Admissions Testing Service

STEP Solutions 2014

Mathematics

STEP 9465/9470/9475

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Test

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STEP Mathematics (9465, 9470, 9475)

STEP 1 2014 Hints and Solutions

Q1 This question is the traditional starter, involving ideas that should certainly be familiar to all candidates. That doesn't necessarily mean it is easy, just that everyone should be able to make a start with it, and make some good progress thereafter. In fact, parts (i), (ii) and (iii) are each especially amenable to all; and (i) is intended to help get you started along this particular road by having you first write out some numerical examples of differences of two squares, before asking you to move on to the algebra.

To begin with, (ii) requires only the observation that each odd number is the difference of consecutive squares, $2k - 1 \equiv k^2 - (k - 1)^2$; and then (iii) relies on noting that multiples of 4 arise from the difference of squares of numbers that are two apart, namely $4k \equiv (k + 1)^2 - (k - 1)^2$. Part (iv) develops these ideas further, although it is important now to deconstruct the problem into its building blocks; in this case, that means examining the four possible cases for $a^2 - b^2$ when a and b are either odd or even, none of which cases yield an even answer that is not automatically a multiple of 4. Alternatively, this is very easily addressed by examining squares modulo 4.

Part (v) draws on ideas of factorisation of (positive) integers into factor-pairs. Here, you are given the product pq where both p and q are odd. Using the *difference of two squares factorisation*, if $pq = a^2 - b^2 = (a - b)(a + b)$, then either $a - b = 1$ and $a + b = pq$ OR $a - b = p$ and $a + b = q$ (taking $p < q$ w.l.o.g.), giving the (exactly) two required factorisations. However, if $p = 2$, then pq is a multiple of 2, but not 4, and we have already shown this case to be impossible.

The final part of the question pulls all these ideas together in a numerical example, and we need only prime-factorise $675 = 3^3 \times 5^2$ and note that this yields $(3 + 1)(2 + 1) = 12$ factors and hence six factor-pairs.

Q2 The main ideas behind this question are relatively straightforward, but there are many difficulties in the execution of them. In (i), an integral of the form $\int \ln(x)\ \mathrm{d}x$ would usually have to be approached by writing it as the product $\int \ln(x) \times 1\ \mathrm{d}x$ and integrating by parts. On this occasion, however, with a given result, it is perfectly possible to verify by differentiation. Curve-sketching (rather than plotting) is a key skill and one that needs practice. The things to look for are crossing-points on the two coordinate axes (which arise here when $\ln(1)$ appears), asymptotes (from when we get ln(0)), symmetries and – if further detail is required – turning points. It should be obvious here, for instance, that the curve is an *even function*, so there is a turning point when $x = 0$. In (iii), you are given the area to be found so that you can check you are doing everything correctly before you go on to answer the rather tougher part (iv). You should first realise (from your sketch-graph) that it is required to integrate $\ln(4 - x^2)$ between $-\sqrt{3}$ and $\sqrt{3}$, though you should always be on the lookout for opportunities to make the working easier – here, you could integrate between 0 and $\sqrt{3}$ and then double the answer. Next, there is a very strong hint supplied by part (i) to split the log. term up as $\ln(2 - x) + \ln(2 + x)$, the first term of which has already been done and the second can be deduced from it with a bit of care.

Part (iv) actually asks nothing new. Curve B is just curve A with all portions drawn above the x-axis and it is only required that you deal with the extra bit(s) of area, and the awkwardness of finding an area up to the asymptote is covered for you by the footnote that follows (iv).

Q3 This question was actually devised to address what happens when students misunderstand or mis-apply a "rule" of mathematics and *it turns out to give the right answer*. Part (i) starts you off gently: integrating both terms, squaring the RHS and solving very quickly gives $b = \frac{4}{3}$. Part (ii) develops in much the same way, but with a non-zero lower limit to the integrals, and we immediately see that the algebra gets much more involved. Importantly, it should be very clear that whatever expression materialises must have (b – 1) as a factor (since setting $b = a$ would definitely give a zero area, thus trivially satisfying the given integral statement). This leads to the required cubic equation.

The final part of (ii) requires a mixture of different ideas (and can be done in a number of different ways). The most basic approach to demonstrating that a cubic curve has only one zero is to illustrate that both of its TPs lie on the same side of the x-axis (or to show there are no TPs). The popular *Change-of-Sign Rule* for continuous functions can be used to identify the position of this zero.

Having got you started with some simple lower limits, part (iii) develops matters more generally, and derives the (perhaps) surprising result that the exploration of this initial "stupid idea" requires b and a to be "not too far apart" to an extent that is easily identifiable.

Q4 This is quite a sweet little question, and deals with the movements of the hands of a clock. Using the *Cosine Rule* and differentiating gives $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{ab\sin\theta}{\sqrt{b^2 + a^2 - 2ab\cos\theta}}$. However, it is important to note that we can only work with $\frac{\mathrm{d}x}{\mathrm{d}\theta}$ and $\frac{\mathrm{d}^2x}{\mathrm{d}\theta^2}$, rather than $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$, since $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ is constant (so that extra terms in the use of the *Chain Rule* simply cancel). Then, differentiating again with respect to θ, equating to zero and solving leads to the quadratic equation in $C = \cos\theta$:

$$abC^2 - \left(b^2 + a^2\right)C + ab = 0$$

which can be factorised (or solved otherwise). Only one of these solutions gives $|\ C\ | < 1$, and substituting $\cos\theta = \frac{a}{b}$ gives the required result. [In the normal course of events, one should justify that this is a maximum value rather than a minimum – but it is rather clear that it must be so in this case.] Finally, $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{11}{6}\pi$ is the constant rate of change of θ, the angle between the two hands, and solving $\cos\left(\frac{11}{6}\pi\right) = \frac{1}{2}$ gives a time of $\frac{2}{11}$ hours ≈ 11 minutes, as required.

Q5 This is another question that asks you to deploy your graph-sketching skills – in association with the supporting algebra, of course – and finding the coordinates of the TPs of the family of cubics given here, at (a, 0) and (–5a, 108a^3) makes it clear that the requested result holds. However, the case $a = 0$ (despite its relative triviality) still needs to be addressed separately.

Using (i)'s result with $a = y$ then immediately gives $27xy^2 \le (x + 2y)^3 \le 3^3$ so that $xy^2 \le 1$. Equality holds (from part (i)) when $x = a = y$ and from when $x + 2y = 3 \Rightarrow x = y = 1$.

Part (iii) requires a little more care and perseverance, but setting $x = p$ and $2a = q + r$ gets you off to a good start. There is now a little bit of "working to one side" needed if you are to convince yourself that $\left(\frac{q+r}{2}\right)^2 \ge qr$. Those students who have previously encountered the *Arithmetic Mean – Geometric Mean Inequality* will have spotted this straightaway; otherwise, this result becomes obvious upon rearrangement, as it is true $\Leftrightarrow (q - r)^2 \ge 0$, which is clearly true. Having to write it out in this way does have the advantage of highlighting that equality holds if and only if $q = r$ here (leading to $p = q = r$ for the main result).

Q6 Applying the given recurrence relation leads to

$$u_1 = 4\sin^2\theta(1 - \sin^2\theta) = 4\sin^2\theta\cos^2\theta = \sin^2(2\theta),$$

though the question becomes very difficult indeed without the use of the two very basic trigonometric identities involved to this point. A moment's thought will help you avoid unnecessary further working at this stage, as the determination of u_2 from $u_1 = \sin^2$(some angle) should obviously give us the result $u_2 = \sin^2$(twice that angle) in exactly the same way. Thus $u_2 = \sin^2(4\theta)$ and, in general, $u_n = \sin^2(2^n\theta)$. Establishing the result inductively requires no more working than that which was given two sentences ago, with the $n = 1$ "baseline" case having been given to you.

In part (ii), we approach a more awkward looking recurrence relation by transforming it using the given substitution, and then comparing it with the required form. This gives the following relationships:

$$p\alpha = 4, \quad q - 2p\beta = 4 \quad \text{and} \quad (q - 1)\beta + r = p\beta^2$$

from which the required result follows. Upon checking, the final r.r. satisfies the appropriate conditions, and since $u_0 = \sin^2\left(\tfrac{1}{4}\pi\right)$, we have $u_n = \sin^2\left(2^n.\tfrac{1}{4}\pi\right)$ and so $v_n = 4\sin^2\left(2^n.\tfrac{1}{4}\pi\right) - 1$. However, it should be noted that $\sin\left(2^n.\tfrac{1}{4}\pi\right) = 0$ for all $n \ge 2$, so that $\{v_n\} = \{1, 3, -1, -1, -1 \ldots\}$.

Q7 When it boils down to it, this vectors question actually involves little more than the use of a standard ratio result, vector equations of lines, and finding intersections of lines. Note first that, if a point T divides a line segment XY in the ratio $p : q = \frac{p}{p+q} : \frac{q}{p+q}$, then the respective position vectors of these points are related thus: $\mathbf{t} = \left(\frac{q}{p+q}\right)\mathbf{x} + \left(\frac{p}{p+q}\right)\mathbf{y}$. Indeed, it is precisely this result that leads to the vector equation of a line in the form $\mathbf{r} = (1 - \alpha)\mathbf{a} + \alpha\mathbf{d}$ (for the line AD here). Writing BE's equation similarly and substituting for $\mathbf{d}$ and $\mathbf{e}$ in terms of r, s, $\mathbf{a}$ and $\mathbf{b}$ then gives the point of intersection of AD and BE by equating the two and comparing terms in $\mathbf{a}$ and $\mathbf{b}$, deriving the given form for $\mathbf{g}$.

The equation of the line OG is then $\mathbf{r} = \lambda\mathbf{g}$, which meets AB at F, which is known to cut AB in the ratio $t : 1$, which gives us two forms for the position vector of F. Comparing terms again yields the required answer.

Incidentally, those students and teachers more familiar with standard results in Euclidean Geometry will spot that this problem is actually an application of, first, *Menelaus' Theorem* and then *Ceva's Theorem*.

Q8 Part (i) directs you to finding the equation of L_a, which is $y = \left(1 - \frac{1}{a}\right)(x - a)$, and it follows immediately that L_b has equation $y = \left(1 - \frac{1}{b}\right)(x - b)$ similarly. Solving simultaneously yields their point of intersection at $\big(ab, (1-a)(1-b)\big)$. As $b \to a$, this point of intersection $\to \left(a^2, (1-a)^2\right)$, and it is clear that $a = \sqrt{x}$, so that $0 < \sqrt{x} < 1$ and $0 < x < 1$, and that $y = \left(1 - \sqrt{x}\right)^2$.

Differentiating this gives $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{1}{\sqrt{x}}$ which, at the point $C\left(\sqrt{c}, (1-\sqrt{c})^2\right)$, gives a tangent with equation $y - \left(1 - \sqrt{c}\right)^2 = \left(1 - \frac{1}{\sqrt{c}}\right)\left(x - \sqrt{c}\right)$. This rearranges into the form $y = \left(1 - \frac{1}{\sqrt{c}}\right)\left(x - \sqrt{c}\right)$, which is simply $L_{\sqrt{c}}$ where $0 < \sqrt{c} < 1$, as required.

Q9 This question provides rather a nice twist to the standard sort of projectiles question. Firstly, there is a non-zero horizontal component of the acceleration. However, since this doesn't affect the vertical motion, it is still the case that $T_H = \dfrac{U\sin\theta}{g}$ and $T_L = \dfrac{2U\sin\theta}{g}$. Next, the time $T = \dfrac{U\cos\theta}{kg}$ is introduced, which is quickly identified as the time when the horizontal component of the velocity is zero. Writing it in the form $T = \dfrac{U\sin\theta}{g}\times\dfrac{1}{k\tan\theta}$ at an early stage is really helpful, partly because it pulls out the common factor of $\dfrac{U\sin\theta}{g}$, but mostly as it identifies the factor of $k\tan\theta$ which is the major determining feature of the question. Having done this, you should realise that the three cases do little more than ask you to sketch what happens depending upon when the wind (for surely that is what is supplying this opposing force on the projectile) causes the particle to "trun round" horizontally, relative to when the maximum height and greatest range are achieved. In the first case, we get a "shortened" parabola compared to the usual shape of a non-wind-resisted projectile (so that T isn't reached before landing); in the second, the particle turns round horizontally before landing; and in the third, the wind is so strong that the particle begins to be blown backwards before reaching its greatest height. In the case of the "afterthought", the particle is constrained to move up and down in a straight line, returning to the point of projection.

Q10 In this question, we examine the dropping of two balls together, smaller on top of larger, which leads to the surprising outcome where the smaller ball bounces so much higher than one would expect it to. To begin with, we examine the bounce of the larger ball when it falls on its own. Since it falls a distance H, it hits the ground with speed u given by $u^2 = 2gH$, either by energy considerations or by using the ("suvat") constant acceleration formulae. It then leaves the ground with speed $v = eu$ according to *Newton's (Experimental) Law of Restitution*, attaining a height H_1 given by $v^2 = 2g(H_1 - R)$. Substituting for v and then u gives the printed answer.

In the extended situation, we again use $u = \sqrt{2gH}$ (for both balls) and $v = eu$ (for the larger ball after it bounces on the ground), though these do not need to be substituted immediately into the "collision" statements that are gained when using the principle of *Conservation of Linear Momentum* and *N*(*E*)*LR* for the subsequent collision between the balls. Solving simultaneously, the upwards speed of the smaller ball is found to be $x = \dfrac{(Me^2 + 2Me - m)}{M + m}u$. Using $x^2 = 2gd$, where d is the distance travelled by the small ball's centre of mass, and $u^2 = 2gH$, we deduce the result that $d = \left(\dfrac{Me^2 + 2Me - m}{M + m}\right)^2 H$; and then $h = 2R + r + d$.

In the final part of the question, substituting in the given numbers yields the answer $\dfrac{M}{m} = 9$.

[Of course, this part of the question has really been asked back-to-front, as the real issue is to explore the speed of the smaller ball after the collision; however, the numerical working would have been much less favourable that way round.]

Q11 Part (i) of this question is a very straightforward single-pulley scenario, with the tension in the string equal throughout its length, which is constant, and the accelerations of the particles equal (relative to the string). Using *Newton's Second Law* for each particle gives the acceleration (as shown) and the tension in the string given by $T = \frac{4Mmg}{M+m}$.

In the second part of the question, we apply exactly the same assumptions and principles in exactly the same way, but there is a complication imposed now by the relative accelerations of the two particles on the P_1 pulley system. If we assume that the P_1 system has the "same" acceleration as the particle on the LHS of P, then (assuming that the m_1 particle accelerates downwards within this sub-system) the two particles on the RHS have accelerations $b - a_2$ and $b + a_2$ respectively. Also, since P_1 is taken to have zero mass, the tension in the main string is twice the tension in the sub-system. Without this set-up, the following working is most unlikely to be meaningful in any mark-scoring capacity. *N*2*L* applied several times, for the different particles, then leads to a set of equations that gives the second printed answer.

With two given answers, the very final part of the question can be done as a "stand alone" piece of work. Notice, however, that this is an *if and only if* proof and thus requires either two separate arguments or one in which every step is reversible. In point of fact, it transpires that $a_1 = a_2$ if and only if $(m_1 - m_2)^2 = 0$, which is equivalent to $m_1 = m_2$.

Q12 As with all such questions, one can make this much easier to deal with by being systematic, and by presenting one's working in a clear and coherent manner. To begin with, for instance, set out a table of the six possible outcomes, along with their associated probabilities. Of course, the reason why you are then asked for $E(X^2)$ rather than $E(X)$ is clearly because squaring negates any concerns about the sign of whatever is inside a pair of "modulus" lines, so that they can simply be replaced by ordinary brackets. You are then told that the (given) answer is a positive integer, which restricts $(k - 1)$ to being a multiple of 6 … namely, 1 or 7. Checking each of these in the possible $E(X)$'s that arise enables you to eliminate $k = 1$ and confirm that $k = 7$.

Now that this information is known, it is possible to draw up the probability distribution for X (again, a table works very well), work out the probabilities $P(X > 25) = \frac{21}{144}$ and $P(X = 25) = \frac{4}{144}$, and then evaluate the expected payout

$$E(W) = \sum \left(w \times P(w)\right) = \left(w \times \tfrac{21}{144}\right) + \left(1 \times \tfrac{4}{144}\right) + 0$$

for the gambler (where W represents "winnings"). For this to be in the casino's favour, this expression must be negative, giving $w < 7$ and so the largest integer value of w is 6.

Q13 Although you are not asked for a sketch, a quick diagram might well help prevent stupid mistakes. Since the area under the triangle's two sloping sides is equal to 1, it follows that the height of the triangle is $h = \dfrac{2}{b-a}$, and so the gradient of the first line is this divided by (c – a) and the equation – that is, g(x) – follows immediately. Without further working, h(x) can be written down immediately by substituting b for a appropriately (i.e. *not* in the bit of the expression for h; and remembering that $c - b$ is negative).

In (i), it is definitely not enough to think "Aha! I recognise that expression" and simply throw the word "centroid" at the problem and hope to get all the points. You won't. The question makes it clear that it is intended for you to do the work to show that this expression is the mean. (For a start, the values a, b and c lie on a line and aren't "masses" placed at the vertices of the triangle.) The value of E(X) must be found by integrating over the two regions, and then sorted out algebraically.

For the final part of the question, it is important first to spend a moment making sure that you can identify the different possible cases that arise. These depend upon whether c is < , = , or > than $\frac{1}{2}(a+b)$. The equality case is the easiest, and the easiest to overlook, since then $m = c$ (by symmetry). In the first case, the median lies under the first of the sloping line segments, and will be gained by setting the area of a triangle equal to $\frac{1}{2}$. In a similar way to earlier, the third case can be dealt with by "working down from b" instead of "working up from a). These give, respectively, $m = a + \sqrt{\frac{1}{2}(b-a)(c-a)}$ and $m = b - \sqrt{\frac{1}{2}(b-a)(b-c)}$.

STEP II 2014

Question 1:

Drawing a diagram and considering the horizontal and vertical distances will establish the relationships for $x\cos\theta$ and $x\sin\theta$ easily. The quadratic equation will then follow from use of the identity $\cos^2\theta + \sin^2\theta \equiv 1$. The same reasoning applied to a diagram showing the case where P and Q lie on AC produced and BC produced will show that the same equation is satisfied.

(*) will be linear if the coefficient of x^2 is 0, so therefore $\cos(\alpha+\beta)$ will need to equal $-\frac{1}{2}$, which gives a relationship between α and β. For (*) to have distinct roots the discriminant must be positive. Using some trigonometric identities it can be shown that the discriminant is equal to $4(1-(\sin\alpha - \sin\beta)^2)$ and it should be easy to explain why this must be greater than 0.

The first case in part (iii) leads to $x = \sqrt{2} \pm 1$ and so there are two diagrams to be drawn. In each case the line joining P to Q will be horizontal.

The second case in part (iii) is an example where (*) is linear. This leads to $x = \frac{\sqrt{3}}{3}$. Therefore Q is at the same point as C and so the point P is the midpoint of AC.

Question 2:

By rewriting in terms of $\cos 2nx$ it can be shown that $\int_0^\pi \sin^2 nx\,dx = \frac{\pi}{2}$ and $\int_0^\pi n^2\cos^2 nx\,dx = \frac{n^2\pi}{2}$. Therefore (*) must be satisfied as n is a positive integer. The function $f(x) = x$ does not satisfy (*) and $f(0) = 0$ but $f(\pi) \neq 0$. The function $g(x) = f(\pi - x)$ will therefore provide a counterexample where $g(\pi) = 0$, but $g(0) \neq 0$.

In part (ii), $f(x) = x^2 - \pi x$ will need to be selected to be able to use the assumption that (*) is satisfied. The two sides of (*) can then be evaluated:

$$\int_0^\pi x^4 - 2\pi x^3 + \pi^2 x^2\,dx = \frac{\pi^5}{30}$$

$$\int_0^\pi 4x^2 - 4\pi x + \pi^2\,dx = \frac{\pi^3}{3}$$

Substitution into (*) then leads to the inequality $\pi^2 \leq 10$.

To satisfy the conditions on $f(x)$ for the second type of function, the values of p, q and r must satisfy $q + r = 0$ and $p + r = 0$. Evaluating the integrals then leads to $\pi \leq \frac{22}{7}$.

Since $\left(\frac{22}{7}\right)^2 < 10$, $\pi \leq \frac{22}{7}$ leads to a better estimate for π^2.

Question 3:

By drawing a diagram and marking the shortest distance a pair of similar triangles can be used to show that $\frac{c/m}{c\sqrt{m^2+1}/m} = \frac{d}{c}$, which simplifies to $d = c/\sqrt{m^2+1}$.

For the second part, the tangent to the curve at the general point (x, y) will have a gradient of y' and so the y-intercept will be at the point $(0, y - xy')$. Therefore the result from part (i) can be applied using $m = y'$ and $c = y - xy'$ to give $a = (y - xy')/\sqrt{(y')^2 + 1}$, which rearranges to give the required result.

Differentiating the equation then gives $y''\big(a^2y' + x(y - xy')\big) = 0$ and so either $y'' = 0$ or

$a^2y' + x(y - xy') = 0$.

If $y'' = 0$ then the equation will be of a straight line and the y-intercept can be deduced in terms of m.

If $a^2y' + x(y - xy') = 0$, then the differential equation can be solved to give the equation of a circle.

Part (iii) then requires combining the two possible cases from part (ii) to construct a curve which satisfies the conditions given. This must be an arc of a circle with no vertical tangents, with straight lines at either end of the arc in the direction of the tangents to the circle at that point.

Question 4:

In part (i), if the required integral is called I then the given substitution leads to an integral which can be shown to be equal to $-I$. This means that $2I = 0$ and so $I = 0$.

In part (ii), once the substitution has been completed, the integral will simplify to $\int_{1/b}^{b} \frac{\arctan\frac{1}{u}}{u} du$. Since $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ the integral can be shown to be equal to $\frac{1}{2}\int_{1/b}^{b} \frac{\pi}{2x} dx$, which then simplifies to the required result.

In part (iii), making with the substitution in terms of k and simplifying will show that the integral is equivalent to

$$\int_0^\infty \frac{ku^2}{(a^2u^2 + k^2)^2} du$$

Therefore choosing $k = a^2$, the integral can be simplified further to

$$\frac{1}{a^2}\int_0^\infty \frac{u^2}{(a^2 + u^2)^2} du = \frac{1}{a^2}\int_0^\infty \frac{1}{a^2 + u^2} du - \frac{1}{a^2}\int_0^\infty \frac{a^2}{(a^2 + u^2)^2} du$$

The result then follows by using the given value for $\int_0^\infty \frac{1}{a^2+x^2} dx$.

Question 5:

Using the substitution $y = xu$, the differential equation can be simplified to

$$x\frac{du}{dx} = \frac{1 + 4u - u^2}{u - 2}$$

This can be solved by separating the variables after which making the substitution $u = \frac{y}{x}$ and substituting the point on the curve gives the required quadratic in x and y.

In part (ii), $\frac{dY}{dX}$ can be shown to be equal to $\frac{dy}{dx}$. The values of a and b need to be chosen so that the right hand side of the differential equation has no constant terms in the numerator or denominator. This leads to the simultaneous equations:

$$a - 2b - 4 = 0$$

$$2a + b - 3 = 0$$

Solving these and substituting the values into the differential equation gives $\frac{dY}{dX} = \frac{X-2Y}{2X+Y}$, and so

$$\frac{dX}{dY} = \frac{2X + Y}{X - 2Y}$$

This is the same differential equation as in part (i), with $x = Y$ and $y = X$. Most of the solution in part (i) can therefore be applied, but the point on the curve is different, so the constant in the final solution will need to be calculated for this case.

Question 6:

One of the standard trigonometric formulas can be used to show that

$$\sin\left(r + \tfrac{1}{2}\right)x - \sin\left(r - \tfrac{1}{2}\right)x = 2\cos rx \sin\tfrac{1}{2}x.$$

Summing these from $r = 1$ to $r = n$ will then give the required result.

In part (i), the definition can be rewritten as $S_2(x) = \sin x + \frac{1}{2}\sin 2x$. The stationary points can then be evaluated by differentiating the function. The sketch is then easy to complete.

For part (ii), differentiating the function gives $S_n'(x) = \cos x + \cos 2x + \cdots + \cos nx$. Applying the result from the start of the question, this can be written as

$$S_n'(x) = \frac{\sin\left(n+\frac{1}{2}x\right) - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$

Since $\sin\frac{1}{2}x \neq 0$ in the given range, the stationary points are where $\sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x = 0$. This can then be simplified to the required form by splitting $\sin\left(n+\frac{1}{2}\right)x$ into functions of nx and $\frac{1}{2}x$ and noting that $\sin\frac{1}{2}x \neq 0$ and $\cos\frac{1}{2}x \neq 0$ in the given range, so both can be divided by. By noting that the difference between $S_{n-1}(x)$ and $S_n(x)$ is $\frac{1}{n}\sin nx$ the result just shown can be used to show the final result of part (ii). Part (iii) then follows by induction.

Question 7:

By considering the regions $x \leq a$, $a < x < b$ and $x \geq b$, $f(x)$ can be written as

$$f(x) \quad = \quad \begin{matrix} a+b-2x & x \leq a \\ b-a & a < x < b \\ 2x-a-b & x \geq b \end{matrix}$$

Therefore the graph of $y = f(x)$ will be made up of two sloping sections (with gradients 2 and -2 and a horizontal section). The graph of $y = g(x)$ will have the same definition in the regions $x \leq a$ and $x \geq b$, with the sloping edges extending to a point of intersection on the x-axis. The quadrilateral with therefore have sides of equal length and right angles at each vertex, so it is a square.

In part (ii), sketches of the cases where $c = a$ and $c = b$ show that these cases give just one solution. If $a < c < b$ there will be no solutions and in the other regions there will be two solutions.

In part (iii) the graphs for the two sides of the equation can be related to graphs of the form of $g(x)$ (apart from the section which is replaced by a horizontal line) in the first part of the question. Since $d - c < b - a$, the horizontal sections of the two graphs must be at different heights so the number of solutions can be seen to be the same as the number of intersections of the graphs of the form of $g(x)$.

Question 8:

The coefficients from the binomial expansion should be easily written down. It can then be shown that

$$\frac{c_{r+1}}{c_r} = \frac{b(n-r)}{a(r+1)}$$

This will be greater than 1 (indicating that the value of c_r is increasing) while $b(n-r) > a(r+1)$, which simplifies to $r < \frac{nb-a}{a+b}$. Similarly, $\frac{c_{r+1}}{c_r} = 1$ if $r = \frac{nb-a}{a+b}$ and $\frac{c_{r+1}}{c_r} < 1$ if $r > \frac{nb-a}{a+b}$. Therefore the maximum value of c_r will be the first integer after $\frac{nb-a}{a+b}$ (and there will be two maximum values for c_r if $\frac{nb-a}{a+b}$ is an integer. The required inequality summarises this information.

In parts (i) and (ii) the values need to be substituted into the inequality. Where there are two possible values, it needs to be checked that they are equal before taking the higher if this has not been justified in the first case.

In part (iii) the greatest value will be achieved when the denominator takes the smallest possible value, therefore $a = 1$, and then in part (iv) the greatest value will be achieved by maximising the numerator. Since the maximum possible value of $G(n, a, b)$ is n, $b \geq n$ will achieve this maximum.

Question 9:

Once a diagram has been drawn the usual steps will lead to the required result:

Resolving vertically:

$$F + T\cos\theta = mg$$

Resolving horizontally:

$$T\sin\theta = R$$

Taking moments about A:

$$mg(a\cos\varphi + b\sin\varphi) = Td\sin(\theta+\varphi)$$

Limiting equilibrium, so $F = \mu R$:

$$\mu T\sin\theta + T\cos\theta = mg$$

Therefore:

$$Td\sin(\theta+\varphi) = T(\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$

And so:

$$d\sin(\theta+\varphi) = (\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$

If the frictional force were acting in the opposite direction, then the only change to the original equations would be the sign of F in the first equation. Therefore the final relationship will change to

$$d\sin(\theta+\varphi) = (-\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$

For the final part, the first and third of the equations above can be used to show that

$$F = \frac{Td\sin(\theta+\varphi)}{a\cos\varphi + b\sin\varphi} - T\cos\theta$$

Since $F > 0$ if the frictional force is upwards, this then leads to the condition $d > \frac{a+b\tan\varphi}{\tan\theta+\tan\varphi}$. Since the string must be attached to the side AB, d cannot be bigger than $2b$, which leads to the final result of the question.

Question 10:

Consideration of the motion horizontally and vertically and eliminating the time variable leads to a Cartesian equation for the trajectory:

$$y = \lambda x - \frac{gx^2}{2u^2}(1+\lambda^2)$$

The maximum value can be found either by differentiation or by completing the square. Completing the square gives:

$$y = -\frac{gx^2}{2u^2}\left(\lambda - \frac{u^2}{gx}\right) + \frac{u^2}{2g} - \frac{gx^2}{2u^2}$$

Which shows that $Y = \frac{u^2}{2g} - \frac{gx^2}{2u^2}$. If this graph is sketched then the region bounded by the graph and the axes will represent all the points that can be reached.

The maximum achievable distance must lie on the curve and the distance, d, of a point on the curve can be shown to satisty $d^2 = \left(\frac{u^2}{2g} + \frac{gx^2}{2u^2}\right)^2$, which must be maximised when x takes the maximum value possible.

Question 11:

A diagram shows that the coordinates of P are $(x + (L - x)\sin\alpha, -(L - x)\cos\alpha)$

Therefore, by differentiating the y-coordinate of P shows that the vertical acceleration of P is $\ddot{x}\cos\alpha$ and applying Newton's Second Law gives

$$T\cos\alpha - kmg = km\ddot{x}\cos\alpha$$

A similar method for the horizontal motion of P and R gives the two equations

$$T\sin\alpha = -km(1 - \sin\alpha)\ddot{x}$$

$$T - T\sin\alpha = -m\ddot{x}$$

For part (ii), eliminating T from the last two equations gives the required relationship. A sketch of the graph of $y = \frac{x}{(1-x)^2}$ will then show that for any value of k there is a possible value between 0 and 1 for $\sin\alpha$.

In part (iii), elimination of T from the two equations formed by considering the motion of P gives the required result.

Question 12:

The required probability in the first part is given by

$$\frac{P(t < T < t + \delta t)}{P(T > t)} = \frac{F(t + \delta t) - F(t)}{1 - F(t)}$$

In the case of small values of δt, $F(t + \delta t) - F(t) \approx f(t)\delta t$, which leads to the correct probability.

In part (ii), differentiation gives $f(t) = \frac{1}{a}$, and substituting into the definition of the hazard function gives $h(t) = \frac{1}{a-t}$. Both graphs are simple to sketch.

In part (iii), using the definition of the hazard function gives $\frac{F\prime(t)}{1-F(t)} = \frac{1}{t}$. Integrating gives $-\ln|1 - F(t)| = \ln|kt|$, and so the probability density function can be found by rearranging to find $F(t)$ and then differentiating.

A similar method in part (iv) shows that if $h(t)$ is of the form stated then $f(t)$ will be of the given form. Similarly, if $f(t)$ has the given form then $h(t)$ can be shown to have the form stated.

In part (v), a differential equation can again be written using the definition of the hazard function and this can again be solved by integrating both sides with respect to t.

Question 13:

Considering the sequence of events for $X = 4$, the 1st, 2nd and 3rd numbers must all be different and then the 4th must be the same as one of the first three. The probability is therefore

$$P(X = 4) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n}$$

The same reasoning applied to $X = r$ gives

$$P(X = r) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{r-2}{n}\right)\frac{r-1}{n}$$

The result of part (i) is then found by observing that the probabilities of all possible outcomes add up to 1.

Substituting the probabilities into the formula for $E(X)$ gives

$$E(X) = \frac{2}{n} + 3\left(1 - \frac{1}{n}\right)\frac{2}{n} + 4\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots + (n+1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$

For part (iii) observe that any case where $X \geq k$ will have the first $k - 1$ numbers all different from each other. Therefore

$$P(X \geq k) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-2}{n}\right)$$

The first formula in part (iv) can be shown by considering $kP(Y = k)$ to be equal to the sum of k copies of $P(Y = k)$ and then regrouping the sum for $E(Y)$. Finally this gives two different expressions for $E(Y)$, which must be equal to each other:

$$\begin{aligned}\frac{2}{n} + 3\left(1 - \frac{1}{n}\right)\frac{2}{n} + 4\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots + (n+1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\ = 1 + 1 + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)\end{aligned}$$

Rearranging and using the result from part (i) then gives the required result.

STEP 3 2014 Hints and Solutions

1. The stem results are obtained through algebraic expansion and equating coefficients. Using the expression $(1+ax)(1+bx)(1+cx)$ for $1+qx^2+rx^3$, manipulating the logarithm of the product, and the series expansions for expressions like $\ln(1+ax)$ yields the displayed result. In parts (ii), (iii), and (iv), it is simplest to find $S_2=-q$, $S_3=r$, $S_5=-qr$, $S_7=q^2r$, and $S_9=\frac{r^3}{3}-q^3r$ by expanding the series for $\ln\big(1+(qx^2+rx^3)\big)$, and choosing a counter-example, selecting a, b and c so that $r\neq 0$.

2. The first part is solved using the given method, the formula $\cosh 2x = 2\cosh^2 x - 1$, and then employing partial fractions or the standard form quoted in the formula book. The second part requires the substitution, $u=\sinh x$, the formula $\cosh 2x = 1 + 2\sinh^2 x$, and a standard form to give $\frac{\sqrt{2}}{2}\tan^{-1}\sqrt{2}\,u + c$. The third part can be approached by making the substitution $u=e^x$ and division of the resulting fraction in the numerator and denominator by e^{2x} to give half the difference of the integrals in the first two parts. Alternatively, a similar style of working with the substitution $u=e^{-x}$ results in a sum instead of a difference.

3. (i) Given that the shortest distance between the line and the parabola will be zero if they meet, investigating the solution of the equations simultaneously , and the discriminant of the resulting quadratic equation, the first result of the question is the case that they do not meet. The closest approach is the perpendicular distance of the point on the parabola where the tangent is parallel to the line, so using the standard parametric form, it is the perpendicular distance of $\left(\frac{a}{m^2},\frac{2a}{m}\right)$ from $=mx+c$, giving the required result with care being taken over the sign of the numerator bearing in mind the inequalities.

(ii) The shortest distance of a point on the axis from the parabola, is either the distance from the vertex to the point, or the distance along one of the normals (which are symmetrically situated) which is not the axis. If the normal at $(at^2,2at)$ passes through $(p,0)$, then $=2a+at^2$. From this it can be simply shown that shortest distance is p if $\frac{p}{a}<2$, and is $2\sqrt{a(p-a)}$ if $\frac{p}{a}\geq 2$.

Then for the circle, the results follow simply, that the shortest distance will be $p-b$ if $p>b$, and 0 otherwise if $\frac{p}{a}<2$, and $2\sqrt{a(p-a)}-b$ if $4a(p-a)>b^2$ or 0 otherwise if $\frac{p}{a}\geq 2$.

4. Expanding the bracket in the integral I_1, and employing $\sec^2 x = 1+\tan^2 x$ yields I plus the integral of a perfect differential which can be evaluated simply. For $I=0$, $y'+y\tan x=0$ over the interval which can be solved using an integrating factor and then the condition $y=0, x=1$ enables the arbitrary constant to be evaluated giving the required result. In part (ii), similar working can be undertaken with the integral which is to be considered, given $b=a$. The argument requires no discontinuity in the interval so $a<\frac{\pi}{2}$. The function $y=\cos ax$ can be shown to meet the requirement.

5. ABCD is a parallelogram if and only if $\overrightarrow{AB} = \overrightarrow{DC}$ which yields the required result. To be a square as well, angle $ABC = 90^o$, and $|AB| = |BC|$, so $c - b = i(b - a)$. Treating the two results as simultaneous equations to be solved for a and c in terms of b and d, the second result of the stem can be shown with reversible logic. For part (i) the same logic can be used for PXQ as just used for ABC. From the stem, XYZT is a square if and only if $i(x - z) = y - t$, and

$x + z = y + t$ and given the working for X in part (i), these can be shown to be true treating Y, Z, and T similarly

6. Starting from $f''(t) > 0$ for $0 < t < x_0 \implies \int_0^{t_0} f''(t)\,dt > 0$ where $0 < t_0 < x_0$, with the given conditions yields $f'(t_0) > 0$, and then repeating the argument with $f'(t)$ instead gives $f(t) > 0$. Choosing $f(x) = 1 - \cos x \cosh x$ and applying the applying the stem of the question for $0 < x < \pi$, gives the required inequality for $0 < x < \pi/2$ in particular. For part (ii), choosing $g(x) = x^2 - \sin x \sinh x$ (in which case $g''(x) = 2f(x)$, where $f(x)$ was the suggested choice for part (i)) and $h(x) = \sin x \cosh x - x$ provide the desired results once care is taken with positivity of functions over the required interval when dividing inequalities.

7. Part (i), the intersecting chords theorem, is basic bookwork relying on angle properties in circles to establish similar triangles and hence the result. Part (ii) can be obtained by considering that Q lies on P_1P_3 and so $\boldsymbol{q} = \boldsymbol{p_1} + \lambda(\boldsymbol{p_3} - \boldsymbol{p_1})$, that Q also lies on P_2P_4 producing a similar result and then equating these two expressions, finally rearranging to give (*). Assuming that $a_1 + a_3 = 0$ and using (*) leads to $a_1(\boldsymbol{p_1} - \boldsymbol{p_3}) = a_2(\boldsymbol{p_4} - \boldsymbol{p_2})$ which, in view of the distinctness of the four points P and the intersection of P_1P_3 and P_2P_4 at Q , leads to the contradiction $a_1 = a_2 = a_3 = a_4 = 0$. Re-writing $\frac{a_1\boldsymbol{p_1}+a_3\boldsymbol{p_3}}{a_1+a_3}$ as $\boldsymbol{p_1} + \frac{a_3(\boldsymbol{p_3}-\boldsymbol{p_1})}{a_1+a_3}$ and similarly, using (*), as $\frac{a_2\boldsymbol{p_2}+a_4\boldsymbol{p_4}}{a_2+a_4}$ and re-writing, the expression can be shown to be the position vector of Q .The final result comes from applying (i) using the information just gained and calculating both expressions by taking scalar products of the vectors whose magnitudes are quoted in (i).

8. The initial result is obtained by extending the given inequality so that each term of the sum is compared with $f(k^n)$ and $f(k^{n+1})$. Part (i) is obtained using the stem, the given function, $= 2$, and summing the sums. The deduction relies on considering the lower limit of the sum. The same approach applies to part (ii), with the new function given and considering the upper limit which is obtained as a geometric progression. Counting the number of elements of $S(1000)$ gives the method for obtaining $\sigma(n)$ using the same function as part (i) except $f(r) = 0$ if r has one or more 2 s in its decimal representation and with $k = 10$, again with the sum of a geometric progression. The final result is particularly attractive, demonstrating how few terms need to be removed from the non-convergent harmonic progression (of part (i)) in order to produce a convergent sequence.

9. $\mathbf{v} = \frac{1-\mathrm{e}^{-\mathrm{kt}}}{\mathrm{k}}\mathbf{g} + \mathrm{e}^{-\mathrm{kt}}\mathbf{u}$ and a further differentiation yields $\mathrm{m}\mathbf{a} = \mathrm{m}\mathbf{g} - \mathrm{mk}\mathbf{v}$ **.** Using $\mathrm{r.j} = 0$

obtains the first displayed result after re-arrangement, as does $\tan\beta = \frac{-\mathbf{v.j}}{\mathbf{v.i}}$ the second.

$\tan\beta - \tan\alpha$ can be shown to be $\frac{2g}{uk\cos\alpha(1-e^{-kT})}(\sinh kT - kT)$ which leads to the two final inequalities.

10. The first result is obtained by considering Newton's second law applied to the mass X under the tension in PX and the thrust of XY. $m\frac{d^2y}{dt^2} = -\frac{\lambda(x+2y)}{a}$ is similarly obtained considering Y. Subtracting the two equations gives a SHM second order differential equation for $-y$, and adding them gives similar for $x + y$. Solving these using initial conditions give $x - y = \frac{1}{2}a\cos\omega t$ and $x + y = -\frac{1}{2}a\cos\sqrt{3}\omega t$. The final result is particularly elegant, and possibly a little surprising that a conservative oscillating system does not return to its starting position. Treating the previous two results as simultaneous equations for x and y, and solving $y = -\frac{1}{2}a$, yields $1 = \cos\sqrt{3}\omega t$ and $1 = \cos\omega t$, so that $\sqrt{3}\omega t = 2n\pi$ and $\omega t = 2m\pi$ for non-zero integers n and m , yielding the contradiction $\sqrt{3} = \frac{n}{m}$.

11. Resolving vertically and horizontally, and solving the resulting simultaneous equations and then tidying up the trigonometric expressions yields $T_A = m\frac{(g\sin\beta+\omega^2 x\sin\alpha\cos\beta)}{\sin(\alpha+\beta)}$ and $T_B = m\frac{(\omega^2 x\sin\alpha\cos\alpha - g\sin\alpha)}{\sin(\alpha+\beta)}$. Trivially, the former is positive, but the same condition applied to the latter, given that its denominator and the common factor of the numerator can be shown to be positive, yields the first required inequality. The geometric inequality could be proved, as candidates tended to, by use of the cosine rule and then completing the square to obtain $(x - h\cos\alpha)^2 = d^2 - h^2 + h^2\cos^2\alpha$. However, use of the sine rule and the maximum of the sine function, or the shortest distance of B from AP gives $d \geq h\sin\alpha$, which along with $\cos^2\alpha + \sin^2\alpha = 1$, give the required inequality. In the particular case, it can be shown that $T_A = \frac{mg}{\cos\alpha}$ and the knowledge of the unattainable maximum value of the cosine function along with the geometric inequality previously obtained leads to the final inequalities. The geometry is that the strings are perpendicular, which can be appreciated by considering the equality case of $d \geq h\sin\alpha$.

12. The first result, $y_m = e^{x_m}$, is obtained merely by considering probabilities, and the given pdf of Y can be obtained by standard techniques or by consideration of changing the variable in the integral of the pdf of X. The mode result relies on differentiation of the pdf of Y equated to zero to give a stationary value. The explanation in part (iii) is simply that the required integral is merely that of the pdf of a Normal variable with mean $\mu + \sigma^2$. The expectation of Y is obtained in the standard manner, using an integral and the pdf of Y, and then a change of variable, in which exponential terms can be combined so as to use the explained result having completed the square in the exponent. Using the three previous parts gives $\lambda = e^{\mu-\sigma^2}$, $y_m = e^{x_m} = e^{\mu}$ because X is symmetric, and, as stated,

$E(Y) = e^{\mu+\frac{1}{2}\sigma^2}$, hence satisfying part (iv).

13. The first result is a trivial application of the definition of a probability generating function, and the second similarly. In order to obtain the first printed result in part (iii), it is necessary to obtain a similar result to those in parts (i) and (ii) giving $tG(t)$ as the score is one higher and then applying the conditionality of the probabilities of these three results which is done

by considering the probability of a score n in the three cases to give the coefficient of t^n. Re-arranging the formula for $G(t)$, either differentiation or the binomial theorem can be used to find the required probability formula. Finding $\mu = G'(1) = c/a$ and the knowledge that $a + b + c = 1$ enables the result of part (iii) to be rearranged to that of part (iv).

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