575e^{-0.25t}}{\left(3+7e^{-0.25t}\right)^2}$$

$$= \frac{225\left(3+7e^{-0.25t}\right) - 675}{\left(3+7e^{-0.25t}\right)^2}$$

$$\frac{810000}{}$$

$$= \frac{225\left(3+7e^{-0.25t}\right)}{\left(3+7e^{-0.25t}\right)^2} - \frac{675}{\left(3+7e^{-0.25t}\right)^2}$$

$$= \frac{\dfrac{27000}{3+7e^{-0.25t}} - 3+7e^{-0.25t}}{1200}$$

$$= \frac{\dfrac{900}{3+7e^{-0.25t}}\left(300 - \dfrac{900}{3+7e^{0.25t}}\right)}{1200}$$

$$= \frac{N(300-N)}{1200}$$

c) $$\frac{dN}{dt} = \frac{N(300-N)}{1200}$$

$$\frac{dN}{dt} = \frac{300N - N^2}{1200}$$

$$\frac{d}{dt}\left(\qquad \right)\frac{d}{dt}$$

$$\frac{d^2N}{dt^2} = \frac{300\frac{dN}{dt} - 2N\frac{dN}{dt}}{1200}$$

Maximal $\Rightarrow$ $\frac{d^2N}{dt^2} = 0$

so

$$\frac{300\frac{dN}{dt} - 2N\frac{dN}{dt}}{1200} = 0$$

$$\Rightarrow \quad 300\frac{dN}{dt} - 2N\frac{dN}{dt} = 0$$

$$\Rightarrow \quad \frac{dN}{dt}(300-2N) = 0$$

$$\Rightarrow \quad 300 - 2N = 0$$

$\Rightarrow \quad N = 150$

so

$$150 = \frac{900}{3+7e^{-0.25t}}$$

$$150(3+7e^{-0.25t}) = 900$$

$$6 = 3+7e^{-0.25t}$$

$$\frac{3}{7} = e^{-0.25t}$$

$$\ln\left(\frac{3}{7}\right) = -0.25t$$

so $t = -4\ln\left(\frac{3}{7}\right)$

d) $0 = \dfrac{N(300-N)}{1200} \Rightarrow N(300-N) = 0$

So max number of mice is 300

11. The curve C, in the standard Cartesian plane, is defined by the equation

$$x = 4 \sin 2y \qquad \frac{-\pi}{4} < y < \frac{\pi}{4}$$

The curve C passes through the origin O

(*a*) Find the value of $\frac{\mathbf{d}y}{\mathbf{d}x}$ at the origin.

(2)

(*b*) (i) Use the small angle approximation for sin $2y$ to find an equation linking x and y for points close to the origin.

(ii) Explain the relationship between the answers to (*a*) and (*b*)(i).

(2)

(*c*) Show that, for all points (x, y) lying on C,

$$\frac{\mathbf{d}y}{\mathbf{d}x} = \frac{1}{a\sqrt{b - x^2}}$$

where a and b are constants to be found.

(3)

(Total for Question 11 is 7 marks)

11a) $x = 4\sin 2y$

$\frac{d}{dx}\left(\qquad \right)\frac{d}{dx}$

$1 = 4 \cdot 2\cos 2y \frac{dy}{dx}$

$1 = 8\cos 2y \frac{dy}{dx}$

at $(0,0)$

$\Rightarrow \quad 1 = 8\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{8}$

b i) $\sin\theta \approx \theta$

$\Rightarrow \sin 2y \approx 2y$

$x = 2y \cdot 4 \Rightarrow x = 8y$

ii) $\frac{y}{x} = \frac{1}{8} = \frac{dy}{dx}$ so the line is straight very near the origin

c) $1 = 8\cos 2y \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{8\cos 2y}$ using $\sin^2 2y + \cos^2 2y = 1$

$\Rightarrow \cos 2y = \sqrt{1-\sin^2 2y}$

$= \frac{dy}{dx} = \frac{1}{8\sqrt{1-\sin^2 2y}}$

Since $x = 4\sin 2y \Rightarrow x^2 = 16\sin^2 2y$

$$\Rightarrow \frac{x^2}{16} = \sin^2 2y$$

So substituting we get:

$$\frac{1}{8\sqrt{1-\frac{x^2}{16}}}$$

$$= \frac{1}{\sqrt{64-4x^2}}$$

$$= \frac{1}{\sqrt{4}\sqrt{16-x^2}}$$

$$= \frac{1}{2\sqrt{16-x^2}}$$

12. The curve C has equation

$$x^2 \tan y = 9 \qquad\qquad 0 < y < \frac{\pi}{2}$$

(*a*) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-18x}{x^4 + 81}$$

(4)

(*b*) Prove that C has a point of inflection at $x = \sqrt[4]{27}$

(3)

(Total for Question 12 is 7 marks)

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a) $x^2 \tan y = 9$

$$\tan y = \frac{9}{x^2}$$

$$\frac{d}{dx} \left(\qquad \right) \frac{d}{dx}$$

$$\sec^2 y \frac{dy}{dx} = 9 \cdot -2x^{-3}$$

$$\sec^2 y \frac{dy}{dx} = -\frac{18}{x^3}$$

$$\frac{dy}{dx} = -\frac{18}{x^3 \sec^2 y}$$

$$\Rightarrow \quad \frac{9}{x^2} = \tan y$$

$$\Rightarrow \quad \frac{81}{x^4} = \tan^2 y$$

So $$\frac{81}{x^4} + 1 = \sec^2 y$$

<u>Substituting</u>

$$\frac{dy}{dx} = -\frac{18}{x^3 \cdot \left(\frac{81}{x^4} + 1\right)}$$

$$= -\frac{18}{\left(\frac{81}{x} + x^3\right)}$$

$$= -\frac{18}{\left(\frac{81}{x} + \frac{x^4}{x}\right)}$$

$$\frac{dy}{dx} = -\frac{18x}{81+x^4}$$

b) <u>Finding</u>: $\frac{d^2y}{dx^2}$ Quotient rule

$f = -18x \qquad g = 81 + x^4$

$f' = -18 \qquad g' = 4x^3$

$$= \frac{-18(81+x^4) + 18x \cdot 4x^3}{(81+x^4)^2}$$

$$= \frac{-1458 - 18x^4 + 72x^4}{(81+x^4)^2}$$

$$\frac{d^2y}{dx^2} = 0$$

$$\frac{-1458 + 54x^4}{(81 + x^4)^2} = 0$$

$$54x^4 - 1458 = 0$$

$$x^4 = 27$$

$$x = \sqrt[4]{27}$$