Admissions Testing Service

STEP Solutions 2015

Mathematics

STEP 9465/9470/9475

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Test

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STEP Mathematics (9465, 9470, 9475)

SI-2015 Hints and Solutions

(See the marking scheme for full details of the expected solution approach)

Q1 This question is intended to be a relatively straightforward entrée into the paper, and thus its demands are fairly routine in nature. That does not mean that it is easy, merely that the appropriate courses of action should be readily accessible to all candidates of a suitable standard. To begin with, the demand for a sketch (of any function) should lead you to consider things such as

- key points (such as where the curve meets either of the coordinate axes);
- asymptotes (note the information given in italics at the end of part (i) regarding what happens as $x \to -\infty$, which indicates that the negative x-axis is an asymptote in this case);
- turning points of the curve, which are clearly flagged as being of significance when considering what happens when $y = k$; i.e. when the curve meets a horizontal line;
- long-term behaviour (you already have sorted for you the "$x \to -\infty$" side of things, so there is only a quick decision to be made about what happens as $x \to +\infty$).

For the key points, first set $x = 0$ and then $y = 0$; the asymptote is effectively given; the TPs come from setting the first derivative to zero and solving for x again (noting, of course, that e^x is always positive); and the curve clearly grows exponentially as x increases positively. The rest of (i) then simply requires a bit of thought as to how many times a horizontal line will cut, or touch, the curve depending upon the value of k.

In part (ii), it is clear that the x in part (i) has now been replaced with an x^2, and this second curve must therefore have reflection symmetry in the y-axis, as all negative values of x are being squared to give the positive counterpart. Previously, when x was equal to zero, we now have$x^2 = 0$, and so each previous crossing-point on the positive x–axis leads to two, one on each side of the y-axis (and at the square-root of its former value). However, the previous y-intercept is unchanged, but must now appear at a TP of the curve (otherwise the symmetry of the gradient would be compromised). Also, the previous TP with positive x-value (the negative one has gone) occurs at the square-roots of the previous value, but again with unchanged y-coordinate.

Q2 It is clear that (i) is an introductory part that requires the use of the $\cos(A - B)$ formula with suitably-chosen values of A and$B$. Using the $\sin(A - B)$ formula then leads to the second result, although there are alternative trig. identities that could be used in both cases, such as a double-angle formula. Repeated use of these, or the double-angle formulae (or *de Moivre's Theorem* for those from a further maths background) lead to the (relatively) well-known 'triple-angle formula' $\cos 3\alpha \equiv 4\cos^3\alpha - 3\cos\alpha$, which gives $x = \cos\alpha$ as a root of the equation $4x^3 - 3x - \cos 3\alpha = 0$. Although there are several possible methods here, a simple division/factorisation leads to $(x - c)\{4(x^2 + cx + c^2) - 3\}$ = 0, and the quadratic formula leads to the remaining two roots which, for their simplest form, requires the use of the most elementary of trig. identities, $s^2 = 1 - c^2$.

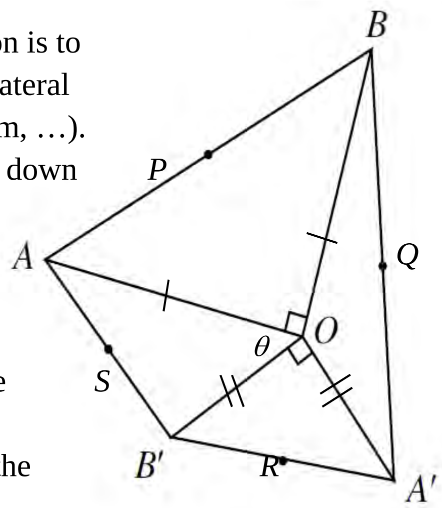
A bit of insight is needed in part (iii), where one should first realise that the constant term is intended be a cosine value (of 3 × some angle), and the most obvious candidate is $\frac{1}{2}\sqrt{2}$ = cos 45°, so that α is 15°. From here, it is now clear that "x" is $\frac{1}{2}y$, and that the three roots are those from part (ii) with the exact numerical forms of the sine and cosine of 15° from part (i) waiting to be deployed in order to find the surd forms requested.

Q3 In this question, it is important to draw suitable diagrams in order to visualise what is going on, and these are not difficult to manage, with the guard either at a corner of the yard (C) or at its middle (M). However, this second case has two possible sub-cases to consider, depending upon whether the 'far' corners of the yard are visible to him/her (which, in fact, turns out to be the $b = 3a$ case which separates the two cases that the question invites you to consider), and it is the extra length of the opposite wall that is visible that makes for different working. These lengths are $\frac{4b^2}{b+a}$ (from C), $\frac{b^2}{a}$ (from M, with the 'far' corners not visible) and $\frac{2b(2b-3a)}{b-a}$ (from M, with the 'far' corners visible). Once obtained, these should be compared in order to find that the guard should stand at C for $b < 3a$ and at M for $b > 3a$ (and at either when $b = 3a$).

Q4 A quick read-through of the question should make it clear that it is the lower end of the rod that is being referred to (as do the subtractions within the given coordinates x and y). The fact that the rod is tangent to the given parabola means that its direction is $\tan\theta = \frac{1}{2}x$, which gives the coordinates of the rod's midpoint as $(\tan\theta, \tan^2\theta)$; a simple right-angled triangle and some accompanying basic trig. then leads to the given answer. The second part of the question is equally straightforward once one realises that when $x_A = 0$, $2\tan\alpha = b\cos\alpha \Rightarrow b = \frac{2\tan\alpha}{\cos\alpha} \Rightarrow y_A = -\tan^2\alpha$. There are several ways to attack the area between two curves – e.g. as $\int (y_1 - y_2)\,dx$, or by translating the bit below the x-axis up by $\tan^2\alpha$ and calculating the difference between area under the "new" curve and a triangle; the key is to eliminate the "b" and then the given answer follows.

Q5 Once one realises that the x within the integral is not the variable, then both integrations are relatively straightforward. In (i), we get $f(x) = \frac{2^x}{x}$, while in (ii) $g(x) = \frac{1}{x}\left(|1+x| - |1-x|\right)$ … remember to use the modulus function when taking square-roots (although one could, alternatively, work out a piece-wise definition for g; that is, in bits). The sketch of f should prompt the solver to differentiate in order to identify the turning point at $\left(\frac{1}{\ln 2},\ e\ln 2\right)$. Noting that $y \to +\infty$ as $x \to 0$ and that $y \to +\infty$ as $x \to +\infty$ gives all else that is needful to draw the graph in (i). In (ii), the piece-wise definition of g is certainly more useful now since its graph is made up of two 'reciprocal' curve bits joined by a horizontal straight-line in the middle.

Q6 The best way to start any geometrically-inclined question is to have a good diagram that doesn't make the shape of the quadrilateral look too specialised in any way (square, rectangle, parallelogram, …). Next, label the midpoints sensibly (see diagram) and then write down their position vectors in terms of **a** and **b**.



It is relatively easy to prove that the opposite sides of this quadrilateral are equal and opposite, but you must then also show that adjacent sides are equal in length and that they are perpendicular. This last outcome is going to follow from the use of the scalar product.

For the final part, you should label one of the angles at the

centre θ (say) and note that the fourth angle at O is thus $180^{\circ} - \theta$.
Having already calculated the squares of the lengths of the square's sides in the form

$$\tfrac{1}{4}\left(a^2 + (a')^2 - 2aa'\cos\left[90^{\circ} + \theta\right]\right)$$

the required result follows from noting that this is maximal when $\cos\left[90^{\circ} + \theta\right] = -1$; i.e. when $\theta = 90^{\circ}$.

Q7 The crucial observation here is that a (continuous) function takes its maximum value on a finite interval either at a maximum turning-point *or* at an endpoint. Differentiating (a 'negative' cubic – so we know what its shape is) gives a MIN. TP at (0, 0) and a MAX. TP at $(\tfrac{1}{3}a, \tfrac{1}{9}a^3)$, and evaluating at the endpoints gives $f(-\tfrac{1}{3}) = \tfrac{1}{9}(3a + 2)$ and $f(1) = 3a - 6$.

Now, a comparison of these possible values for f then yields that $\tfrac{1}{9}(3a + 2) \geq \tfrac{1}{9}a^3 \Leftrightarrow a \geq 0, a \leq 2$; and that $\tfrac{1}{9}a^3 \geq 3a - 6$ holds for all $a \geq 0$; and also that $\tfrac{1}{9}(3a + 2) \geq 3a - 6 \Leftrightarrow a \leq \tfrac{7}{3}$ (which, actually, affects nothing, but the working should be done anyhow). Thus $M(a) = \begin{cases} \tfrac{1}{9}(3a+2) & 0 \leq a \leq 2 \\ \tfrac{1}{9}a^3 & 2 \leq a \leq 3 \\ 3a - 6 & a \geq 3 \end{cases}$.

Q8 The standard "bookwork" approach to this opening part is to write the sum (S) both forwards and backwards, add the terms in pairs (n pairs, each of value $n + 1$) and then to half this to get $S = \tfrac{1}{2}n(n + 1)$. As with any such invitation to establish a result, one should not simply seek to quote a result and thus merely "write down" the given answer. When looking at part (ii)'s question, the *binomial theorem* should really be screaming at you from the page, and all that is needed is to observe that the binomial expansion of $(N - m)^k$ consists of $k + 1$ terms, the first k of which contain a factor of (at least one) N. The final term, since k is odd must be $-m^k$ which then conveniently cancels with the $+ m^k$ term to leave something that is clearly divisible by N.

In the next part of the question, you are invited to explore the cases n odd and n even separately (indeed the results that follow are slightly different). To begin with,

$$S = 1^k + 2^k + \ldots + n^k \text{ (an odd no. of terms)} = 0^k + 1^k + 2^k + \ldots + n^k \text{ (an even no. of terms)}$$

So these terms can now be paired up:

$$n \text{ with } 0, \quad n - 1 \text{ with } 1, \quad \ldots, \quad (\tfrac{1}{2}n + \tfrac{1}{2}) \text{ with } (\tfrac{1}{2}n - \tfrac{1}{2}),$$

so that all pairs are of the form $(n - m)^k + m^k$, which was just established as being divisible by n. Next, in the case when

$$S = 1^k + 2^k + \ldots + n^k \text{ (an even no. of terms)} = 0^k + 1^k + 2^k + \ldots + n^k \text{ (an odd no. of terms)},$$

the pairs are now

$$n \text{ with } 0, \quad n - 1 \text{ with } 1, \quad \ldots, \quad (\tfrac{1}{2}n + 1) \text{ with } (\tfrac{1}{2}n - 1),$$

but with an odd term, $(\tfrac{1}{2}n)^k$, left over. This gives us (from the same previous result as before) a sum consisting of terms divisible by n and one that is divisible by $\tfrac{1}{2}n$, giving the second result.

Then, for n even, so that $(n + 1)$ is odd, $S + (n + 1)^k$ is divisible by $n + 1$ (by the previous result) $\Rightarrow S$ is divisible by $n + 1$; and for n odd, so that $(n + 1)$ is even, $S + (n + 1)^k$ is divisible by $\tfrac{1}{2}(n + 1)$. Thus, since $\text{hcf}(n, n + 1) = 1 \Rightarrow \text{hcf}(\tfrac{1}{2}n, n + 1) = 1$ for n even, and $\text{hcf}(n, \tfrac{1}{2}(n + 1)) = 1$ for n odd, it follows that S is divisible by $\tfrac{1}{2}n(n + 1)$ for all positive integers n.

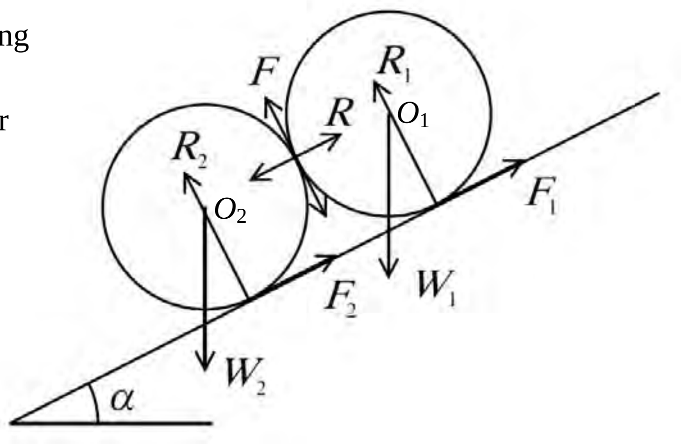
Q9 The standard time taken to land (at the level of the projection) of a projectile is $t = \dfrac{2u\sin\alpha}{g}$. Thus, a bullet fired at time t, $0 \le t \le \dfrac{\pi}{6\lambda}$, lands at time $T_L = t + \dfrac{2u}{g}\sin\left(\dfrac{\pi}{3} - \lambda t\right)$. Differentiating this w.r.t. t and setting it equal to zero, gives $k = \cos\left(\dfrac{\pi}{3} - \lambda t\right)$. The horizontal range is then given by $R = \dfrac{2u^2 \sin\alpha\cos\alpha}{g}$ and this gives the required answer. Moreover, substituting the endpoints of the given time interval $0 \le t \le \dfrac{\pi}{6\lambda}$ into $k = \cos\left(\dfrac{\pi}{3} - \lambda t\right)$ gives $\dfrac{1}{2} \le k \le \dfrac{\sqrt{3}}{2}$. However, if $k < \frac{1}{2}$, then one sees that $\dfrac{dT_L}{dt} < 0$ throughout the gun's firing, so that T_L is a (strictly) decreasing function. Hence its maximum value occurs at t = 0, i.e. $\alpha = \dfrac{\pi}{3}$, whence $R = \dfrac{2u^2}{g} \times \dfrac{1}{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{u^2\sqrt{3}}{2g}$.

Q10 The difficulty in this question lies in ignoring unnecessary information (not given). Firstly, then, note that the speed of the rain relative to the bus is $v\cos\theta - u$ (or $u - v\cos\theta$ if negative), and when $u = 0$, the area of the bus getting wet, A, is such that $A \propto hv\cos\theta + av\sin\theta$. Now the given result follows from observing that when $v\cos\theta - u > 0$, the rain hitting the top of the bus is the same, whilethe rain hits the back of the bus as before, but with speed $v\cos\theta - u$ instead of $v\cos\theta$; and when $v\cos\theta - u < 0$, the rain hitting the top of the bus is the same, while the rain hits the front of the bus as before, but with $u - v\cos\theta$ instead of $v\cos\theta$.

Next, as the journey time $\propto \dfrac{1}{u}$, we need to minimise $J = \dfrac{av\sin\theta}{u} + \dfrac{h\,|\,v\cos\theta - u\,|}{u}$. For $v\cos\theta - u > 0$ and $w \le v\cos\theta$, we minimise $J = \dfrac{av\sin\theta}{u} + \dfrac{hv\cos\theta}{u} - h$, and this decreases as u increases, and this is done by choosing u as large as possible; i.e. $u = w$. For $u - v\cos\theta > 0$, we minimise $J = \dfrac{av\sin\theta}{u} - \dfrac{hv\cos\theta}{u} + h$, and this decreases as u increases if $a\sin\theta > h\cos\theta$, so we again choose u as large as possible; i.e. $u = w$. Next, if $a\sin\theta < h\cos\theta$, then J increases with u when u exceeds $v\cos\theta$, so we choose $u = v\cos\theta$ in this case. Finally, if $a\sin\theta = h\cos\theta$ then J is independent of u, so we may as well take $u = w$.

For the return journey, simply replace θ by $180^\circ - \theta$ to give $J = \dfrac{av\sin\theta}{u} + \dfrac{hv\cos\theta}{u} + h$, which always decreases as u increases, so take $u = w$ again.

Q11 As with all statics problems, the key to getting a good start, and to making life as easy as possible for the working that follows, is to have a good, clear diagram with all relevant forces, in appropriate directions, marked on it (see alongside).



To begin with, take moments about the respective cylinders' axes yields $F = F_1 = F_2$, as required. Next, write down the four equations that arise from resolving for each cylinder in the directions parallel and perpendicular to the plane.

These are:-

$F_1 + R = W_1 \sin\alpha$ ①; $R_1 + F = W_1 \cos\alpha$ ②; $F_2 - R = W_2 \sin\alpha$ ③ and $R_2 - F = W_2 \cos\alpha$ ④.

(Note that one could replace some of these with equivalent equations gained from resolving for the whole system.) Replacing F_1 and F_2 by F, equating for $\sin\alpha$, re-arranging for F in terms of R and using the Friction Law, $F \le \mu R$, appropriately leads to the first given answer in (ii). A bit more determination is needed to gain the second given answer, however. Firstly, $\tan\alpha$ can be gained by division in at least two ways, and both F and R must be eliminated from any equations being used. Thereafter, it is simply a matter of forcing the working through correctly and, hopefully, concisely.

Q12 Here, you are given the relevant Poisson result at the outset, and this is intended to guide your thinking later on in the question. To begin with, though, part (i) is actually a Binomial situation … requiring just a single general term. In part (ii), you were asked to prove *algebraically* a result that you might usually be required to quote and use. This requires a good understanding of the use of the sigma-notation and a clear grasp as to which of the various terms are constant relative to the summation, and then combining the remaining terms together appropriately to give the requested Poisson answer. Most important of all, of course, it is essential to have the first line of working correct; this is

$$\mathrm{P}(S = r) = \sum_{n=r}^{\infty} \frac{\mathrm{e}^{-8} 8^n}{n!} \times \frac{n!}{r!(n-r)!} \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{n-r}$$

and one follows this through to the point where the result $\dfrac{\mathrm{e}^{-8} \times 2^r}{r!} \sum_{n=r}^{\infty} \dfrac{6^{n-r}}{(n-r)!}$ is obtained. At this stage another simple trick is required – effectively a re-labelling of the starting-point, using $m = n - r$ to re-write this as $\dfrac{\mathrm{e}^{-8} \times 2^r}{r!} \sum_{m=0}^{\infty} \dfrac{6^m}{m!}$. The required result follows immediately since the infinite sum is just e^6.

Having established this, the final part of the question is relatively straightforward, requiring only the use of the conditional probability formula applied to $\mathrm{P}(M = 8 \mid M + T = 12)$.

Q13 The first three parts of this question are very easy indeed, if looked at in the right way. In part (i) it is not necessary at all that you recognise the Geometric Distribution (indeed, some of you may not have encountered it at all), but the result asked for is simply "(n – 1) failures followed by 1 success", and one can write down immediately, and without explanation, the answer $\mathrm{P}(A) = \left(\frac{5}{6}\right)^{n-1} \left(\frac{1}{6}\right)$. In (ii), you have a situation in which one can apply the principle of symmetry: either a 5 arises before a 6, or *vice versa*, so the required probability is just $\mathrm{P}(B) = \frac{1}{2}$. Part (iii) can be approached similarly, in that the first 4s, 5s, 6s can arise in the orders **<u>456</u>**, 465, **<u>546</u>**, 564, 645, 654 $\Rightarrow \mathrm{P}(B \cap C) = \frac{1}{3}$ (i.e. also "by symmetry", but with three pairings to consider).

Parts (iv) and (v), however, each turn out to require the use of the result given at the end of the question, as the outcomes (theoretically) stretch off to infinity. For (iv), it is best to consider only on which throw the first 6 occurs (since we stop at that point). It cannot occur on the first throw, so we have the sum of the situations:

a 5 occurs on the first throw, followed by a 6 on the second;

one 5 and a 1-4 occur, in either order, followed by the 6 on the third;

one 5 and two 1-4s occur, in any of three possible orders, followed by a 6 on the fourth;

etc.

Thus P(D) = $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)+\binom{2}{1}\left(\frac{1}{6}\right)\left(\frac{4}{6}\right)\left(\frac{1}{6}\right)+\binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{4}{6}\right)^2\left(\frac{1}{6}\right)+\ldots$, and this factorises as $\left(\frac{1}{36}\right)\left\{1+2\left(\frac{2}{3}\right)+3\left(\frac{2}{3}\right)^2+\ldots\right\}$, and the big bracket is just the given result with $x = \frac{2}{3}$ and $n = 2$.

Before getting too deeply into part (v), a couple of simple results should be noted. Firstly, we use the fact that P(E) = P(D) = $\frac{1}{4}$, the answer to (iv); and then that we will need to use the basic probability result P(D∪E) = P(D) + P(E) – P(D∩E) = $\frac{1}{2}$ – P(D∩E). Turning this around, since it is far easier to calculate the probability, P(D∩E), that *both* one 4 and one 5 occur before the first 6. Again, looking at this from the viewpoint of finishing after the first 6 is thrown, we see that

$$P(D\cap E) = \left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)+\binom{3}{1}\left(\tfrac{3}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)+\binom{4}{2}\left(\tfrac{3}{6}\right)^2\left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)+\ldots = \left(\tfrac{1}{108}\right)\left\{1+3\left(\tfrac{1}{2}\right)+6\left(\tfrac{1}{2}\right)^2+\ldots\right\}$$

and the big bracket is the given result with $x = \frac{1}{2}$ and $n = 3$, leading to the answer P(D∪E) = $\frac{23}{54}$.

STEP 2 2015 Hints and Solutions

Question 1

For the first result, show that the gradient of the function is positive for all positive values of x (by differentiating) and also that $f(0) \geq 0$. Once this result has been established sum a set of the terms, using $x = \frac{1}{k}$, note that $\ln\left(1 + \frac{1}{k}\right)$ can be written as $\ln(k+1) - \ln(k)$ and then the required result follows.

For the second part, first show that $x + \ln(1-x)$ is *negative* for $0 < x < 1$ and then use the substitution $x = \frac{1}{k^2}$, noting that $\ln\left(1 - \frac{1}{k^2}\right)$ can be written as $\ln(k-1) - 2\ln(k) + \ln(k+1)$. Deal with the sum starting with $k = 2$ and then add the initial 1 afterwards.

Question 2

As with all geometric questions a good diagram of the information given makes the solution to this question much easier to reach. The first result in this question follows from an application of the sine rule with applications of the relevant formulae for $\sin(A+B)$ and the double angle formulae. From a diagram of the triangle it should then be an easy application of trigonometry to show that $DE = \frac{x}{2}$. There are a number of different methods for establishing that FC trisects the angle ACB – one method is to show that $\sin(\angle FCE) = \frac{1}{2}$, following which it is relatively straightforward to work out the sizes of angles ACB and ACF in terms of α and show that they must satisfy the correct relationship.

Question 3

For the first part note that $T_8 - T_7$ can be interpreted as the triangles that can be made using the rod of length 8 and two other, shorter rods. These can then be counted by noting that there are 6 possibilities if the length 7 rod is used, 4 possibilities if the length 6 (but not the length 7) rod is used and 2 possibilities if the length 5 (but not 6 or 7) rod is used. It is clear that at least one rod longer than length 4 must be used. To evaluate $T_8 - T_6$ note that it is equal to $(T_8 - T_7) + (T_7 - T_6)$ and then evaluate $T_7 - T_6$ in a similar manner to $T_8 - T_7$. Similar reasoning easily gives formulae for $T_{2m} - T_{2m-1}$ and $T_{2m} - T_{2m-2}$.

For the induction, the rule for $T_{2m} - T_{2m-2}$ deduced in the previous part can be used to show the inductive step, while the easiest way to show the base case is to list the possibilities. The easiest way to establish the result for an odd number of rods is to use the formula for $T_{2m} - T_{2m-1}$ and the formula for T_{2m} that was just proven.

Question 4

For the first part, note that the graph of $\arctan x$ satisfies the requirement of being continuous, but does not satisfy $f(0) = \pi$. Since $\tan(x + \pi) = \tan x$, a translation of the graph of $y = \arctan x$ vertically by a distance of π gives the required graph.

It should be clear that the graph of $y = \frac{x}{1+x^2}$ has no vertical asymptotes, approaches the x-axis as $x \to \pm\infty$ and passes through the origin. Identifying the stationary points should be the next task after which a graph should be easy to sketch. The graph of $y = g(x)$ should then be easy to sketch by considering the fact that $f(x)$ is an increasing function and $g(x)$ is obtained by composing the two functions already sketched.

To sketch the graph of $y = \frac{x}{1-x^2}$ first note that there must be two vertical asymptotes. Once stationary points have been checked for it should be straightforward to complete the sketch. In this case, the asymptotes need to be considered to deduce the shape of the graph for $y = h(x)$ as the composition with $f(x)$ will lead to discontinuities. Noting again that $\tan(x + \pi) = \tan x$ the discontinuities can be resolved by translating sections of that graph vertically by a distance of π.

Question 5

The initial proof by induction is a straightforward application of the $\tan(A + B)$ formula. The final part of section (i) requires recognition that there are many possible values of x to give a particular value of $\tan x$, but only one of them is the value that would be obtained by applying the arctan function. The result can therefore be shown by establishing that the difference between consecutive terms of the sequence is never more than π.

For the second part of the question a diagram of the triangle and application of the $\tan 2A$ formula shows that the value of α_n must be of the form used in the first part of the question. All that remains is then to show that the limit of the sum must give the required value.

Question 6

The first part of the question requires use of the $\cos(A + B)$ formula. Following this the integral should be easy to evaluate given that $\int \sec^2 x\, dx = \tan x + c$. In the second part, apply the substitution and note that the limits of the integral are reversed, which is equivalent to multiplying by -1. Following this a simple rearrangement (noting that the variable that the integration is taken over can be changed from y to x) should establish the required result. The integral at the end of this part can then be evaluated simply by applying this result along with the integral evaluated in part (i).

In the final part of the question it is tempting to make repeated applications of the result proven in part (ii). However, this is not valid as it would require the use of a function satisfying $f(\sin x) = x$, which is not possible on the interval over which the integral is defined. Instead, application of a similar substitution to part (ii) to $\int_0^\pi x^3 f(\sin x)\, dx$ will simplify to allow this integral to be evaluated based on the integration of $\frac{1}{(1+\sin x)^2}$. An application of the result from part (ii) will also be required.

Question 7

For part (i) note that the lines joining the centres of the two circles and one of the points where the bisection occurs form a right-angled triangle, so the radius of the new circle can be calculated. To show that no such circle can exist when $r < a$ note that the diametrically opposite points on C must be a distance of $2a$ apart, and no two points on a circle of radius r can be that far apart. For the case $r = a$ note that the new circle would be the same as C (and so would have more than two intersection points).

For part (ii) a similar method can be used to deduce the distances between the centre of the new circle and each of C_1 and C_2. From these distances equations can be formed relating the x and y coordinates of the centre of the new circle. It is then an easy task to eliminate the y-coordinate of the centre of the circle from the equations to get the given value of the x-coordinate.

The expression for y can easily be found by substituting back into the equations obtained from the distance between the centres of two of the circles. Once this is done, note that $y^2 \geq 0$ to obtain the final inequality.

Question 8

The first part of the question follows from consideration of similar triangles in the diagram if the line through P and the centres of the circles is added. For the second part, expressions can be written down for the position vectors of Q and R by noting that the same method as in part (i) will still apply. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ can then be compared to show that one is a multiple of the other.

For the final part of the question, note that Q will lie halfway between P and R if $\overrightarrow{PQ} = \overrightarrow{QR}$.

Question 9

A diagram to represent this situation will show the angles that will be required to calculate the moments of each of the particles about A in terms of θ. Following this, simple trigonometric manipulation should lead to a relationship between $\sin\theta$ and $\cos\theta$. From this, either a right-angled triangle or one of the basic trigonometric identities can be used to reach the required result.

For the second part of the question the amount of potential energy that needs to be gained by the system should be easy to calculate and this must be equal to the initial kinetic energy of the system.

Question 10

The component of the velocity of the particle in the direction of the string at any moment must be equal to V, which leads to $V \text{ cosec } \theta$ as the speed of the particle along the floor. Alternatively, introduce a variable to represent the length of string still in the room or the height of the room and then differentiate x, the distance of the particle from the point directly beneath the hole, with respect to time. The length of the string (to the hole in the ceiling) is decreasing at a rate of $V\ ms^{-1}$, which then allows the introduced variable to be eliminated to reach an expression for the speed of the particle.

Differentiation of the speed of the particle allows the acceleration to be calculated. Finally, note that the particle will remain on the floor as long as the vertical component of the tension is less than the weight of the particle and then the point at which the particle leaves the floor can be identified.

Question 11

For the first part, the coordinates of A are found by applying simple trigonometric ratios and differentiation with respect to time gives the velocity of A. In the second part, the first equation results from consideration of conservation of momentum and the second results from conservation of energy (with a substitution based on the first equation made to eliminate one variable).

Since no energy is lost in any collisions the relationships from part (ii) must continue to hold and this shows that $\dot{\theta}$ cannot be 0 which means that the direction in which θ changes remains the same unless there is a collision. Since the first collision occurs when $\theta = 0$, the second one must be when $\theta = \pi$.

For the final part, note that the equations in part (ii) must still hold, and if $v = 0$, the kinetic energy of B must be 0. Since the kinetic energies of A and C must be equal (by symmetry) it must be the case that the kinetic energy of A is $\frac{1}{4}mu^2$ and can also be calculated from the expression for the velocity of A shown in part (i). Since $\dot{\theta}^2 > 0$, this can then be used to find the values of θ. Finally, note that given these values of θ, v will only be 0 on the occasions when $\dot{\theta}$ is positive.

Question 12

For the first part, note that A can only win the game if the first two tosses result in heads, since once there has been a tail, B will win as soon as two consecutive heads have been tossed and A cannot win until there have been two consecutive heads and one further toss. In the second part, note that this logic still applies to the game for A and similar reasoning can be applied to the game for C. For the other two players switching heads and tails in any sequence that results in a win for B will give a sequence that results in a win for D, and vice versa, so the probabilities must be equal. Since only sequences which alternate between heads and tails forever (and the probabilities of such sequences tend to zero as the lengths of the sequences increase) the probabilities must both also be ¼ .

For the final part, note that C must win if the first two tosses are TT. Since only the previous two tosses are important in determining what could happen on the next toss, each case can be analysed by a tree diagram which shows the outcomes after one further toss.

For example, following HT:

- H gives the position if the last two tosses were TH, and so a probability of winning of q,
- T gives the position if the last two tosses were TT and so a probability of winning of 1.

The total probability is therefore $\frac{1}{2}q + \frac{1}{2}$, but this must also be equal to p.

This yields three equations in the three unknowns which allows all of the individual probabilities to be calculated. Once this is done the overall probability can be calculated.

Question 13

To calculate the expected value of the total cost, note that there is a constant component of ky and then the expected value of the $a(X - y)$ given that $X > y$ must be added, which can be calculated by integration of $(x - y)\lambda e^{-\lambda x}$ with respect to x, between y and ∞. Differentiating the expression for $E(C)$ with respect to y allows the position of the stationary point to be found. If this is at a negative value then y should be chosen to be 0 and otherwise the value of y for the stationary point should be used.

A slightly more complicated integration is needed to establish the formula for $Var(C)$ and then differentiation of this gives a value that is clearly negative for positive values of y, which shows that the variance is decreasing as y increases.

STEP 3 2015 Hints and solutions

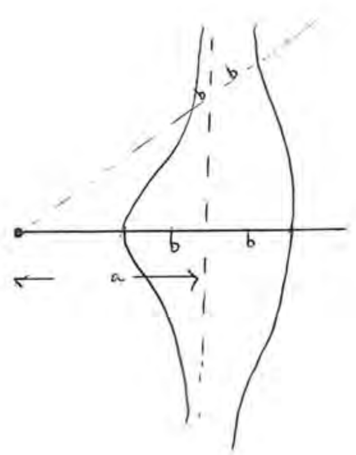
1. The first result can be obtained by simplifying the LHS and then writing it as $\int_0^\infty u \frac{u}{(1+u^2)^{n+1}} du$ and integrating this by parts. To obtain the evaluation of I_{n+1}, the first result can be re-arranged to make I_{n+1} the subject, and then iterating the result to express it in terms of I_1 which is a standard integral. The expression can be tidied by multiplying numerator and denominator by $(2n)(2n-2)\ldots(2)$. The first result for (ii) is obtained by means of the substitution $u = x^{-1}$, the second by adding the two versions of J, and the third by the substitution $u = x - x^{-1}$, being careful with limits of integration and employing symmetry. Part (iii) is solved by expressing the integrand as $\frac{x^{-2}}{((x-x^{-1})^2+1)^n}$ and then employing first part (ii) then part (i) to obtain I_n, which is $\frac{(2n-2)!\pi}{2^{2n-1}\left((n-1)!\right)^2}$.

2. Part (ii) is the only false statement, and a simple counter-example is $s_n = 1$ and $t_n = 2$ for n odd, and $s_n = 2$ and $t_n = 1$ for n even. Part (i) $m = 1000$ is a suitable value, then $1000 \leq n$ and as n is positive, the inequality can be multiplied by it giving the required result. Part (iii) requires the use of the definition twice with values m_1 and m_2 say, and then using $m = \max(m_1, m_2)$. For part (iv), we can choose $m = 4$, and an inductive argument such as

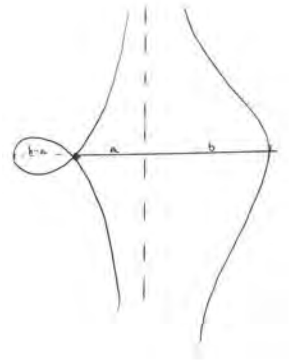
$(k+1)^2 = \left(1+\frac{1}{k}\right)^2 k^2 \leq \left(1+\frac{1}{4}\right)^2 k^2 < 2k^2 \leq 2 \times 2^k = 2^{k+1}$ works.

3. The part (i) inequality for $\sec\theta$ can be obtained by making r the subject of the formula as $r = a\sec\theta \pm b$ and invoking $a > b$ remembering that $r < 0$ is not permitted.

Then the points lie on a conchoid of Nicomedes with *A* being the pole (origin), *d* being *b*, and L being the line $r = a\sec\theta$ ("x" $= a$). A sketch is



In part (ii), the extra feature is the loop as specified with end-points at the pole corresponding to $\sec\theta = \frac{-b}{a}$. A sketch is



So in the given case, the area is given by $2 \times \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi}(\sec\theta + 2)^2 d\theta$ which is $\frac{4\pi}{3} + \sqrt{3} - 4\ln\left|2 + \sqrt{3}\right|$.

4. Part (i) is imply shown by considering the image of the function $f(z) = z^3 + az^2 + bz + c$ as $z \to \pm\infty$ and then observing that the function is continuous and exhibits a sign-change. Part (ii) can be approached by writing $z^3 + az^2 + bz + c = (z - z_1)(z - z_2)(z - z_3)$ giving $a = -S_1$, $= \frac{{S_1}^2 - S_2}{2}$, which can be obtained by considering $(z_1 + z_2 + z_3)^2$ and the required result for $6c$ which can be neatly obtained by considering $f(z_1) + f(z_2) + f(z_3) = 0$.

Writing $z_k = r_k(\cos\theta_k + i\sin\theta_k)$ for $k = 1, 2, 3$, employing de Moivre's theorem, the three sums imply the reality of S_1 , S_2 , and S_3 , and hence a , b , and c which by virtue of the result of part (i) yields the reality of z_1 , z_2 , or z_3 and hence the required result. The final result can be considered as two cases, the trivial one of all three roots being real, and the one where the other two are complex. The latter can be shown to give the required result by considering the real and imaginary parts of the roots of a real quadratic.

5. (i) Step 3 is straightforward on the basis of steps 1 and 2, noting that no lowest terms restriction need be made in part 1. Step 5 requires that the given expression is a positive integer as well as well as being integer when multiplied by root two. Step 6 requires justification that $\sqrt{2} - 1 < 1$.

(ii) The rationality of $2^{2/3}$ on the basis of $2^{1/3}$ being rational is simply obtained by squaring the latter, and the opposite implication can be made by squaring the former or dividing 2 by the former. To construct the similar argument, let the set T be the set of positive integers with the following property: n is in T if and only if $n2^{1/3}$ and $n2^{2/3}$ are integers, and taking t to be the smallest positive integer in that set, consider $t\left(2^{2/3} - 1\right)$ to produce the argument.

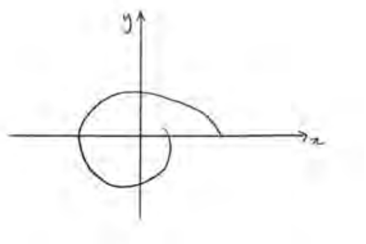
6. Treating the equations for u and v as simultaneous equations for w and z , one finds that $w = \frac{u \pm \sqrt{2v - u^2}}{2}$ and $z = \frac{u \mp \sqrt{2v - u^2}}{2}$ which demonstrates that if $u \in \mathbb{R}$ and $u^2 \leq 2v$, i.e. $v \in \mathbb{R}$, w and z are real. If w and z are real, then u and v are (trivially) and $2v - u^2 = (w - z)^2 \geq 0$.

In (ii), the first two equations yield $3wz = 1$, making it possible to write the third equation as $u(u^2 - 1) = \lambda(u - 1)$ which has an obvious factor of $(u - 1)$ leading to $u = 1$ or $u = \frac{-1 \pm \sqrt{1 + 4\lambda}}{2}$ from the quadratic equation. If one of the solutions of the quadratic equation gives the same root $u = 1$, then there are not three possible values, i.e. if $\lambda = 2$. From the first part of the question, for w and z to be real, we would want u to be real, $u^2 - \frac{2}{3}$ to be real, and $u^2 \leq 2\left(u^2 - \frac{2}{3}\right)$, in other words $u^2 \geq \frac{4}{3}$. So a counter-example could be $u = 1$ giving $2w^2 - 2w + \frac{2}{3} = 0$ which has a negative discriminant.

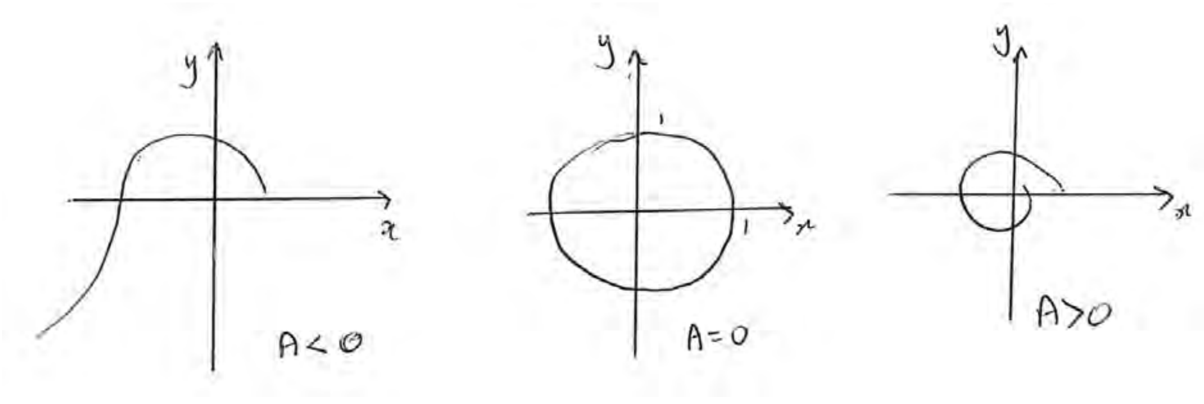
7. The opening result is simply achieved by following the given explanation for $D^2 f(x)$ with $f(x) = x^a$. Parts (i) and (ii) can both be shown usual the principle of mathematical induction with initial statements

"Suppose $D^k P(x)$ is a polynomial of degree r i.e. $D^k P(x) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_0$ for some integer k." and "Suppose $D^k (1-x)^m$ is divisible by $(1-x)^{m-k}$ i.e. $D^k (1-x)^m = f(x)(1-x)^{m-k}$ for some integer k, with $k < m - 1$." Part (iii) is obtained by expressing $(1-x)^m$ in sigma notation (by the binomial theorem), then carrying out $D^n (1-x)^m$ using the idea in the stem, and finally invoking the result of part (ii) and then substituting $x = 1$.

8. Transforming the differential equation in part (i) is made by substituting for x and y as given, for $\frac{dy}{dx}$ using $\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$ and a similar result for $\frac{dx}{d\theta}$, and then simplifying the algebra by multiplying out and collecting like terms bearing in mind that a factor r can be cancelled as $r \neq 0$. The transformed equation can be solved by separating variables or using an integrating factor, to give $r = ke^{-\theta}$, the sketch of which is



The same techniques for part (ii) yields a differential equation $r - r^3 + \frac{dr}{d\theta} = 0$ which is solved by separating the variables and then employing partial fractions giving a variety of possible solution sketches



($A \neq -1$ but it is possible to consider $A < -1$ in which case $\theta < 0$)

9. Whilst the first part can be obtained otherwise, the simplest approach is by conserving energy, when $\frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 + \frac{\lambda}{2a}\left(\sqrt{a^2 + x^2} - a\right)^2$ leads to the required answer simply. x_0 is found by setting $\dot{x} = 0$ leading to $x_0 = \sqrt{\frac{v^2}{k^2} + \frac{2av}{k}}$. The acceleration can be found by applying

Newton's 2nd Law or by differentiating the equation found in the first part, and substituting leads to the result $-\frac{kv\sqrt{v^2+2akv}}{v+ak}$ for the acceleration when $x = x_0$. Treating the equation found in the first part as a differential equation for x in terms of , the expression for the period is

$\tau = 4\int_0^{x_0} \frac{1}{\left[v^2-k^2\left(\sqrt{a^2+x^2}-a\right)^2\right]^{\frac{1}{2}}} dx$. Making the substitution $u^2 = \frac{k\left(\sqrt{a^2+x^2}-a\right)}{v}$, leads to

$x = \sqrt{2kav}\frac{u}{k}\left(1+\frac{v}{2ka}u^2\right)^{\frac{1}{2}}$, which making a binomial expansion and using the given condition to approximate $x \approx \sqrt{2kav}\frac{u}{k}$ results in the final given expression.

10. The position vector of the upper particle is $\begin{pmatrix} x + a\sin\theta \\ y + a\cos\theta \end{pmatrix}$ so differentiating with respect to time yields the correct velocity and acceleration which gives the second result when used in Newton's second law resolving horizontally and vertically. The corresponding equations are $m\begin{pmatrix} \ddot{x} - a\,\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta \\ \ddot{y} + a\,\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta \end{pmatrix} = T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the other particle by merely swapping the direction of the tension and the displacement from the midpoint. The deductions are obtained by adding the two equations of motion, and in the case of $\ddot{\theta}$, subtracting the two equations and then eliminating T between the equations for each component. Using $\begin{pmatrix} \dot{x} + a\,\dot{\theta}\cos\theta \\ \dot{y} - a\,\dot{\theta}\sin\theta \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ and a similar equation for the lower particle, initial values of $\dot{y}$ and $\dot{\theta}$ can be found and then the time for the rod to rotate by $\frac{1}{2}\pi$ can be obtained and substituted in the displacement equation under uniform acceleration to obtain the final result.

11. (i) The first result is obtained, as the question prompts, by considering a component of force on the rod due to P, and taking moments about the hinge to find that that component is zero with the consequence that any force exerted on the rod by P must be parallel to the rod. Bearing in mind the horizontal acceleration of P towards the centre of the circle it describes, resolving perpendicular to the rod and writing the equation of motion for P leads directly to the given equation with the stated substitution being made. The force exerted by the hinge on the rod is along the rod towards P and resolving vertically for forces on P and rearranging gives $F = mg\sec\alpha$.

(ii) Taking moments for the whole system about the hinge gives

$$m_1 g\, d_1 \sin\beta + m_2 d_2 g \sin\beta = m_1 d_1 (r - d_1\sin\beta)\omega^2\cos\beta + m_2 d_2 (r - d_2\sin\beta)\omega^2\cos\beta$$

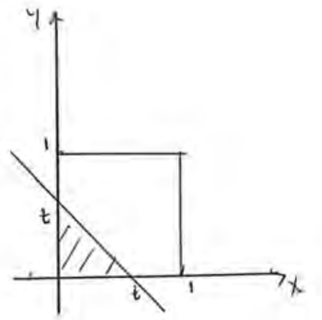
which can be rearranged into the required form with $b = \frac{m_1 {d_1}^2 + m_2 {d_2}^2}{m_1 d_1 + m_2 d_2}$.

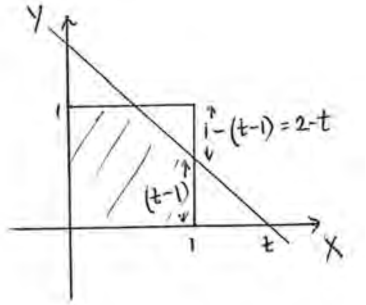
12. (i) The required probability generating function is $G(x) = \frac{1}{6}(1 + x + x^2 + x^3 + x^4 + x^5)$ and it is simple to write down the probability distribution function of S_2 and hence of R_2 and arrive at the same pgf. As a consequence, it can be argued that the pgf for R_n is also $G(x)$ and so the required probability is ${}^1\!/_6$.

(ii) $G_1(x) = \frac{1}{6}(1 + 2x + x^2 + x^3 + x^4) = \frac{1}{6}(x + y)$. $G_2(x)$ would be $\left(G_1(x)\right)^2$ except that the powers must be multiplied congruent to modulus 5 , and it can be shown that $xy = y$ and $y^2 = 5y$ so obtaining the required result for $G_2(x)$. Obtaining $G_n(x) = \frac{1}{6^n}\left(x^{n-5p} + \frac{6^n-1}{5}y\right)$

where p is an integer such that $0 \le n - 5p \le 4$, and the probability that S_n is divisible by 5 will be the coefficient of x^0 which in turn is the coefficient of y as required. If n is divisible by 5, the probability that S_n is divisible by 5 will be $\frac{1}{5}\left(1 + \frac{4}{6^n}\right)$ as $x^{n-5p} = x^0$.

13. (i)

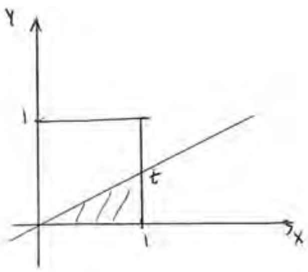


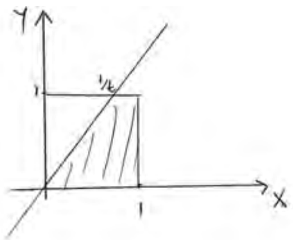


lead to $P(X + Y < t) = \frac{1}{2}t^2$ if $0 \le t \le 1$, $P(X + Y < t) = 1 - \frac{1}{2}(2 - t)^2$ if $1 < t \le 2$, $P(X + Y < t) = 0$ if $t < 0$ and $P(X + Y < t) = 1$ if $t > 2$.

From this, the cumulative distribution function of $(X + Y)^{-1}$ by means of the logic $P((X + Y)^{-1} < t) = P\left(X + Y > \frac{1}{t}\right) = 1 - P\left(X + Y < \frac{1}{t}\right)$ and the required probability density function can be found by differentiation. From that, by integration, $E\left(\frac{1}{X+Y}\right) = 2\ln 2$

(ii)





lead to

$$P\left(\frac{Y}{X} < t\right) = \begin{cases} \frac{1}{2}t \ \ for\ 0 \le t \le 1 \\ 1 - \frac{1}{2}t^{-1} \ \ for\ t > 1 \end{cases}$$

That $P\left(\frac{X}{X+Y} < t\right) = P\left(\frac{X+Y}{X} > \frac{1}{t}\right) = P\left(\frac{Y}{X} > \frac{1}{t} - 1\right) = 1 - P\left(\frac{Y}{X} < \frac{1}{t} - 1\right)$ and differentiation leads to

the probability density function $f(t) = \begin{cases} \frac{1}{2}t^{-2} \ \ for\ \frac{1}{2} \le t \le 1 \\ \frac{1}{2}(1-t)^{-2} \ \ for\ 0 \le t < \frac{1}{2} \end{cases}$.

$E\left(\frac{X}{X+Y}\right) = \frac{1}{2}$ can be written down because by symmetry, $E\left(\frac{X}{X+Y}\right) = E\left(\frac{Y}{X+Y}\right)$

and $E\left(\frac{X}{X+Y}\right) + E\left(\frac{Y}{X+Y}\right) = E\left(\frac{X+Y}{X+Y}\right) = E(1) = 1$. This is simply verified by integration using the probability density function found.

[An earlier potential version of the question had X , Y , and Z independently uniformly distributed on $[0,1]$, considered the distribution of $\ln X$,

went on to find the pdf of U , where $U = -\ln(XY)$ and finished by showing that $(XY)^Z$ is also uniformly distributed on $[0,1]$.]

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