2021 ASSESSMENT MATERIALS

# A-level
# MATHS
## Differentiation (Topic G)

model answers checked

Total number of marks: 44

**2** A curve has equation $y = x^5 + 4x^3 + 7x + q$ where $q$ is a positive constant.

Find the gradient of the curve at the point where $x = 0$

Circle your answer.

**[1 mark]**

0 4 7 $q$

**13** A curve, $C$, has equation

$$y = \frac{e^{3x-5}}{x^2}$$

Show that $C$ has exactly one stationary point.

Fully justify your answer.

**[7 marks]**

$y = \dfrac{e^{3x-5}}{x^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{(3e^{3x-5})x^2 - 2x(e^{3x-5})}{x^4}$

Stationary point $\Rightarrow \dfrac{dy}{dx} = 0$ so $3x^2e^{3x-5} - 2xe^{3x-5} = 0$

$xe^{3x-5}(3x-2) = 0$

$x(3x-2) = 0$ ($e^{3x-5} \neq 0$ for all $x$ so we can divide by it)

$x = 0, \ x = \frac{2}{3}$

As $y = \dfrac{e^{3x-5}}{x^2}$, we divide by $x$ so $y$ is not defined for $x=0$ so there is only one stationary point: $x = \frac{2}{3}$

**15** A curve has equation $y = x^3 - 48x$

The point $A$ on the curve has $x$ coordinate $-4$

The point $B$ on the curve has $x$ coordinate $-4 + h$

**15 (a)** Show that the gradient of the line $AB$ is $h^2 - 12h$

**[4 marks]**

$$\text{gradient} = \frac{\Delta y}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{(-4+h)^3 - 48(-4+h) - \left((-4)^3 - 48(-4)\right)}{-4+h-(-4)}$$

$$= \frac{(-4)^3 + 3(-4)^2h + 3(-4)h^2 + h^3 + 192 - 48h - (-4)^3 - 192}{h}$$

$$= \frac{-12h^2 + h^3}{h}$$

$$= h^2 - 12h$$

**15 (b)** Explain how the result of part **(a)** can be used to show that $A$ is a stationary point on the curve.

**[2 marks]**

As $h \to 0$, $\frac{\Delta y}{\Delta x} \to 0$ and as $h$ gets smaller, the line AB approaches the tangent at A This shows that the gradient at A is 0 which means A is a stationary point

**10** The volume of a spherical bubble is increasing at a constant rate.

Show that the rate of increase of the radius, $r$, of the bubble is inversely proportional to $r^2$

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

**[4 marks]**

$\frac{dv}{dt} = k$ where $k$ is a constant

$$\frac{dv}{dr} = \frac{d}{dr}V = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$$

$$\frac{dv}{dt} \times 1/\frac{dv}{dr} = \frac{dv}{dt} \times \frac{dr}{dv} = \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = k \times \frac{1}{4\pi r^2}$$

$$= \frac{k}{4\pi} \times \frac{1}{r^2}$$

$\Rightarrow \frac{dr}{dt} \propto \frac{1}{r^2}$ so the rate of change of the radius is inversely proportional to $r^2$

**12** A curve $C$ has equation

$$x^3 \sin y + \cos y = Ax$$

where $A$ is a constant.

$C$ passes through the point $P\left(\sqrt{3}, \frac{\pi}{6}\right)$

**12 (a)** Show that $A = 2$

**[2 marks]**

Sub A and P into the equation for C:

$$(\sqrt{3})^3 \sin \frac{\pi}{6} + \cos \frac{\pi}{6} = A\sqrt{3}$$

$$\Rightarrow 3\sqrt{3}\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2} = A\sqrt{3}$$

$$\Rightarrow 3 + 1 = 2A \quad \text{so} \quad A = 2$$

**12 (b) (i)** Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 - 3x^2 \sin y}{x^3 \cos y - \sin y}$

**[5 marks]**

$$\frac{d}{dx}(x^3 \sin y + \cos y) = \frac{d}{dx}(2x)$$

$$\Rightarrow 3x^2 \sin y + x^3 \cos y \frac{dy}{dx} - \sin y \frac{dy}{dx} = 2$$

(Product and chain rule and implicit differentiation)

$$\Rightarrow x^3 \cos y \left(\frac{dy}{dx}\right) - \sin y \left(\frac{dy}{dx}\right) = 2 - 3x^2 \sin y$$

$$\Rightarrow \frac{dy}{dx}(x^3 \cos y - \sin y) = 2 - 3x^2 \sin y$$

$$\Rightarrow \frac{dy}{dx} = \frac{2 - 3x^2 \sin y}{x^3 \cos y - \sin y}$$

**12 (b) (ii)** Hence, find the gradient of the curve at *P*.

**[2 marks]**

$$\left.\frac{dy}{dx}\right|_{x=\sqrt{3},\ y=\frac{\pi}{6}} = \frac{2 - 3(\sqrt{3})^2 \sin\frac{\pi}{6}}{(\sqrt{3})^3 \cos\frac{\pi}{6} - \sin\frac{\pi}{6}}$$

$$= \frac{2 - \frac{9}{2}}{\frac{9}{2} - \frac{1}{2}} = \frac{-\frac{5}{2}}{4} = -\frac{5}{8}$$

**12 (b) (iii)** The tangent to *C* at *P* intersects the $x$-axis at *Q*.

Find the exact $x$-coordinate of *Q*.

**[4 marks]**

$$y - y_1 = m(x - x_1) \Rightarrow y - \frac{\pi}{6} = -\frac{5}{8}(x - \sqrt{3})$$

$x$-axis $\Rightarrow$ $y$-coordinate $= 0$

$$0 - \frac{\pi}{6} = -\frac{5}{8}(x - \sqrt{3}) \Rightarrow \frac{8\pi}{30} = x - \sqrt{3}$$

$$\Rightarrow x = \sqrt{3} + \frac{4\pi}{15}$$

**6** A function f is defined by $f(x) = \dfrac{x}{\sqrt{2x-2}}$

**6 (a)** State the maximum possible domain of f.

**[2 marks]**

$2x - 2 > 0 \Rightarrow x - 1 > 0$

$\Rightarrow x > 1$

**6 (b)** Use the quotient rule to show that $f'(x) = \dfrac{x-2}{(2x-2)^{\frac{3}{2}}}$

**[3 marks]**

$$f'(x) = \frac{1(\sqrt{2x-2}) - x\left(\frac{1}{2\sqrt{2x-2}} \times 2\right)}{2x-2} = \frac{(2x-2)^{\frac{1}{2}} - x(2x-2)^{-\frac{1}{2}}}{2x-2}$$

$$= \frac{(2x-2)^{-\frac{1}{2}}\left[(2x-2) - x\right]}{2x-2} = \frac{2x-2-x}{(2x-2)^{3/2}} = \frac{x-2}{(2x-2)^{3/2}}$$

**6 (c)** Show that the graph of $y = f(x)$ has exactly one point of inflection.

**[7 marks]**

$$f''(x) = \frac{(2x-3)^{3/2} - (x-2)\left(\frac{3}{2}(2)(2x-2)^{\frac{1}{2}}\right)}{(2x-2)^3}$$

Point of inflection $\Rightarrow f''(x) = 0$

So $(2x-2)^{3/2} - (x-2)\left(3(2x-2)^{\frac{1}{2}}\right) = 0$

$(2x-2) - 3(x-2) = 0$

$2x - 2 - 3x + 6 = 0$

$-x - 4 = 0$

$x = 4$

So the only point of inflection is at $x = 4$

**6 (d)** Write down the values of $x$ for which the graph of $y = \mathrm{f}(x)$ is convex.

**[1 mark]**

$1 < x < 4$