A- level
MATHS

Algebra and Functions (Topic B)

Total number of marks: 42

2 The graph of $y = 5^x$ is transformed by a stretch in the y-direction, scale factor 5

State the equation of the transformed graph.

Circle your answer.

[1 mark]

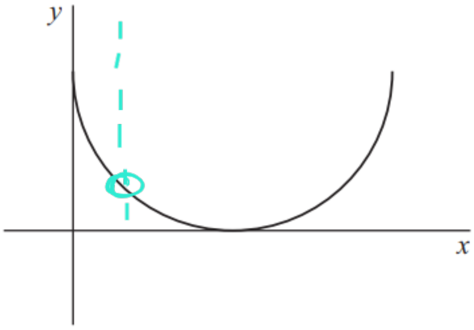
$y = 5 \times 5^x$ $\qquad$ $y = 5^{\frac{x}{5}}$ $\qquad$ $y = \frac{1}{5} \times 5^x$ $\qquad$ $y = 5^{5x}$

3 Determine which one of these graphs does **not** represent y as a function of x.

Tick (✓) **one** box.

For each x there corresponds a <u>unique</u> y.

[1 mark]



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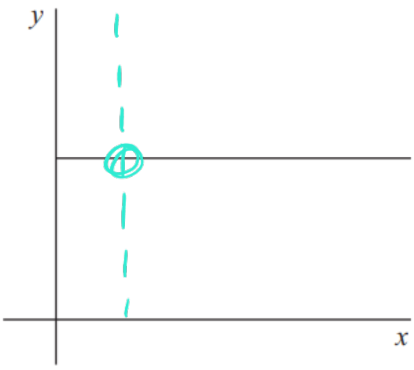

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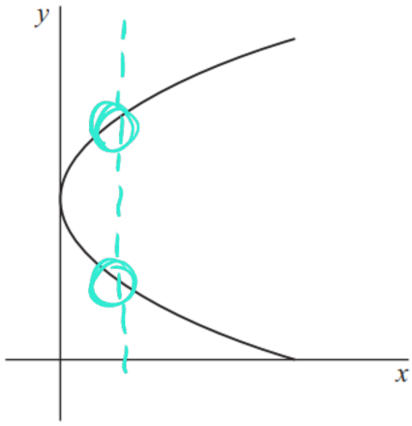

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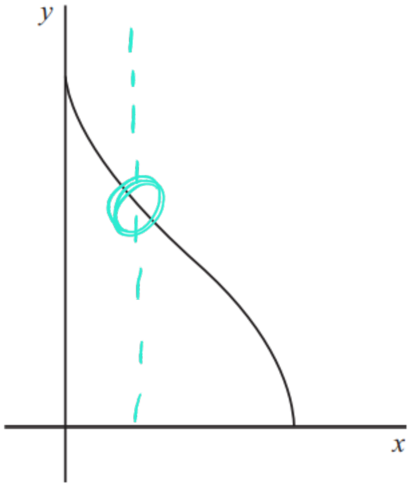

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4 Sketch the region defined by the inequalities

$$y \leq (1-2x)(x+3) \text{ and } y - x \leq 3$$

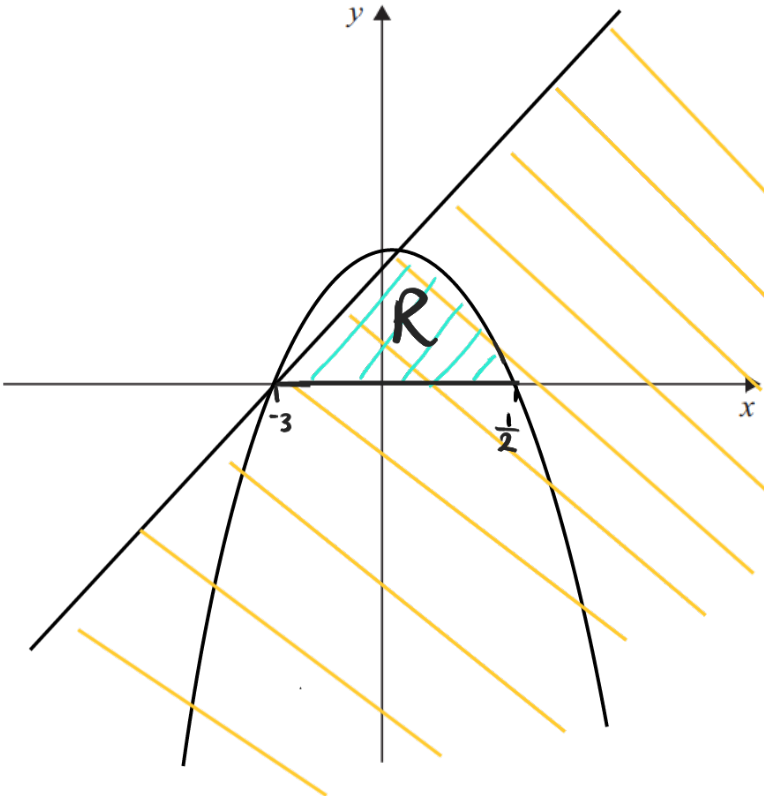
$y \leqslant x+3$

Clearly indicate your region by shading it in and labelling it R.

[3 marks]

$y \leqslant -2x^2 - 5x + 3$

roots: $x = -3$, $x = \frac{1}{2}$



4 $p(x) = 4x^3 - 15x^2 - 48x - 36$

4 (a) Use the factor theorem to prove that $x-6$ is a factor of $p(x)$.

[2 marks]

$x - 6 = 0 \quad x = 6$

when $x = 6$:

$p(6) = 4(6)^3 - 15(6)^2 - 48(6) - 36 = 0 \quad \therefore x-6$ is a factor of $p(x)$

4 (b) (i) Prove that the graph of $y = p(x)$ intersects the x-axis at exactly one point.

[4 marks]

$$\begin{array}{r} 4x^2 + 9x + 6 \\ x-6 \overline{) 4x^3 - 15x^2 - 48x - 36} \\ -\ 4x^3 - 24x^2 \\ \hline 9x^2 - 48x \\ 9x^2 - 54x \\ \hline 6x - 36 \\ 6x - 36 \\ \hline 0 \end{array}$$

$(4x^2 + 9x + 6)(x-6) = 0$

$4x^2 + 9x + 6 = 0$

using $b^2 - 4ac$ (discriminant):

$81 - (4 \times 4 \times 6) = -15$

$-15 < 0 \quad \therefore$ no roots

$\therefore x = 6$ is the only root.

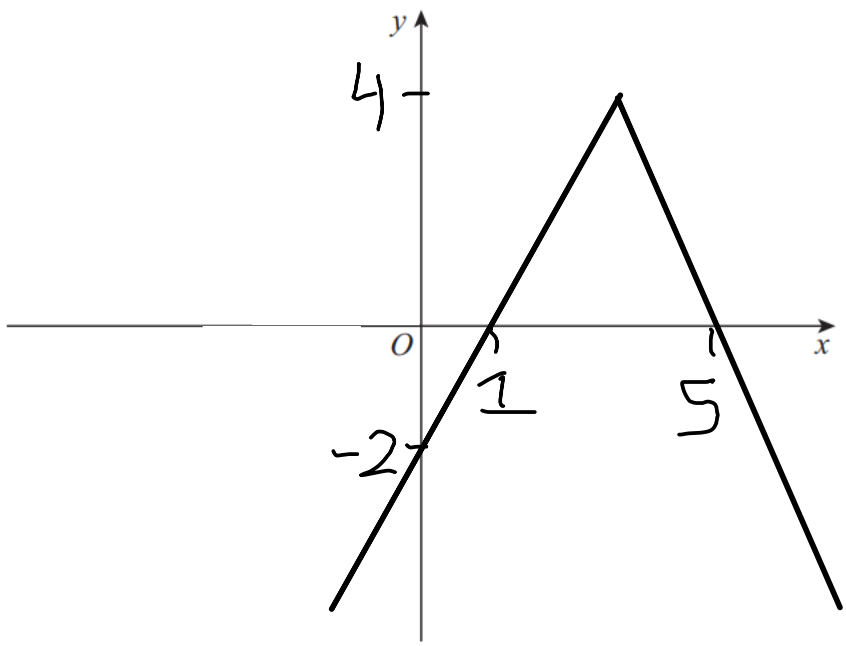
4 (b) (ii) State the coordinates of this point of intersection.

(6,0)

[1 mark]

4 (a) Sketch the graph of

$$y = 4 - |2x - 6|$$



[3 marks]

4 (b) Solve the inequality

$$4 - |2x - 6| > 2$$

[2 marks]

$2 > |2x - 6|$ so $2 > 2x - 6$

and $2 > -2x + 6$

$2 > 2x - 6 \Rightarrow 8 > 2x$ so $x < 4$

$2 > -2x + 6 \Rightarrow 2x > 4$ so $x > 2$

Thus $2 < x < 4$

9 Chloe is attempting to write $\frac{2x^2+x}{(x+1)(x+2)^2}$ as partial fractions, with constant numerators.

Her incorrect attempt is shown below.

Step 1 $\quad \frac{2x^2+x}{(x+1)(x+2)^2} \equiv \frac{A}{x+1} + \frac{B}{(x+2)^2}$

Step 2 $\quad 2x^2+x \equiv A(x+2)^2 + B(x+1)$

Step 3 $\quad$ Let $x=-1 \Rightarrow A=1$
Let $x=-2 \Rightarrow B=-6$

Answer $\quad \frac{2x^2+x}{(x+1)(x+2)^2} \equiv \frac{1}{x+1} - \frac{6}{(x+2)^2}$

9 (a) (i) By using a counter example, show that the answer obtained by Chloe cannot be correct.

[2 marks]

let $x=1$:

$$\frac{1}{1+1} - \frac{6}{(1+2)^2} = -\frac{1}{6}$$

$$\frac{2(1)^2+1}{(1+1)(1+2)^2} = \frac{1}{6}$$

$$\Rightarrow -\frac{1}{6} \neq \frac{1}{6}$$

$$\therefore \frac{2x^2+x}{(x+1)(x+2)^2} \neq \frac{1}{x+1} - \frac{6}{(x+2)^2}$$

9 (a) (ii) Explain her mistake in Step 1.

[1 mark]

She should split it as $\frac{A}{x+1} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)}$

9 (b) Write $\frac{2x^2+x}{(x+1)(x+2)^2}$ as partial fractions, with constant numerators.

[4 marks]

$x(2x+1)$

$$\frac{2x^2+x}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)}$$

$$2x^2+x = A(x+2)^2 + B(x+1) + C(x+1)(x+2)$$

when $x=-2$:
$6 = B \quad B = -6$

when $x=-1$
$1 = A$

when $x=0$
$0 = 4A + B + 2C$
$0 = 4(1) - 6 + 2C$
$2 = 2C$
$C = 1$

$$\boxed{\therefore \frac{2x^2+x}{(x+1)(x+2)^2} = \frac{1}{(x+1)} + \frac{1}{(x+2)} - \frac{6}{(x+2)^2}}$$

checking
let $x=1$:

$$\frac{2(1)^2+1}{(1+1)(1+2)^2} = \frac{1}{6} \qquad \frac{1}{(1+1)} + \frac{1}{(1+2)} - \frac{6}{(1+2)^2}$$

$$= \frac{1}{6}$$

6 The function f is defined by

$$f(x) = \frac{1}{2}(x^2 + 1),\ x \geq 0$$

6 (a) Find the range of f. **[1 mark]**

when $x = 0$:

$\frac{1}{2}(0^2+1) = \frac{1}{2}$ $\quad f(x) \geq \frac{1}{2}$

6 (b) (i) Find $f^{-1}(x)$ **[3 marks]**

let $x = \frac{1}{2}(y^2+1)$ $\quad \sqrt{2x-1} = y$

$2x = y^2 + 1$

$2x - 1 = y^2$ $\quad \therefore f^{-1}(x) = \sqrt{2x-1}$

6 (b) (ii) State the range of $f^{-1}(x)$ **[1 mark]**

$f^{-1}(x) \geq 0$

6 (c) State the transformation which maps the graph of $y = f(x)$ onto the graph of $y = f^{-1}(x)$ **[1 mark]**

Reflection in the line $y = x$

6 (d) Find the coordinates of the point of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ **[2 marks]**

Method 1

$\sqrt{2x-1} = \frac{1}{2}(x^2+1)$

$2x - 1 = \frac{1}{4}(x^2+1)^2$

$8x - 4 = x^4 + 2x^2 + 1$

$0 = x^4 + 2x^2 - 8x + 5 = 0$

$x = 1$ $\quad x =$ imaginary roots

$\therefore y = \frac{1}{2}(1^2+1)$

$= 1$

$(1, 1)$

Method 2

$y = x$

$y = \sqrt{2x-1}$

$x = \sqrt{2x^2-1}$

$x^2 = 2x^2 - 1$

$0 = x^2 - 1$

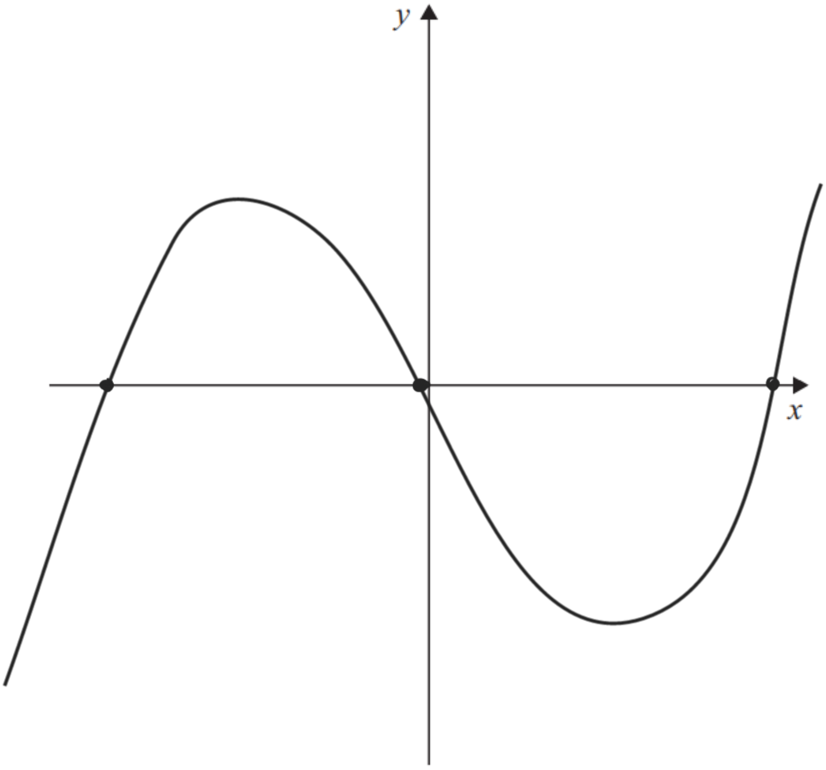
$0 = (x+1)(x-1)$

$x = -1$ $\quad x = 1$ $\quad$ As $x = y$

domain: $x \geq \frac{1}{2}$ $\therefore$ reject $\quad y = 1$ $\quad \therefore (1, 1)$

7 (a) Sketch the graph of any cubic function that has **both** three distinct real roots **and** a positive coefficient of x^3

[2 marks]



7 (b) The function $f(x)$ is defined by

$$f(x) = x^3 + 3px^2 + q$$

where p and q are constants and $p > 0$

7 (b) (i) Show that there is a turning point where the curve crosses the y-axis.

[3 marks]

$f'(x) = 3x^2 + 6px$ (gradient (m) = 0 ∴ turning point)

when $x = 0$

$f'(0) = 0$ ∴ turning point where curve crosses the y-axis ($x = 0$)

7 (b) (ii) The equation $f(x) = 0$ has three distinct real roots.

By considering the positions of the turning points find, in terms of p, the range of possible values of q.

[5 marks]

$f'(x) = 3x^2 + 6px$

when $f'(x) = 0$

$0 = 3x^2 + 6px$

$0 = 3x(x + 2p)$

Turning point when $x = 0$ and when $x = -2p$

$f''(x) = 6x + 6p$

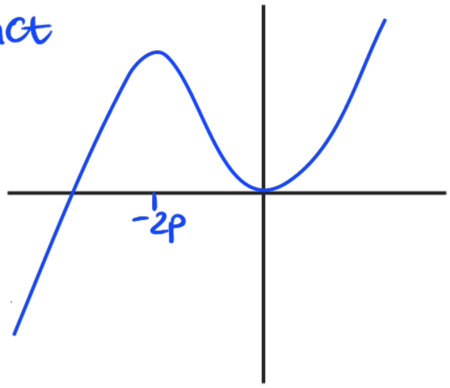
when $x = 0$, $f''(x) = 6p > 0$ (local minimum)

when $x = -2p$

$f''(x) = -12p + 6p = -6p < 0$ (local maximum)

if $q = 0$:

(2 distinct roots)



$q < 0$ (due to needed 3 distinct roots) * look on sketched graph *

when $x = -2p$

$f(-2p) = (-2p)^3 + 3p(-2p)^2 + q$

$= -8p^3 + 12p^3 + q$

$= 4p^3 + q$

if $f(-2p) = 0$,

$4p^3 + q = 0$

$-4p^3 = q$

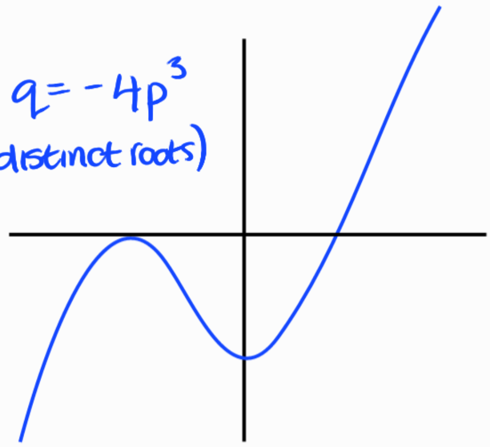
($\therefore$ only 2 distinct roots)

* look on sketched graph *

$\therefore q > -4p^3$

if $q = -4p^3$

(2 distinct roots)



$\therefore -4p^3 < q < 0$