Model Solutions

Additional Assessment Materials

Summer 2021

Pearson Edexcel GCE in Mathematics

9MA0 (Public release version)

Resource Set 1: Topic 4

Sequences

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Additional Assessment Materials, Summer 2021

General guidance to Additional Assessment Materials for use in 2021

Context

- Additional Assessment Materials are being produced for GCSE, AS and A levels (with the exception of Art and Design).
- The Additional Assessment Materials presented in this booklet are an optional part of the range of evidence teachers may use when deciding on a candidate's grade.
- 2021 Additional Assessment Materials have been drawn from previous examination materials, namely past papers.
- Additional Assessment Materials have come from past papers both published (those materials available publicly) and unpublished (those currently under padlock to our centres) presented in a different format to allow teachers to adapt them for use with candidate.

Purpose

- The purpose of this resource to provide qualification-specific sets/groups of questions covering the knowledge, skills and understanding relevant to this Pearson qualification.
- This document should be used in conjunction with the mapping guidance which will map content and/or skills covered within each set of questions.
- These materials are only intended to support the summer 2021 series.

1. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $\sqrt{1+4x}$, giving each coefficient in its simplest form.

(4)

The expansion can be used to find an approximation to $\sqrt{26}$.

(b) Explain why $x = \frac{25}{4}$ **should not** be used in the expansion to find an approximation to $\sqrt{26}$.

(1)

(c) Explain how you could use $x = \frac{1}{100}$ to find an approximation to $\sqrt{26}$.

There is no need to carry out the calculation.

(2)

(Total for Question 1 is 7 marks)

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a) $\sqrt{1+4x} = (1+4x)^{\frac{1}{2}}$

So $n = \frac{1}{2}$

$x = 4x$

$$= 1 + \frac{1}{2} \cdot 4x + \frac{\frac{1}{2} \cdot \left(\frac{1}{2}-1\right)}{2!}(4x)^2 + \frac{\frac{1}{2} \cdot \left(\frac{1}{2}-1\right) \cdot \left(\frac{1}{2}-2\right)}{3!}(4x)^3$$

$$= 1 + 2x + \frac{\frac{1}{2} \cdot -\frac{1}{2}}{2} \cdot 16x^2 + \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{6} \cdot 64x^3$$

$$= 1 + 2x - \frac{1}{8} \cdot 16x^2 + \frac{1}{16} \cdot 64x^3$$

$$= 1 + 2x - 2x^2 + 4x^3 + \ldots$$

b) because the binomial is only defined for $|x| < 1$ and for us we have $4x = \frac{25}{4}$

$x = 25$ so the binomial is not valid

c) $\sqrt{1 + \frac{4}{100}} = \sqrt{\frac{26}{25}} = \frac{\sqrt{26}}{5}$

So we would need to multiply our answer by 5

2. The sequence $u_1, u_2, u_3, ...$ is defined by

$$u_{n+1} = \frac{4}{2 - u_n}, \quad u_1 = 1 .$$

(a) Show that this sequence is periodic, stating the period.

(3)

(b) Hence find $\sum_{n=1}^{50} u_n$.

(2)

(Total for Question 2 is 5 marks)

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a) First few terms

$u_1 = 1$
$u_2 = 4$
$u_3 = -2$
$u_4 = 1$
$u_5 = 4$
$\vdots$

3 terms repeat so period of 3

b) $$\sum_{n=1}^{50} u_n = 16\sum_{n=1}^{3} u_n + u_{49} + u_{50}$$

$$= 16 \cdot 3 + 1 + 4 = 53$$

3. In the binomial expansion of

$$(a + 2x)^7 \qquad \text{where } a \text{ is a constant}$$

the coefficient of x^4 is 15 120

Find the value of a.

(3)

(Total for Question 3 is 3 marks)

$(a + 2x)^7$ Since $n = 7$ is positive integer we can use the standard binomial.

Only need the x^4 term which is:

$$\binom{7}{3} \cdot a^3 \cdot (2x)^4$$

$$= 35a^3 \cdot 16x^4$$

$$= 560a^3x^4$$

So $560a^3 = 15120$

$$a^3 = 27$$

$$a = 3$$

4. A car has six forward gears.

The fastest speed of the car

- in 1^{st} gear is 28 km h^{-1}
- in 6^{th} gear is 115 km h^{-1}

Given that the fastest speed of the car in successive gears is modelled by an **arithmetic sequence**,

(*a*) find the fastest speed of the car in 3^{rd} gear.

(3)

Given that the fastest speed of the car in successive gears is modelled by a **geometric sequence**,

(*b*) find the fastest speed of the car in 5^{th} gear.

(3)

(Total for Question 4 is 6 marks)

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a) $g_1 = 28$ $\qquad g_6 = 115$

$$g_n = a + (n-1)d$$

So $\quad 28 = a$

$$g_6 = 28 + 5d$$

$$115 = 28 + 5d$$

$$\Rightarrow \quad d = \frac{87}{5}$$

$$g_3 = 28 + 2 \cdot \frac{87}{5} \quad \Rightarrow \quad g_3 = 62.8$$

b) $g_1 = 28$ $\quad g_6 = 115$

$g_n = ar^{n-1}$

so $\quad a = 28$

$g_6 = 115 = 28r^5$

$\sqrt[5]{\frac{115}{28}} = r \Rightarrow r \approx 1.327$

so

$g_5 = 28 \cdot 1.327^4$

$g_5 = 86.69$

5. The value, £V, of a vintage car t years after it was first valued on 1st January 2001, is modelled by the equation

$$V = Ap^t, \quad \text{where } A \text{ and } p \text{ are constants.}$$

Given that the value of the car was £32 000 on 1st January 2005 and £50 000 on 1st January 2012,

(a) (i) find p to 4 decimal places,

(ii) show that A is approximately 24 800.

(4)

(b) With reference to the model, interpret

(i) the value of the constant A,

(ii) the value of the constant p.

(2)

Using the model,

(c) find the year during which the value of the car first exceeds £100 000.

(4)

(Total for Question 5 is 10 marks)

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ai) $V = Ap^t$ when $t = 4 \quad V = 32000$

$t = 11 \quad V = 50000$

So

$$\begin{cases} 32000 = Ap^4 & (1) \\ 50000 = Ap^{11} & (2) \end{cases}$$

$$(2) \div (1) = \frac{50000}{32000} = \frac{p^{11}}{p^4} \Rightarrow \frac{50}{32} = p^7$$

$$\Rightarrow p = \sqrt[7]{\frac{50}{32}}$$

$$\Rightarrow p = 1.0658 \text{ to } 4 \text{ d.p}$$

ii) Using p in equation (1)

$$32000 = A \cdot 1.0658^4$$

$$\Rightarrow A = 24796 \approx 24800$$

b) i) A is the initial value of the car

ii) p is the rate of increase in price of the car

c) $10000 = Ap^t$

$10000 = 24800 \cdot 1.0658^t$

$\Rightarrow \frac{125}{31} = 1.0658^t$

$a^b = c$

$\Rightarrow \log_a c = b$

$\log_{1.0658}\left(\frac{125}{31}\right) = t$

$\Rightarrow t = 21.88\ldots$

so $t = 22$ (integer)

So 2023 is when the car exceeds 100,000

6. (*a*) Use the binomial expansion, in ascending powers of x, to show that

$$\sqrt{(4-x)} = 2 - \frac{1}{4}x + kx^2 + \ldots$$

where *k* is a rational constant to be found.

(4)

A student attempts to substitute $x = 1$ into both sides of this equation to find an approximate value for $\sqrt{3}$.

(*b*) State, giving a reason, if the expansion is valid for this value of *x*.

(1)

(Total for Question 6 is 5 marks)

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a) $\sqrt{4-x} = (4-x)^{\frac{1}{2}}$

$= \left(4\left(1-\frac{x}{4}\right)\right)^{\frac{1}{2}}$

$= 4^{\frac{1}{2}}\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$

$= 2\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$

So Binomial expansion on $\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$

$n = \frac{1}{2}$

$x = -\frac{x}{4}$

$2\left(1 + \frac{1}{2}\cdot -\frac{x}{4} + \frac{\frac{1}{2}\cdot -\frac{1}{2}}{2!}\left(-\frac{x}{4}\right)^2 + \ldots\right)$

$$= 2\left(1 - \frac{x}{8} - \frac{x^2}{128} + \dots\right)$$

$$= 2 - \frac{x}{4} - \frac{x^2}{64} + \dots$$

so $k = -\frac{1}{64}$

b) $x = -\frac{1}{4}x$ when $x = 1 = -\frac{1}{4}$

$\left|-\frac{1}{4}\right| < 1$ so the expansion is valid

7. (i) Show that $\sum_{r=1}^{16}(3+5r+2^r) = 131\,798$.

(4)

(ii) A sequence $u_1, u_2, u_3, \ldots,$ is defined by

$$u_{n+1} = \frac{1}{u_n}, \quad u_1 = \frac{2}{3}.$$

Find the exact value of $\sum_{r=1}^{100} u_r$.

(3)

(Total for Question 7 is 7 marks)

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i) $\sum_{r=1}^{16}(3+5r+2^r) = \sum_{r=1}^{16} 3 + 5\sum_{r=1}^{16} r + \sum_{r=}^{16} 2^r$

$5\sum_{r=1}^{16} r$

Using $\frac{1}{2}n(n+1) \Rightarrow 5\cdot\left(\frac{1}{2}\cdot 16(16+1)\right) = 5\cdot 8\cdot 17$

$= 680$

$\sum_{n=1}^{16} 2^r$: Using $S_n = \frac{a(1-r^n)}{1-r}$ with $a=1$ $r=2$

$n = 16$

$S_n = \frac{2(1-2^{16})}{1-2} = 131070$

$u_n = ar^{n-1}$

So $\sum_{n=1}^{16} = 48 + 680 + 131070 = 131798$

b) $u_{n+1} = \frac{1}{u_n}$

$u_1 = \frac{2}{3}$

$u_2 = \frac{3}{2}$

$u_3 = \frac{2}{3}$

$\vdots$

$\left.\begin{matrix} u_1 \\ u_2 \end{matrix}\right\}$ period of 2

$$\sum_{r=1}^{100} u_r = 50\left(\frac{2}{3} + \frac{3}{2}\right) = \frac{325}{3}$$

8. A company extracted 4500 tonnes of a mineral from a mine during 2018. The mass of the mineral which the company expects to extract in each subsequent year is modelled to decrease by 2% each year.

(a) Find the total mass of the mineral which the company expects to extract from 2018 to 2040 inclusive, giving your answer to 3 significant figures.

(2)

(b) Find the mass of the mineral which the company expects to extract during 2040, giving your answer to 3 significant figures.

(2)

The costs of extracting the mineral each year are assumed to be:

- £800 per tonne for the first 1500 tonnes,
- £600 per tonne for any amount in excess of 1500 tonnes.

The expected cost of extracting the mineral from 2018 to 2040 inclusive is £x million.

(c) Find the value of x, giving your answer to 3 significant figures.

(3)

(Total for Question 8 is 7 marks)

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a) $u_n = 4500 \cdot 0.98^{n-1}$

$\sum_{n=1}^{22} u_n$ Using $S_n = \dfrac{a(1-r^n)}{1-r}$ with

$a = 4500$ $r = 0.98$ $n = 22$

So $S_n = 80730$

So total mass is 807,000 tonnes (3SF)

b) $u_n = ar^{n-1}$ $\qquad n = 22 \quad a = 4500$

$r = 0.98$

$u_n = 4500 \cdot 0.98^{21}$

$u_{22} = 2940$ $\quad$ So mass = 2940 tonnes

c) $80370 - 1500 = 78870$

so $\quad 800 \times 1500 + 78870 \times 600$

so $\quad x = 47.4$

9. (i) Find the value of

$$\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r$$

(3)

(ii) Show that

$$\sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right) = 2$$

(3)

(Total for Question 9 is 6 marks)

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9) i) $\displaystyle\sum_{r=4}^{\infty} 20\left(\frac{1}{2}\right)^r = \sum_{r=1}^{\infty} 20\left(\frac{1}{2}\right)^r - \sum_{r=1}^{3} 20\left(\frac{1}{2}\right)^r$

$S_\infty = \dfrac{a}{1-r}$

$$\sum_{r=0}^{\infty} 20\left(\frac{1}{2}\right)^r - \sum_{r=0}^{1} 20\left(\frac{1}{2}\right)^r - \sum_{r=1}^{3} 20\left(\frac{1}{2}\right)^r$$

$$\frac{20}{\frac{1}{2}} - 20 - \frac{20\left(1-\left(\frac{1}{2}\right)^3\right)}{1-\frac{1}{2}} \times \frac{1}{2}$$

$$= 20 - \frac{35}{2} = \frac{5}{2}$$

b)

$$\sum_{n=1}^{48} \log_5 \left| \frac{n+2}{n+1} \right| = \log_5 \left| \frac{3 \cdot 4 \cdot 5 \cdot \ldots \cdot 49 \cdot 50}{2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot 49} \right|$$

$$= \log_5 \left| \frac{50}{2} \right| = \log_5 |25|$$

$$= 2$$

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A geometric series has common ratio r and first term a.

Given $r \neq 1$ and $a \neq 0$

(a) prove that

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

(4)

Given also that S_{10} is four times S_5

(b) find the exact value of r.

(4)

(Total for Question 10 is 8 marks)

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a)

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$$

$$r \cdot S_n = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$S_{10} = \frac{a(1-r^{10})}{1-r}$$

$$S_5 = \frac{a(1-r^5)}{(1-r)}$$

$$S_{10} = 4S_5 \Rightarrow \frac{a(1-r^{10})}{1-r} = \frac{4a(1-r^5)}{1-r}$$

$$\Rightarrow a(1-r^{10}) = 4a(1-r^5)$$

$$\Rightarrow 1-r^{10} = 4(1-r^5)$$

$$\Rightarrow 1-r^{10} = 4-4r^5$$

$$\Rightarrow r^{10}-4r^5+3=0$$

set $X = r^5$ $\Rightarrow X^2-4X+3=0$

$$(X-3)(X-1)=0$$

$$\Rightarrow r^5 = 3 \text{ or } 1$$

so $r = \sqrt[5]{3}$

11. A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by

$$a_{n+1} = \frac{k(a_n + 2)}{a_n} \qquad n \in ¥$$

where k is a constant.

Given that

- the sequence is a periodic sequence of order 3
- $a_1 = 2$

(*a*) show that

$$k^2 + k - 2 = 0$$

(3)

(*b*) For this sequence explain why $k \neq 1$

(1)

(*c*) Find the value of

$$\sum_{r=1}^{80} a_r$$

(3)

(Total for Question 11 is 7 marks)

$$a_2 = \frac{k(2+2)}{2} = 2k$$

$$a_3 = \frac{k(2k+2)}{2k} \Rightarrow a_3 = k+1$$

$$a_4 = a_1 = 2 = \frac{k(k+1+2)}{k+1} = \frac{k^2+3k}{k+1}$$

$$2 = \frac{k^2+3k}{k+1}$$

$2(k+1) = k^2 + 3k$

$2k + 2 = k^2 + 3k$

$\Rightarrow \quad k^2 + k - 2 = 0$

b) $k \neq 1$ as this would mean the seqence is not periodic and diverges

c) $k^2 + k - 2 = 0$

$(k-1)(k+2) = 0$

$k = -2$

$\Rightarrow a_2 = 2k = -4 \quad \Leftarrow 2(-2) = -4$

$a_3 = k + 1 = -1 \quad (-2) + 1 = -1$

$$\sum_{r=1}^{80} a_r = 26 \sum_{r=1}^{3} a_r + a_{79} + a_{80}$$

$$= 26 \times (2 - 4 + 1) + 2 - 4$$

$$= -28$$