

A-level MATHS

Differentiation (Topic G)

Total number of marks: 44

- 2 A curve has equation $y = x^5 + 4x^3 + 7x + q$ where q is a positive constant.

Find the gradient of the curve at the point where $x = 0$

Circle your answer.

[1 mark]

0

4

7

q

- 13 A curve, C , has equation

$$y = \frac{e^{3x-5}}{x^2}$$

Show that C has exactly one stationary point.

Fully justify your answer.

[7 marks]

$$y = \frac{e^{3x-5}}{x^2} \Rightarrow \frac{dy}{dx} = \frac{(3e^{3x-5})x^2 - 2xe^{3x-5}}{x^4}$$

$$\text{Stationary point} \Rightarrow \frac{dy}{dx} = 0 \quad \text{so} \quad 3x^2e^{3x-5} - 2xe^{3x-5} = 0$$

$$xe^{3x-5}(3x-2) = 0$$

$$x(3x-2) = 0 \quad (e^{3x-5} \neq 0 \text{ for all } x \text{ so we can divide by it})$$

$$x = 0, \quad x = \frac{2}{3}$$

As $y = \frac{e^{3x-5}}{x^2}$, we divide by x so y is not defined

for $x = 0$ so there is only one stationary point: $x = \frac{2}{3}$

- 15 A curve has equation $y = x^3 - 48x$

The point A on the curve has x coordinate -4

The point B on the curve has x coordinate $-4 + h$

15 (a) Show that the gradient of the line AB is $h^2 - 12h$

[4 marks]

$$\text{gradient} = \frac{\Delta y}{\Delta x}$$

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{(-4+h)^3 - 48(-4+h) - ((-4)^3 - 48(-4))}{-4+h - (-4)} \\ &= \frac{(-4)^3 + 3(-4)^2h + 3(-4)h^2 + h^3 + 192 - 48h - (-4)^3 - 192}{h} \\ &= \frac{-12h^2 + h^3}{h} \\ &= h^2 - 12h \end{aligned}$$

15 (b) Explain how the result of part (a) can be used to show that A is a stationary point on the curve.

[2 marks]

As $h \rightarrow 0$, $\frac{\Delta y}{\Delta x} \rightarrow 0$ and as h gets smaller, the line AB approaches the tangent at A. This shows that the gradient at A is 0 which means A is a stationary point.

10 The volume of a spherical bubble is increasing at a constant rate.

Show that the rate of increase of the radius, r , of the bubble is inversely proportional to r^2

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

[4 marks]

$$\frac{dv}{dt} = k \quad \text{where } k \text{ is a constant}$$

$$\frac{dv}{dr} = \frac{d}{dr} v = \frac{d}{dr} \left(\frac{4}{3}\pi r^3 \right) = 4\pi r^2$$

$$\frac{dv}{dt} \times \frac{1}{\frac{dv}{dr}} = \frac{dv}{dt} \times \frac{dr}{dv} = \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = k \times \frac{1}{4\pi r^2} \quad 4$$

$$= \frac{k}{4\pi} \times \frac{1}{r^2}$$

$\Rightarrow \frac{dr}{dt} \propto \frac{1}{r^2}$ so the rate of change of the radius is inversely proportional to r^2

12 A curve C has equation

$$x^3 \sin y + \cos y = Ax$$

where A is a constant.

C passes through the point $P(\sqrt{3}, \frac{\pi}{6})$

12 (a) Show that $A = 2$

[2 marks]

Sub A and P into the equation for C:

$$(\sqrt{3})^3 \sin \frac{\pi}{6} + \cos \frac{\pi}{6} = A\sqrt{3}$$

$$\Rightarrow 3\sqrt{3} \left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2} = A\sqrt{3}$$

$$\Rightarrow 3 + 1 = 2A \quad \text{so } A = 2$$

12 (b) (i) Show that $\frac{dy}{dx} = \frac{2 - 3x^2 \sin y}{x^3 \cos y - \sin y}$

[5 marks]

$$\frac{d}{dx} (x^3 \sin y + \cos y) = \frac{d}{dx} (2x)$$

$$\Rightarrow 3x^2 \sin y + x^3 \cos y \frac{dy}{dx} - \sin y \frac{dy}{dx} = 2$$

(Product and chain rule and implicit differentiation)

$$\Rightarrow x^3 \cos y \left(\frac{dy}{dx}\right) - \sin y \left(\frac{dy}{dx}\right) = 2 - 3x^2 \sin y$$

$$\Rightarrow \frac{dy}{dx} (x^3 \cos y - \sin y) = 2 - 3x^2 \sin y$$

$$\Rightarrow \frac{dy}{dx} = \frac{2 - 3x^2 \sin y}{x^2 \cos y - \sin y}$$

12 (b) (ii) Hence, find the gradient of the curve at P .

[2 marks]

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=\sqrt{3}, y=\frac{\pi}{6}} &= \frac{2 - 3(\sqrt{3})^2 \sin \frac{\pi}{6}}{(\sqrt{3})^3 \cos \frac{\pi}{6} - \sin \frac{\pi}{6}} \\ &= \frac{2 - \frac{9}{2}}{\frac{9}{2} - \frac{1}{2}} = \frac{-\frac{5}{2}}{4} = -\frac{5}{8} \end{aligned}$$

12 (b) (iii) The tangent to C at P intersects the x -axis at Q .

Find the exact x -coordinate of Q .

[4 marks]

$$y - y_1 = m(x - x_1) \Rightarrow y - \frac{\pi}{6} = -\frac{5}{8}(x - \sqrt{3})$$

x -axis $\Rightarrow y$ -coordinate = 0

$$\begin{aligned} 0 - \frac{\pi}{6} &= -\frac{5}{8}(x - \sqrt{3}) \Rightarrow \frac{8\pi}{30} = x - \sqrt{3} \\ &\Rightarrow x = \sqrt{3} + \frac{4\pi}{15} \end{aligned}$$

6 A function f is defined by $f(x) = \frac{x}{\sqrt{2x-2}}$

6 (a) State the maximum possible domain of f .

[2 marks]

$$\begin{aligned} 2x-2 > 0 &\Rightarrow x-1 > 0 \\ &\Rightarrow x > 1 \end{aligned}$$

6 (b) Use the quotient rule to show that $f'(x) = \frac{x-2}{(2x-2)^{\frac{3}{2}}}$

$$\begin{aligned} f'(x) &= \frac{1(\sqrt{2x-2}) - x\left(\frac{1}{2\sqrt{2x-2}} \times 2\right)}{2x-2} = \frac{(2x-2)^{\frac{1}{2}} - x(2x-2)^{-\frac{1}{2}}}{2x-2} \quad [3 \text{ marks}] \\ &= \frac{(2x-2)^{-\frac{1}{2}} \left[(2x-2) - x \right]}{2x-2} = \frac{2x-2-x}{(2x-2)^{\frac{3}{2}}} = \frac{x-2}{(2x-2)^{\frac{3}{2}}} \end{aligned}$$

6 (c) Show that the graph of $y = f(x)$ has exactly one point of inflection.

[7 marks]

$$f''(x) = \frac{(2x-2)^{\frac{3}{2}} - (x-2)\left(\frac{3}{2}(2)(2x-2)^{\frac{1}{2}}\right)}{(2x-2)^3}$$

Point of inflection $\Rightarrow f''(x) = 0$

$$\text{So } (2x-2)^{\frac{3}{2}} - (x-2)(3(2x-2)^{\frac{1}{2}}) = 0$$

$$(2x-2) - 3(x-2) = 0$$

$$2x-2-3x+6 = 0$$

$$-x+4 = 0$$

$$x = 4$$

So the only point of inflection is at $x=4$

6 (d) Write down the values of x for which the graph of $y = f(x)$ is convex.

[1 mark]

$$1 < x < 4$$