STEP Solutions
2012

Mathematics
STEP 9465/9470/9475

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Contents

STEP Mathematics (9465, 9470, 9475)

STEP I - Hints and Solutions

Question 1

The relationships for p and q can be obtained by substituting the coordinates of the three known points into the equation of the line, or by using the formula for calculating the gradient from a pair of points.

The formula for the sum of the distances is then easy to find and differentiation with respect to m will allow the minimum value to be found.

Similarly, the distance for the second part can be written as $\sqrt{p^2 + q^2}$ and again differentiation can be used to find the value of m for which the minimum occurs. Since the minimum value of $\sqrt{p^2 + q^2}$ occurs at the same value of m as the minimum value of $(p^2 + q^2)$, the differentiation can be simplified by just differentiating $(p^2 + q^2)$ with respect to m. It is then a simple matter to write this answer in a form similar to that of the first part.

Question 2

To sketch the graph it is important to know where the stationary points are. Either by considering the graph of $y = x^2 - 3$ or by differentiating it can be seen that there are two minima and one maximum.

The equation in the second part can be rearranged to show that the solutions correspond to the intersections of the graph of $y = (x^2 - 3)^2$ and a straight line. The case requiring care is the two solution case as this must include the straight line which touches the two minima.

For the next part of the question, differentiation of the equation twice leads quickly to the two possible values of x. Both cases then need to be considered, but it should be clear that one graph is a reflection of the other in the y-axis, so the sets of values for b will be the same for both cases.

The final graph should clearly have a minimum for some negative value of x. $\frac{d^2y}{dx^2}$ will still be 0 at $x = \pm 1$, so there will be two points of inflexion.

Question 3

The sketch of the graph, including a chord and tangent should not cause much difficulty. Adding the line $x = b$ should show that the area under the graph lies between two triangles, both with a base of length b and with heights $\sin b$ and b. Integrating the function between the two limits and then rearranging will give the correct relationship.

For the second part of the question a different diagram is needed, this time showing the area under a curve contained within a trapezium and with a trapezium contained within it. The differentiation and integration of $y = a^x$ will produce the $\ln a$ expressions required in the final answer and so the vertical lines $x = 0$ and $x = 1$ can be used to define the regions.

Question 4

The equations of the tangents at P and Q should be easy to find and then the solution of simultaneous equations will give the required coordinates for T. Similarly, the equations for the normal should be easy to find, but it is more difficult to find simplified expressions for the solution to the simultaneous equations (which are useful for the final part of the question). The factorisation of the difference between two cubes, $a^3 - b^3 \equiv (a - b)(a^2 + ab + b^2)$ will be useful for avoiding a lot of algebraic manipulation. If this expression is simplified correctly then the final result becomes straightforward.

Question 5

The most obvious approaches to the first integration are to integrate by parts or to use the identity $\sin 2x \equiv 2 \sin x \cos x$ and then make a substitution. The second integration can also be evaluated by integrating by parts, but the identity for $\cos 2x$ is not as useful.

For the third integration it is necessary to rewrite the integral in a form from which the previous results can be applied. The first point to note is that the expression within the logarithm is not a simple cosine function and so the first step to making the expression similar to those used previously is to rewrite it in the form $R\cos(x - \alpha)$. Once this is done, the substitution of $u = x - \frac{1}{4}\pi$, with some knowledge of the relationship between sine and cosine graphs should reduce the integral to a combination of the two previous ones.

Question 6

By writing down expressions for the height of the pole using the tangent of each of the angles of elevation, the problem is quickly reduced to a two dimensional problem about lines within a circle. The simplest way to tackle this is to observe that the triangles in the diagram are similar to each other, but approaches working with various right angled triangles also lead to the correct solution.

The proof of the identity should be straight forward for those familiar with the commonly used trigonometric relationships and the inequality is then easily found by considering the consequence of the constraint placed on *p* and *q*. Given that cot is a decreasing function in the required range of values, the final result follows easily.

Question 7

By substituting x for all of the terms in the recurrence relation the result for the first part should follow easily. For the second part, the two values x and y can be substituted in both orders into the relation, giving two equations in x and y. Solving these two equations then leads to the correct set of possibilities.

Following the same principle, the substitution of x , y and z into the relation can be done in three different ways, leading to three simultaneous equations. Solving these equations gives an equation in p and q , which can be solved to give the two different cases. The case where $p + q = 1$ is easy to check, and for the case where $(p - q)^2 + (p + 1)^2 + (q + 1)^2 = 0$ it should be noticed that this can only occur if $p = q = -1$.

Question 8

The new differential equation follows quite easily once the substitution given has been followed. It is an example of a differential equation that can be solved by separating the variables and so by evaluating the two integrals that are reached the required form can be obtained.

The same substitution will also reduce the second differential equation to a simpler form and it is again an example that can be solved by separating the variables. It should be clear that partial fractions are an appropriate method for evaluating the integral that is required.

Question 9

The first part of the question is a straightforward calculation involving the equations for motion under uniform acceleration. It is important to explain the reason for choosing the positive square root however. The final result follows from correct use of an identity for $\cos 2\theta$.

It is also easy to find an expression for the range once the time has been calculated and further application of an identity for $\cos 2\theta$ will give an expression for this in terms of c . Differentiation with respect to c will then give the result that the maximum value occurs when $c = \frac{1}{5}$. The final part of the question can be solved by substituting the appropriate values of c into the formula for the range.

Question 10

Although the equation looks complicated, calculations for the time that it takes the stone to drop and the time for the sound to return allow the first relationship to be deduced quite easily. The second relationship can be shown by simplifying the expression for T and showing that it is equal to $\sqrt{\frac{2D}{g}}$. This can then be rearranged to give $D = \frac{1}{2}gT^2$. The final part of the question is then a substitution of the values given into the formula to obtain the estimate.

Question 11

While the diagram may look a little more complicated than the standard questions on this topic, the first section of this question requires the usual steps to establish a pair of simultaneous equations. The difference in the second part of the question is that the acceleration cannot be assumed to be constant (as the pulley in the middle of the diagram is able to move) and the important extra relationship that is needed is the relationship between the accelerations at the three points (the two particles and the pulley).

Question 12

The probabilities for a failure in each year long period need to be calculated by evaluating the integral and from these it is possible to construct a tree diagram from which the probability can be calculated. The final part of the question is simply the calculation of a conditional probability. As always with conditional probability the important step is to deduce which two probabilities need to be calculated.

Question 13

There are a number of ways to approach this problem. The most obvious is to work out how many possibilities there are for each number of digits. A clear method for categorising these is needed to work out the number of possibilities in each case. For example, if there are 4 different digits then there are five choices for the digits to be used and four choices for the digit to be repeated. There are then ten choices for the positions of the repeated digits and 3! choices for the order of the remaining digits. This gives 1200 altogether.

Hints & Solutions for Paper 9470 (STEP II) June 2012

Question 1

To be honest, the binomial expansions of $(1 \pm x)^n$, in the cases $n = 1, 2$, are used so frequently within AS- and A-levels that they should be familiar to all candidates taking STEPs. Replacing x by x^k is no great further leap.

The general term in $\left(1 - x^6\right)^{-2}$ is easily seen to be $(n + 1)\, x^{6n}$ and the x^{24} term in $\left(1 - x^6\right)^{-2}\left(1 - x^3\right)^{-1}$ comes from $1.x^{24} + 2x^6.x^{18} + 3x^{12}.x^{12} + 4x^{18}.x^6 + 5x^{24}.1$, so that the coefficient of x^{24} is $1 + 2 + 3 + 4 + 5 = 15$, arising from a sum of triangular numbers. Thus, the coefficient of x^n is

$$\begin{cases} 0 & \text{if } n = 6k + \{1, 2, 4, 5\} \\ \tfrac{1}{2}(k+1)(k+2) & \text{if } n = 6k + 3 \\ \tfrac{1}{2}(k+1)(k+2) & \text{if } n = 6k \end{cases}$$

which is most easily described without using n directly, as here.

In (ii), f(x) = $\left(1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \ldots\right)\left(1 + x^3 + x^6 + x^9 + \ldots\right)\left(1 + x + x^2 + x^3 + \ldots\right)$ and the x^{24} term comes from

$$\begin{aligned} &1.1.5x^{24} + 1.x^6.4x^{18} + 1.x^{12}.3x^{12} + 1.x^{18}.2x^6 + 1.x^{24}.1 \\ &+ x^3.x^3.4x^{18} + x^3.x^9.3x^{12} + x^3.x^{15}.2x^6 + x^3.x^{21}.1 \\ &+ x^6.1.4x^{18} + x^6.x^6.3x^{12} + x^6.x^{12}.2x^6 + x^6.x^{18}.1 \\ &+ x^9.x^3.3x^{12} + x^9.x^9.2x^6 + x^9.x^{15}.1 \\ &+ x^{12}.1.3x^{12} + x^{12}.x^6.2x^6 + x^{12}.x^{12}.1 \\ &+ x^{15}.x^3.2x^6 + x^{15}.x^9.1 \\ &+ x^{18}.1.2x^6 + x^{18}.x^6.1 \\ &+ x^{21}.x^3.1 \\ &+ x^{24}.1.1 \end{aligned}$$

giving the coefficient of x^{24} as $15 + 2 \times (10 + 6 + 3 + 1) = 55$.

However, there are lots of ways to go about doing this. For instance ...
Note that, because every non-multiple-of-3 power in bracket 3 is redundant, the x^{24} term comes from considering f(x) = $\left(1 - x^6\right)^{-2}\left(1 - x^3\right)^{-2}$ = $\left(1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \ldots\right)\left(1 + 2x^3 + 3x^6 + 4x^9 + \ldots\right)$.
Again, every non-multiple-of-6 power in *this* 2nd bracket is also redundant, so one might consider only

$$f(x) = \left(1 + 3x^6 + 5x^{12} + 7x^{18} + 9x^{24} + \ldots\right)\left(1 + 2x^6 + 3x^{12} + 4x^{18} + 5x^{24} + \ldots\right)$$

from which the coefficient of x^{24} is simply calculated as $1 \times 5 + 3 \times 4 + 5 \times 3 + 7 \times 2 + 9 \times 1 = 55$. This result, in some form or another, gives the way of working out the coefficient of x^{6n} for any non-negative integer n. It is immediately obvious that it is $\sum_{r=0}^{n}(n + 1 - r)(2r + 1)$ which turns out to be the same as $\sum_{r=1}^{n+1} r^2 = \tfrac{1}{6}(n+1)(n+2)(2n+3)$. The proof of this result could be by induction or direct manipulation of the standard results for Σr and Σr^2.

The coefft. of x^{25} is 55, the same as for x^{24}, since the extra x only arises from replacing 1 by x, x^3 by x^4, etc., in the first bracket's term (at each step of the working) and the coefficients are equal in each case.

In the case when $n = 11$, the coefficient of x^{66} is $12 \times 1 + 11 \times 3 + 10 \times 5 + \ldots + 2 \times 21 + 1 \times 23 = 650$.

Question 2

Firstly, $\mathrm{p}\big(\mathrm{q}(x)\big)$ has degree mn.

(i) Deg[p(x)] $= n \Rightarrow$ Deg[p(p(x))] $= n^2$ & Deg[p(p(p(x)))] $= n^3$.

Deg[LHS] $\le \max\left(n^3, n\right)$ while RHS is of degree 1. Therefore the LHS is not constant so $n = 1$ and p(x) is linear. Setting $\mathrm{p}(x) = ax + b \Rightarrow \mathrm{p}(\mathrm{p}(x)) = a(ax + b) + b = a^2x + (a + 1)b$ and

$$\mathrm{p}(\mathrm{p}(\mathrm{p}(x))) = a[a^2x + (a + 1)b] + b = a^3x + (a^2 + a + 1)b.$$

Then $a^3x + (a^2 + a + 1)b - 3ax - 3b + 2x \equiv 0 \Rightarrow (a^3 - 3a + 2)x + (a^2 + a - 2)b \equiv 0$

$\Rightarrow (a - 1)(a^2 + a - 2)x + (a^2 + a - 2)b \equiv 0$

$\Rightarrow (a^2 + a - 2)[(a - 1)x + b] \equiv 0$

$\Rightarrow (a + 2)(a - 1)[(a - 1)x + b] \equiv 0$

We have, then, that $a = -2$ or 1. In either case, b takes any (arbitrary) value and the solutions are thus

$$\mathrm{p}_1(x) = -2x + b \quad \text{and} \quad \mathrm{p}_2(x) = x + b.$$

(ii) Deg[RHS] = 4 while Deg[LHS] $\le \max\left(n^2, 2n, n\right)$, so it follows that $n = 2$ and p(x) is quadratic. Setting $\mathrm{p}(x) = ax^2 + bx + c$, we have

$$2\mathrm{p}(\mathrm{p}(x)) = 2a(ax^2 + bx + c)^2 + 2b(ax^2 + bx + c) + 2c$$
$$= 2a\left\{a^2x^4 + 2abx^3 + 2acx^2 + b^2x^2 + 2bcx + c^2\right\} + 2b(ax^2 + bx + c) + 2c$$

$$3\big(p(x)\big)^2 = 3\left[a^2x^4 + 2abx^3 + \left(2ac + b^2\right)x^2 + 2bcx + c^2\right] \text{ and } -4\mathrm{p}(x) = -4ax^2 - 4bx - 4c.$$

Thus, LHS = $\left(2a^3 + 3a^2\right)x^4 + \left(4a^2b + 6ab\right)x^3 + \left(2ab^2 + 4a^2c + 2ab + 3b^2 + 6ac - 4a\right)x^2$
$+ \left(4abc + 2b^2 + 6bc - 4b\right)x + \left(2ac^2 + 2bc + 2c + 3c^2 - 4c\right)$,

while the RHS = x^4.

Equating terms gives

x^4) $\quad 2a^3 + 3a^2 - 1 = 0 \Rightarrow (a+1)^2(2a-1) \Rightarrow a = -1$ or $\frac{1}{2}$

x^3) $\quad 2ab(2a + 3) = 0 \Rightarrow b = 0$

x^2) $\quad 2a(2ac + 3c - 2) = 0 \Rightarrow c = 2$ when $a = -1$; i.e. $\mathrm{p}_1(x) = -x^2 + 2$

OR $c = \frac{1}{2}$ when $a = \frac{1}{2}$; i.e. $\mathrm{p}_2(x) = \frac{1}{2}\left(x^2 + 1\right)$.

Note that there are two sets of conditions yet to be used, so the results obtained need to be checked (visibly) for consistency:

x^1) $2b(2ac + b + 3c - 2) = 0$ checks and x^0) $c(2ac + 3c - 2) = 0$ checks also.

Question 3

It helps greatly to begin with, to note that if $t = \sqrt{x^2 + 1} + x$, then $\dfrac{1}{t} = \sqrt{x^2 + 1} - x$. These then give the result $x = \frac{1}{2}t - \frac{1}{2}t^{-1}$, from which we find $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2} + \frac{1}{2}t^{-2}$ and (changing the limits) $x : (0, \infty) \to t : (1, \infty)$,

so that $\displaystyle\int_0^\infty \mathrm{f}\left(\sqrt{x^2+1} + x\right)\,\mathrm{d}x = \int_1^\infty \mathrm{f}(t) \times \tfrac{1}{2}\left(1 + \frac{1}{t^2}\right)\,\mathrm{d}t = \tfrac{1}{2}\int_1^\infty \mathrm{f}(x)\left(1 + \frac{1}{x^2}\right)\,\mathrm{d}x$, as required.

For the first integral, $I_1 = \displaystyle\int_0^\infty \frac{1}{\left(\sqrt{x^2+1} + x\right)^2}\,\mathrm{d}x$, we are using $\mathrm{f}(x) = \dfrac{1}{x^2}$ in the result established initially.

Then $I_1 = \frac{1}{2}\displaystyle\int_1^\infty \left(1 + \frac{1}{x^2}\right).\frac{1}{x^2}\,\mathrm{d}x = \frac{1}{2}\int_1^\infty \left(x^{-2} + x^{-4}\right)\,\mathrm{d}x = \frac{1}{2}\left[-\frac{1}{x} - \frac{1}{3x^3}\right]_1^\infty = \frac{1}{2}\left(0 + 1 + \frac{1}{3}\right) = \frac{2}{3}$.

In the case of the second integral, the substitution $x = \tan\theta \Rightarrow \mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. Also $\sqrt{1+x^2} = \sec\theta$ and the required change of limits yields $\left(0, \tfrac{1}{2}\pi\right) \to (0, \infty)$. We then have

$$I_2 = \int_0^{\frac{1}{2}\pi} \frac{1}{(1+\sin\theta)^3}\,\mathrm{d}\theta = \int_0^{\frac{1}{2}\pi} \left(\frac{\sec\theta}{\sec\theta + \tan\theta}\right)^3 \mathrm{d}\theta \quad \text{[Note the importance of changing to sec and tan]}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{\sec\theta}{(\sec\theta + \tan\theta)^3}.\sec^2\theta\,\mathrm{d}\theta = \int_0^{\infty} \frac{\sqrt{x^2+1}}{\left(\sqrt{x^2+1}+x\right)^3}\,\mathrm{d}x .$$

We now note, matching this up with the initial result, that we are using $\mathrm{f}(t) = \dfrac{\frac{1}{2}\left(t + \frac{1}{t}\right)}{t^3} = \dfrac{t^2+1}{2t^4}$, so that

$$I_2 = \tfrac{1}{2}\int_1^{\infty} \left(\frac{t^2+1}{t^2}\right).\left(\frac{t^2+1}{2t^4}\right) \mathrm{d}t = \tfrac{1}{4}\int_1^{\infty} \left(t^{-2} + 2t^{-4} + t^{-6}\right) \mathrm{d}t = \tfrac{1}{4}\left[-\frac{1}{t} - \frac{2}{3t^3} - \frac{1}{5t^5}\right]_1^{\infty} = \tfrac{1}{4}\left(0 + 1 + \tfrac{2}{3} + \tfrac{1}{5}\right) = \tfrac{7}{15} .$$

Question 4

(i) This first result is easily established: For $n, k > 1$, $n^{k+1} > n^k$ and $k + 1 > k$ so $(k+1) \times n^{k+1} > k \times n^k$

$\Rightarrow \dfrac{1}{(k+1)n^{k+1}} < \dfrac{1}{kn^k}$ (since all terms are positive).

Then $\ln\left(1 + \dfrac{1}{n}\right) = \dfrac{1}{n} - \dfrac{1}{2n^2} + \dfrac{1}{3n^3} - \dfrac{1}{4n^4} + \dfrac{1}{5n^5} - \ldots$ (a result which is valid since $0 < \dfrac{1}{n} < 1$)

$= \dfrac{1}{n} - \left(\dfrac{1}{2n^2} - \dfrac{1}{3n^3}\right) - \left(\dfrac{1}{4n^4} - \dfrac{1}{5n^5}\right) - \ldots < \dfrac{1}{n}$ since each bracketed term is positive, using

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the previous result. Exponentiating then gives $1 + \dfrac{1}{n} < \mathrm{e}^{\frac{1}{n}} \Rightarrow \left(1 + \dfrac{1}{n}\right)^n < \mathrm{e}$.

(ii) A bit of preliminary log. work enables us to use the $\ln(1 + x)$ result on

$$\ln\left(\frac{2y+1}{2y-1}\right) = \ln\left(1 + \frac{1}{2y}\right) - \ln\left(1 - \frac{1}{2y}\right) = \left(\frac{1}{2y} - \frac{1}{2(2y)^2} + \frac{1}{3(2y)^3} - \frac{1}{4(2y)^4} + \frac{1}{5(2y)^5} - \ldots\right)$$

$$- \left(-\frac{1}{2y} - \frac{1}{2(2y)^2} - \frac{1}{3(2y)^3} - \frac{1}{4(2y)^4} - \frac{1}{5(2y)^5} - \ldots\right)$$

$$= 2\left(\frac{1}{2y} + \frac{1}{3(2y)^3} + \frac{1}{5(2y)^5} + \ldots\right) > \frac{1}{y} \quad \text{(since all terms after the first are positive).}$$

Again, note that we should justify that the series is valid for $0 < \dfrac{1}{2y} < 1$ i.e. $y > \tfrac{1}{2}$ in order to justify the use of the given series. It then follows that $\ln\left(\dfrac{2y+1}{2y-1}\right)^y > 1$, and setting $y = n + \tfrac{1}{2}$ (the crucial final step)

gives $\ln\left(\dfrac{2n+2}{2n}\right)^{n+\frac{1}{2}} > 1 \Rightarrow \left(1 + \dfrac{1}{n}\right)^{n+\frac{1}{2}} > \mathrm{e}$.

(iii) This final part only required a fairly informal argument, but the details still required a little bit of care in order to avoid being too vague.

As $n \to \infty$, $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} = \left(1+\frac{1}{n}\right)^{n} \times \left(1+\frac{1}{n}\right)^{\frac{1}{2}} \to \left(1+\frac{1}{n}\right)^{n} \times 1+ \;\to \left(1+\frac{1}{n}\right)^{n}$ from above and e is squeezed into the same limit from both above and below.

Question 5

With any curve-sketching question of this kind, it is important to grasp those features that are important and ignore those that aren't. For instance, throughout this question, the position of the *y*-axis is entirely immaterial: it could be drawn through any branch of the curves in question or, indeed, appear as an asymptote. So the usually key detail of the *y*-intercept, at $\left(0, \frac{1}{a^2-1}\right)$ in part (i), does not help decide what the function is up to. The asymptotes, turning points (clearly important in part (ii) since they are specifically requested), and any symmetries are important. The other key features to decide upon are the "short-term" (when x is small) and the "long-term" (as $x \to \pm\infty$) behaviours.

In (i), there are vertical asymptotes at $x = a - 1$ and $x = a + 1$; while the *x*-axis is a horizontal asymptote. There is symmetry in the line $x = a$ (a consequence of which is the maximum TP in the "middle" branch) and the "long-term" behaviour of the curve is that it ultimately resembles the graph of $y = \frac{1}{x^2}$.

(ii) Differentiating the function in (ii) gives

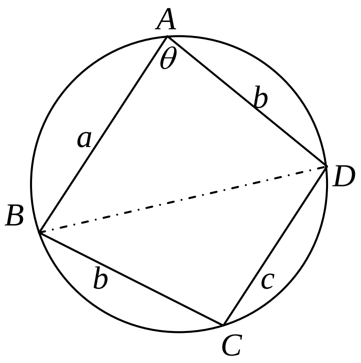
$$g'(x) = \frac{-2}{\left[(x-a)^2-1\right]^2\left[(x-b)^2-1\right]^2}\left\{(x-b)\left[(x-a)^2-1\right]+(x-a)\left[(x-b)^2-1\right]\right\}$$

and setting the numerator = 0 $\Rightarrow$ $(x-a)(x-b)\left[x-a+x-b\right]+\left[x-a+x-b\right]=0$. Factorising yields $(2x-a-b)\left(x^2-(a+b)x+(ab-1)\right)=0$, so that $x=\tfrac{1}{2}(a+b)$ or $\dfrac{a+b\pm\sqrt{(a+b)^2-4ab+4}}{2}$.

In the first case, where $b > a + 2$ (i.e. $a + 1 < b - 1$), there are five branches of the curve, with 4 vertical asymptotes: $x = a \pm 1$ and $x = b \pm 1$. As the function changes sign as it "crosses" each asymptote, and the "long-term" behaviour is still to resemble $y = \frac{1}{x^2}$, these branches alternate above and below the *x*-axis, with symmetry in $x = \tfrac{1}{2}(a+b)$.

In the second case, where $b = a + 2$ (i.e. $a + 1 = b - 1$), the very middle section has collapsed, leaving only the four branches, but the curve is otherwise essentially unchanged from the previous case.

Question 6



A quick diagram helps here, leading to the important observation, from the GCSE geometry result "*opposite angles of a cyclic quad. add to* 180°", that $\angle BCD = 180^\circ - \theta$. Then, using the Cosine Rule twice (and noting that $\cos(180^\circ - \theta) = -\cos\theta$):

<u>in ΔBAD</u>: $BD^2 = a^2 + d^2 - 2ad\cos\theta$

<u>in ΔBCD</u>: $BD^2 = b^2 + c^2 + 2bc\cos\theta$

Equating for BD^2 and re-arranging gives $\cos\theta = \dfrac{a^2-b^2-c^2+d^2}{2(ad+bc)}$

Next, the well-known formula for triangle area, $\Delta = \frac{1}{2}ab\sin C$, twice, gives $Q = \frac{1}{2}ad\sin\theta + \frac{1}{2}bc\sin\theta$, since $\sin(\pi - \theta) = \sin\theta$. Rearranging then gives $\sin\theta = \dfrac{2Q}{ad+bc}$ or $\dfrac{4Q}{2(ad+bc)}$.

Use of $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \dfrac{16Q^2}{4(ad+bc)^2} + \dfrac{\left(a^2 - b^2 - c^2 + d^2\right)^2}{4(ad+bc)^2} = 1$ and this then gives the printed result, $16Q^2 = 4(ad+bc)^2 - \left(a^2 - b^2 - c^2 + d^2\right)^2$.

Then, $16Q^2 = \left(2ad + 2bc - a^2 + b^2 + c^2 - d^2\right)\left(2ad + 2bc + a^2 - b^2 - c^2 + d^2\right)$ by the *difference-of-two-squares* factorisation

$$= \left([b+c]^2 - [a-d]^2\right)\left([a+d]^2 - [b-c]^2\right)$$
$$= \left([b+c]-[a-d]\right)\left([b+c]+[a-d]\right)\left([a+d]-[b-c]\right)\left([a+d]+[b-c]\right)$$

using the *difference-of-two-squares* factorisation in each large bracket

$$= (b+c+d-a)(a+b+c-d)(a+c+d-b)(a+b+d-c).$$

Splitting the 16 into four 2's (one per bracket) and using $2s = a + b + c + d$

$$\Rightarrow Q^2 = \frac{(2s-2a)}{2}\frac{(2s-2b)}{2}\frac{(2s-2c)}{2}\frac{(2s-2d)}{2} = (s-a)(s-b)(s-c)(s-d).$$

Finally, for a triangle (guaranteed cyclic), letting $d \to 0$ (**Or** $s - d \to s$ **Or** let $D \to A$), we get the result known as *Heron's Formula*: $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

Question 7

Many of you will know that this point G, used here, is the centroid of the triangle, and has position vector $\mathbf{g} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$.

Then $\overrightarrow{GX_1} = \mathbf{x}_1 - \mathbf{g} = \frac{1}{3}(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3)$ and so $\overrightarrow{GY_1} = -\frac{1}{3}\lambda_1(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3)$, where $\lambda_1 > 0$.

Also $\overrightarrow{OY_1} = \overrightarrow{OG} + \overrightarrow{GY_1} = \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) - \frac{1}{3}\lambda_1(2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3) = \frac{1}{3}\left([1-2\lambda]\mathbf{x}_1 + [1+\lambda_1](\mathbf{x}_2 + \mathbf{x}_3)\right)$, the first printed result.

The really critical observation here is that the circle centre O, radius 1 has equation $|\mathbf{x}|^2 = 1$ or $\mathbf{x} \, . \, \mathbf{x} = 1$, where $\mathbf{x}$ can be the p.v. of any point on the circle.

Thus, since $\overrightarrow{OY_1} \bullet \overrightarrow{OY_1} = 1$, we have

$$1 = \tfrac{1}{9}\left\{(1-2\lambda_1)^2 + 2(1+\lambda_1)^2 + 2(1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1 \bullet (\mathbf{x}_2 + \mathbf{x}_3) + 2(1+\lambda_1)^2 \mathbf{x}_2 \bullet \mathbf{x}_3\right\}$$
$$\Rightarrow \quad 9 = 1 - 4\lambda_1 + 4\lambda_1^2 + 2 + 4\lambda_1 + 2\lambda_1^2 + 2(1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1 \bullet (\mathbf{x}_2 + \mathbf{x}_3) + 2(1+\lambda_1)^2 \mathbf{x}_2 \bullet \mathbf{x}_3$$
$$\Rightarrow \quad 0 = -3(1-\lambda_1)(1+\lambda_1) + (1-2\lambda_1)(1+\lambda_1)\mathbf{x}_1 \bullet (\mathbf{x}_2 + \mathbf{x}_3) + (1+\lambda_1)^2 \mathbf{x}_2 \bullet \mathbf{x}_3$$

As $\lambda_1 > 0$, $0 = -3(1-\lambda_1) + (1-2\lambda_1)\mathbf{x}_1 \bullet (\mathbf{x}_2 + \mathbf{x}_3) + (1+\lambda_1)\mathbf{x}_2 \bullet \mathbf{x}_3$

$$\Rightarrow \quad 0 = -3 + 3\lambda_1 + \left(\mathbf{x}_1 \bullet \mathbf{x}_2 + \mathbf{x}_2 \bullet \mathbf{x}_3 + \mathbf{x}_3 \bullet \mathbf{x}_1\right) + \lambda_1(\mathbf{x}_2 \bullet \mathbf{x}_3) - 2\lambda_1\left(\mathbf{x}_1 \bullet \mathbf{x}_2 + \mathbf{x}_1 \bullet \mathbf{x}_3\right)$$

$\Rightarrow \lambda_1 = \dfrac{3 - (\alpha + \beta + \gamma)}{3 + \alpha - 2\beta - 2\gamma}$, using $\alpha = \mathbf{x}_2 \bullet \mathbf{x}_3$, $\beta = \mathbf{x}_3 \bullet \mathbf{x}_1$ and $\gamma = \mathbf{x}_1 \bullet \mathbf{x}_2$.

Similarly, $\lambda_2 = \dfrac{3 - (\alpha + \beta + \gamma)}{3 + \beta - 2\alpha - 2\gamma}$ and $\lambda_3 = \dfrac{3 - (\alpha + \beta + \gamma)}{3 + \gamma - 2\alpha - 2\beta}$.

Using $\dfrac{GX_i}{GY_i} = \dfrac{1}{\lambda_i}$ (i = 1, 2, 3), $\dfrac{GX_1}{GY_1} + \dfrac{GX_2}{GY_2} + \dfrac{GX_3}{GY_3} = \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2} + \dfrac{1}{\lambda_3} = \dfrac{9 + (\alpha+\beta+\gamma) - 4(\alpha+\beta+\gamma)}{3 - (\alpha+\beta+\gamma)}$

$$= \frac{9 - 3(\alpha+\beta+\gamma)}{3 - (\alpha+\beta+\gamma)} = 3.$$

[Interestingly, this result generalises to n points on a circle: $\sum_{i=1}^{n} \dfrac{GX_i}{GY_i} = n$.]

Question 8

$\beta - \alpha > q \ (> 0) \Rightarrow \beta^2 - 2\alpha\beta + \alpha^2 > q^2 \Rightarrow \alpha^2 + \beta^2 - q^2 > 2\alpha\beta \Rightarrow \dfrac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} > 2 \Rightarrow$ the opening result, $\dfrac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} - 2 > 0$.

$u_{n+1} = \dfrac{u_n{}^2 - q^2}{u_{n-1}}$ etc. $\Rightarrow u_n{}^2 - u_{n+1}u_{n-1} = q^2 = u_{n+1}{}^2 - u_{n+2}u_n$ (since the result is true at all stages) and equating for $q^2 \Rightarrow u_n(u_n + u_{n+2}) = u_{n+1}(u_{n-1} + u_{n+1})$.

Now this gives $\dfrac{u_n + u_{n+2}}{u_{n+1}} = \dfrac{u_{n-1} + u_{n+1}}{u_n}$ which $\Rightarrow \dfrac{u_{n-1} + u_{n+1}}{u_n}$ is constant (independent of n). Calling this constant p gives $u_{n+1} - pu_n + u_{n-1} = 0$, as required. In order to determine p, we only need to use the fact that $p = \dfrac{u_{n-1} + u_{n+1}}{u_n}$ for all n, so we choose the first few terms to work with.

$$u_2 = \frac{\beta^2 - q^2}{\alpha} \text{ and } p = \frac{u_0 + u_2}{u_1} = \frac{\alpha + \dfrac{\beta^2 - q^2}{\alpha}}{\beta} = \frac{\alpha^2 + \beta^2 - q^2}{\alpha\beta}.$$

Alternatively, $u_2 = \gamma = \dfrac{\beta^2 - q^2}{\alpha} = p\beta - \alpha \Leftrightarrow p = \dfrac{\alpha^2 + \beta^2 - q^2}{\alpha\beta}$

and $u_3 = \dfrac{\gamma^2 - q^2}{\beta} = p\gamma - \beta \Leftrightarrow p = \dfrac{\gamma^2 + \beta^2 - q^2}{\beta\gamma} = \dfrac{\left(\dfrac{\beta^2 - q^2}{\alpha}\right)^2 + \beta^2 - q^2}{\beta\left(\dfrac{\beta^2 - q^2}{\alpha}\right)}$

$$= \frac{(\beta^2 - q^2)^2 + \alpha^2(\beta^2 - q^2)}{\alpha\beta(\beta^2 - q^2)} = \frac{\beta^2 - q^2 + \alpha^2}{\alpha\beta}$$

since $\beta^2 - q^2 \neq 0$ as u_2 non-zero (given). Since p is consistent for any chosen α, β, the proof follows inductively on any two consecutive terms of the sequence.

Finally, on to the given cases.

If $\beta > \alpha + q$, $u_{n+1} - u_n = (p-1)u_n - u_{n-1} = \left(\dfrac{\beta^2 + \alpha^2 - q^2}{\alpha\beta} - 1\right)u_n - u_{n-1}$

$> (2-1)u_n - u_{n-1}$ by the initial result

$> u_n - u_{n-1}$

Hence, if $u_n - u_{n-1} > 0$ then so is $u_{n+1} - u_n$. Since $\beta > \alpha$, $u_2 - u_1 > 0$ and proof follows inductively.

If $\beta = \alpha + q$ then $p = 2$ and $u_{n+1} - u_n = u_n - u_{n-1}$ so that the sequence is an AP.
Also, $u_0 = \alpha$, $u_1 = \alpha + q$, $u_2 = \alpha + 2q$, … ⇒ the common difference is q (and we still have a strictly increasing sequence, since $q > 0$ given).

Question 9

In the standard way, we use the constant-acceleration formulae to get

$$x = ut\cos\alpha \text{ and } y = 2h - ut\sin\alpha - \tfrac{1}{2}gt^2 .$$

When $x = a$, $t = \dfrac{a}{u\cos\alpha}$. Substituting this into the equation for y ⇒ $y = 2h - a\tan\alpha - \dfrac{ga^2}{2u^2}\sec^2\alpha$.
As $y > h$ at this point (the ball, assuming it to be "a particle", is above the net), we get
$h - a\tan\alpha > \dfrac{ga^2}{2u^2}\sec^2\alpha \;\Rightarrow\; \dfrac{1}{u^2} < \dfrac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}$, as required.

For the next part, we set $y = 0$ in $y = 2h - ut\sin\alpha - \tfrac{1}{2}gt^2$ and solve as a quadratic in t to get

$t = \dfrac{-2u\sin\alpha + \sqrt{4u^2\sin^2\alpha + 16gh}}{2g}$ ... (the positive root is required).

Setting $x = (u\cos\alpha)t$ and noting that $x < b$, $u\cos\alpha\left(\dfrac{\sqrt{u^2\sin^2\alpha + 4gh} - u\sin\alpha}{g}\right) < b$

$\Rightarrow \sqrt{u^2\sin^2\alpha + 4gh} < \dfrac{bg}{u\cos\alpha} + u\sin\alpha$.

There are several ways to proceed from here, but this is (perhaps) the most straightforward.
Squaring ⇒ $u^2\sin^2\alpha + 4gh < \dfrac{b^2g^2\sec^2\alpha}{u^2} + 2bg\tan\alpha + u^2\sin^2\alpha$

Cancelling $u^2\sin^2\alpha$ both sides & dividing by g ⇒ $4h < \dfrac{b^2g\sec^2\alpha}{u^2} + 2b\tan\alpha$

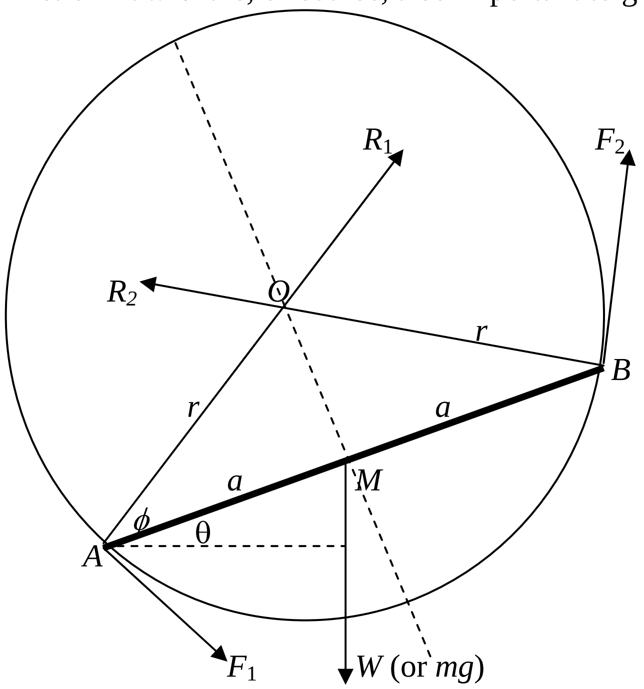
Re-arranging for $\dfrac{1}{u^2}$ ⇒ $\dfrac{2(2h - b\tan\alpha)}{b^2g\sec^2\alpha} < \dfrac{1}{u^2}$

Using the first result, $\dfrac{1}{u^2} < \dfrac{2(h - a\tan\alpha)}{a^2g\sec^2\alpha}$, in here ⇒ $\dfrac{2(2h - b\tan\alpha)}{b^2g\sec^2\alpha} < \dfrac{2(h - a\tan\alpha)}{a^2g\sec^2\alpha}$

Re-arranging for $\tan\alpha$ ⇒ $ab(b-a)\tan\alpha < h\left(b^2 - 2a^2\right)$, which leads to the required final answer
$\tan\alpha < \dfrac{h\left(b^2 - 2a^2\right)}{ab(b-a)}$. However, it is necessary (since we might otherwise be dividing by a quantity that could be negative) to explain that $b > a$ (we are now on the other side of the net to the projection point) else the direction of the inequality would reverse.

Question 10

As with many statics problems, a good diagram is essential to successful progress. Then there are relatively few mechanical principles to be applied ... resolving (twice), taking moments, and the standard "Friction Law". It is, of course, also important to get the angles right.



Taking moments about M :
$R_1\, a \sin\phi = R_2\, a \sin\phi + F_1\, a \cos\phi + F_2\, a \cos\phi$

Using the Friction Law : $F_1 = \mu R_1$ and $F_2 = \mu R_2$

Dividing by $\cos\phi$ and re-arranging
$R_1 \tan\phi = R_2 \tan\phi + \mu R_1 + \mu R_2$
$\Rightarrow (R_1 - R_2) \tan\phi = \mu (R_1 + R_2)$

For the second part, it seems likely that we will have to resolve twice (not having yet used this particular set of tools), though we could take moments about some other point in place of one resolution. There is also the question of which directions to resolve in – here, it should be clear very quickly that "horizontally and vertically" will only yield some very messy results.

Moments about O : $\mu (R_1 - R_2)\, r = W r \sin\phi \sin\theta$
Resolving // AB : $(R_1 - R_2) \cos\phi + \mu (R_1 + R_2) \sin\phi = W \sin\theta$
(Give one **A1** here if all correct apart from a – sign)
Resolving $\perp^{r}$ AB : $(R_1 + R_2) \sin\phi - \mu (R_1 - R_2) \cos\phi = W \cos\theta$
Note that only two of these are actually required, but it may be easier to write them all down first and *then* decide which two are best used.

Dividing these last two eqns. $\Rightarrow \tan\theta = \dfrac{(R_1 - R_2)\cos\phi + \mu(R_1 + R_2)\sin\phi}{(R_1 + R_2)\sin\phi - \mu(R_1 - R_2)\cos\phi}$

Using first result, $\mu (R_1 + R_2) = (R_1 - R_2) \tan\phi \Rightarrow \tan\theta = \dfrac{(R_1 - R_2)\cos\phi + (R_1 - R_2)\tan\phi\sin\phi}{(R_1 - R_2)\dfrac{\tan\phi}{\mu}\sin\phi - \mu(R_1 - R_2)\cos\phi}$

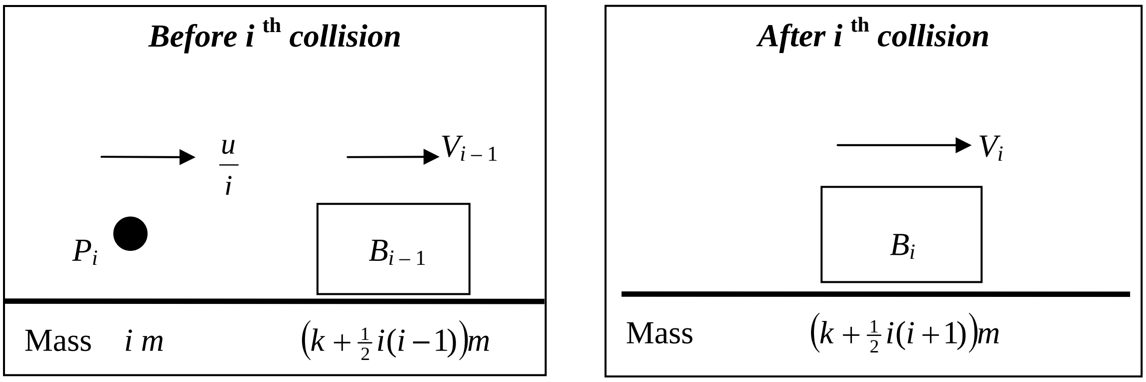
$\Rightarrow \tan\theta = \dfrac{\cos\phi + \tan\phi\sin\phi}{\dfrac{\tan\phi}{\mu}\sin\phi - \mu\cos\phi}$. (There is no need to note that $R_1 \neq R_2$ for then the rod would hve to be positioned symmetrically in the cylinder.)

Multiplying throughout by $\mu\cos\phi \Rightarrow \tan\theta = \dfrac{\mu(\cos^2\phi + \sin^2\phi)}{\sin^2\phi - \mu^2\cos^2\phi} = \dfrac{\mu}{1 - \cos^2\phi - \mu^2\cos^2\phi}$ and, using

$\cos\phi = \dfrac{a}{r}$ gives $\tan\theta = \dfrac{\mu}{1 - \dfrac{a^2}{r^2} - \mu^2\left(\dfrac{a^2}{r^2}\right)} = \dfrac{\mu r^2}{r^2 - a^2(1 + \mu^2)}$.

Finally, $\tan\lambda = \mu = \left(\dfrac{R_1 - R_2}{R_1 + R_2}\right)\tan\phi$, from the first result, $< \tan\phi \Rightarrow \lambda < \phi$.

Question 11
Again, a diagram is really useful for helping put ones thoughts in order; also, we are going to have to consider what is going on generally (and not just "pattern-spot" our way up the line).



Using the principle of *Conservation of Linear Momentum*,
CLM → $\quad m\,u + M\,V_{i-1} = (M + im)\,V_i \quad$ (NB $V_0 = 0$) leads to

$$V_1 = \frac{u}{k+1},\; V_2 = \frac{2u}{k+1+2},\; V_3 = \frac{3u}{k+1+2+3},\;\dots,\; V_n = \frac{nu}{k+\frac{1}{2}n(n+1)} = \frac{2nu}{2k+n(n+1)}.$$

Alternatively, CLM → for *all* particles gives $mu + 2m\left(\frac{u}{2}\right) + 3m\left(\frac{u}{3}\right) + \dots\dots + nm\left(\frac{u}{n}\right) = \left(k+\frac{1}{2}n(n+1)\right)mV$,

and rearranging for $V = V_n$ yields $V_n = \dfrac{2nu}{2k+n(n+1)}$.

The last collision occurs when $V_n \ge \dfrac{u}{n+1}$, i.e. $\dfrac{2nu}{N(N+1)+n(n+1)} \ge \dfrac{u}{n+1}$

$\Rightarrow\; 2n(n+1) \ge N(N+1) + n(n+1) \;\Rightarrow\; n(n+1) \ge N(N+1) \;\Rightarrow$ there are N collisions.

Now, the total KE of all the P_i's is $\sum_{i=1}^{N} \frac{1}{2}(i\,m)\left(\frac{u}{i}\right)^2 = \frac{1}{2}mu^2 \sum_{i=1}^{N} \frac{1}{i}$.

The final KE of the block is $\frac{1}{2}N(N+1)mV_N^{\,2} = \frac{1}{2}N(N+1)m\left(\frac{u}{N+1}\right)^2 = \frac{1}{2}mu^2\left(\frac{N}{N+1}\right)$.

Therefore, the loss in KE is the difference: $\frac{1}{2}mu^2 \sum_{i=1}^{N} \frac{1}{i} - \frac{1}{2}mu^2\left(\frac{N}{N+1}\right)$.

Since $\dfrac{N}{N+1} = 1 - \dfrac{1}{N+1}$, the loss in KE is $\frac{1}{2}mu^2\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N} - 1 + \frac{1}{N+1}\right) = \frac{1}{2}mu^2 \sum_{i=2}^{N+1}\left(\frac{1}{i}\right)$.

Question 12

This *can* be broken down into more (four) separate cases, but there is no need to:

P(light on) = $p \times \frac{3}{4} \times \frac{1}{2} + (1-p) \times \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}(1+2p)$, and then the conditional probability

P(Hall | on) = $\dfrac{\frac{1}{8}(1-p)}{\frac{1}{8}(1+2p)} = \dfrac{(1-p)}{(1+2p)}$.

To make progress with this next part of the question, it is important to recognise the underlying binomial distribution, and that each day represents one such (Bernouilli) trial. We are thus dealing with B(7, p_1), where $p_1 = \dfrac{(1-p)}{(1+2p)}$ is the previously given answer.

For the modal value to be 3, we must have P(2) < P(3) < P(4); that is,

$$\binom{7}{2}(p_1)^2(1-p_1)^5 < \binom{7}{3}(p_1)^3(1-p_1)^4 \quad \textit{and} \quad \binom{7}{4}(p_1)^4(1-p_1)^3 < \binom{7}{3}(p_1)^3(1-p_1)^4.$$

Using $p_1 = \dfrac{(1-p)}{(1+2p)}$ gives

$$21\left(\frac{1-p}{1+2p}\right)^2\left(\frac{3p}{1+2p}\right)^5 < 35\left(\frac{1-p}{1+2p}\right)^3\left(\frac{3p}{1+2p}\right)^4 \Rightarrow 3(3p) < 5(1-p) \Rightarrow p < \tfrac{5}{14}$$

and

$$35\left(\frac{1-p}{1+2p}\right)^4\left(\frac{3p}{1+2p}\right)^3 < 35\left(\frac{1-p}{1+2p}\right)^3\left(\frac{3p}{1+2p}\right)^4 \Rightarrow (1-p) < (3p) \Rightarrow p > \tfrac{1}{4}.$$

Question 13

Working with the distribution $P_o(\lambda = k\pi y^2)$, P(no supermarkets) = $e^{-k\pi y^2}$ and $P(Y < y) = 1 - e^{-k\pi y^2}$.

Differentiating w.r.t. y to find the pdf of $Y \Rightarrow f(y) = 2k\pi\, y\, e^{-k\pi y^2}$, as given. Then

$E(Y) = \int_0^\infty 2k\pi\, y^2\, e^{-k\pi y^2}\, dy$. Using *Integration by Parts* and writing $2k\pi\, y^2\, e^{-k\pi y^2}$ as $y\left(2k\pi\, y\, e^{-k\pi y^2}\right)$

gives $E(Y) = \left[y\left(e^{-k\pi y^2}\right)\right]_0^\infty + \int_0^\infty e^{-k\pi y^2}\, dy = 0 + \int_0^\infty e^{-k\pi y^2}\, dy$. It is useful (but not essential) to use the

simplifying substitution $x = y\sqrt{2k\pi}$ at this stage to get $\dfrac{1}{\sqrt{2k\pi}}\int_0^\infty e^{-\frac{1}{2}x^2}\, dx = \dfrac{1}{\sqrt{2k\pi}}\sqrt{\dfrac{\pi}{2}} = \dfrac{1}{2\sqrt{k}}$ (by the given result, relating to the standard normal distribution's pdf, at the very beginning of the question).

Next, $E(Y^2) = \int_0^\infty 2k\pi\, y^3\, e^{-k\pi y^2}\, dy$, and using *Integration by Parts* and, in a similar way to earlier,

writing $2k\pi\, y^3\, e^{-k\pi y^2}$ as $y^2\left(2k\pi\, y\, e^{-k\pi y^2}\right)$, $E(Y^2) = \left[y^2\left(e^{-k\pi y^2}\right)\right]_0^\infty + \int_0^\infty 2y\, e^{-k\pi y^2}\, dy$

$= 0 + \dfrac{1}{k\pi}\int_0^\infty 2k\pi\, y\, e^{-k\pi y^2} = \dfrac{-1}{k\pi}\left[e^{-k\pi y^2}\right]_0^\infty$ (using a previous result, or by substitution) = $\dfrac{1}{k\pi}$

$\Rightarrow \text{Var}(Y) = \dfrac{1}{k\pi} - \dfrac{1}{4k} = \dfrac{4-\pi}{4k\pi}$, the given answer, as required.

STEP 3 2012 Hints and Solutions

1. The stem integrates to give $z = y^n \left(\frac{dy}{dx}\right)^2$ and for part (i) using $n = 1$, the stem gives $\frac{dz}{dx} = \sqrt{y}\frac{dy}{dx}$ which can solved for *z* using the initial conditions, and the integrated stem is a first order differential equation for *y*, which when solved, again with the initial conditions, produces the required result. Part (ii) follows the same pattern, with $n = -2$ instead, which has solution $y = e^{\frac{1}{2}x^2}$.

2. The simplification in the opening is $\left(1 - x^{2^{n+1}}\right)$, obtained by repeated use of the difference of two squares. A simple algebraic rearrangement, followed by taking a limit, the logarithm of both expressions, and differentiation produces the other three results in part (i). Part (ii) can be obtained by replacing x by x^3 in $\ln(1-x) = -\sum_{r=0}^{\infty} \ln\left(1 + x^{2^r}\right)$ from part (i), factorising the difference and the sums of cubes and subtracting that part (i) result before differentiating. An alternative is to replicate part (i) using instead the product

$(1 + x + x^2)(1 - x + x^2)(1 - x^2 + x^4)(1 - x^4 + x^8)\ldots\left(1 - x^{2^n} + x^{2^{n+1}}\right)$, but then a little extra care is required with the rearrangement, and consideration of the limit.

3. The two parabolas, with vertices oriented in the direction of the positive axes, touch in the third quadrant in case (a), and in the first quadrant in the other three cases. In case (b), there are intersections in the second and third quadrants, in (c) in the third and fourth, and in (d), the trickiest case, they are in the third quadrant and in the first, between the touching point and the vertex of the parabola on the *x* axis. The first result of part (ii)is obtained by eliminating y between the two equations, the second by differentiating and equating gradients, and the third, by eliminating x^4 , $(a^4 = aa^3)$, from the first result using the second.

The cases that arise are $a = 1$, $= 21$, $\frac{k}{m} = \frac{21}{12} < 2$ (d), $a = \frac{-1+\sqrt{13}}{2}$, $k = \frac{13}{2}\sqrt{13} - \frac{5}{2}$, $\frac{k}{m} < 2$ (d), and $a = \frac{-1-\sqrt{13}}{2}$, $k = \frac{-13}{2}\sqrt{13} - \frac{5}{2}$, $k < 0$ (a).

4. Writing $\frac{n+1}{n!}$ as $\frac{n}{n!} + \frac{1}{n!}$ and then cancelling the first fraction, give exponential series. Similarly, $(n+1)^2$ can be written as $n(n-1) + 3n + 1$, and $(2n-1)^3$ as $8n(n-1)(n-2) + 12n(n-1) + 2n - 1$, the latter giving the result $21e + 1$. Using partial fractions, $\frac{n^2+1}{(n+1)(n+2)}$ can be written as $1 + \frac{2}{n+1} - \frac{5}{n+2}$, the first term giving a GP, and the other two, log series. The result for part (ii) is thus $12 - 16\ln 2$.

5. For non-integer rational points, it makes sense to use values of $\cos\theta$ and $\sin\theta$ based on Pythagorean triples such as 3, 4, 5 or 5, 12, 13, The technique for (i) (b) can be used for (ii) (a), merely by changing the value of m, whereas for (ii) (b), a slightly more involved expression is needed such as $x = a\cos\theta + b\sqrt{m}\sin\theta$, $y = a\sin\theta - b\sqrt{m}\cos\theta$. For (ii) (c), there are two alternatives that work sensibly, $= a\cosh\theta + \sqrt{m}\sinh\theta$, $y = a\sinh\theta + \sqrt{m}\cosh\theta$
or $x = a\sec\theta + \sqrt{m}\ \tan\theta$, $y = a\tan\theta + \sqrt{m}\ \sec\theta$.

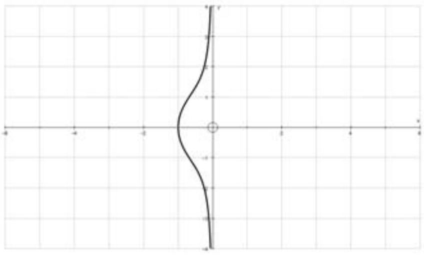
A completely different approach for the last part of the question is to write $x^2-y^2=(x+y)(x-y)=7$ and to choose $x+y=a+b\sqrt{2}$ with nearly any choices of rational *a* and *b* possible. Then, as $x-y=\frac{7}{x+y}$, and numbers of the form $a+b\sqrt{2}$ are a field over the operations $\times$ and $+$, $x-y$ has to be of the correct form, and then solving for *x* and *y*, they likewise have to be of the required form.

Some possible solutions are

(i)(a) $(1,0)$ and $\left(\frac{3}{5},\frac{4}{5}\right)$, (b) $(1,1)$, $1+m$, and $\left(\frac{7}{5},\frac{1}{5}\right)$ (using $m=1$, $\cos\theta=\frac{3}{5}$)

(ii)(a) $(1,\sqrt{2})$ and $\left(\frac{3}{5}+\frac{4}{5}\sqrt{2},\frac{4}{5}-\frac{3}{5}\sqrt{2}\right)$ (using $m=2$, $\cos\theta=\frac{3}{5}$), (b) $\left(\frac{9}{5}+\frac{4}{5}\sqrt{2},\frac{12}{5}-\frac{3}{5}\sqrt{2}\right)$ (using $a=3$, $b=1$, $\cos\theta=\frac{3}{5}$), (c) $\left(\frac{39}{5}+\frac{12}{5}\sqrt{2},\frac{36}{5}+\frac{13}{5}\sqrt{2}\right)$ (using $m=2$, $a=3$ and $\cosh\theta=\frac{13}{5}$, $\sinh\theta=\frac{12}{5}$ or $\sec\theta=\frac{13}{5}$, $\tan\theta=\frac{12}{5}$)

6. Substituting $x+iy$ for *z* in the quadratic equation and equating real and imaginary parts yields the first two results, the imaginary gives two situations, one as required and the other substituted into the real gives the second. The first of these two results substituted into the real gives a circle radius 1, centre the origin, whilst the second gives the real axis without the origin. The second quadratic equation succumbs to the same approach giving the real axis without the origin (again), and a circle centre $(-1,0)$ radius 1 also omitting the origin. The same approach in the third case yields the real axis with $p=\frac{-x^2\pm\sqrt{x^4-8x}}{2x}$ and considering the discriminant, $x<0$ and $x\geq 2$. On the other hand, $p=-2x$ produces $y^2=-\frac{x^3+1}{x}$



7. The second order differential equation $\ddot{y}+7\dot{y}+6y=0$ may be obtained by differentiating the first equation, then substituting for $\dot{z}$ using the other equation, and then doing likewise for z using the first again. The solutions for y and z follow in the usual manner. Parts (i) and (ii) yield $z_1(t)=2e^{-t}-2e^{-6t}$ and $z_2(t)=\frac{c(e^6-1)}{(e^5-1)}e^{-t}-\frac{ce^5(e-1)}{(e^5-1)}e^{-6t}$ respectively. Part (iii) merely requires the sum to be expanded and two geometric progressions emerge so that $c=\frac{2e}{e-1}\frac{(e^5-1)}{(e^6-1)}$.

8. $F_2=1$, $F_3=2$, $F_4=3$, $F_5=5$ so both expressions in part (i) equal -1. Considering either $[(F_nF_{n+3}-F_{n+1}F_{n+2})-(F_{n-2}F_{n+1}-F_{n-1}F_n)]$ or $(F_nF_{n+3}-F_{n+1}F_{n+2})+(F_{n-1}F_{n+2}-F_nF_{n+1})$, and applying the recurrence relation, both can be found to be zero. \so the given expression is shown to be 1 if *n* is odd and -1 if *n* is even. The tan compound angle formula enables the proof in part (iii) to be completed once the initial recurrence relation and the result from part (ii) have been applied to the expression obtained following algebraic simplification. Rearranging the result and substituting into the required sum gives, by the method of differences, $\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{1}{F_{2r+1}}\right)=\frac{\pi}{4}$

9. Eliminating T_1, T_2 , a , and α between the five equations $m_1 g - T_1 = m_1 a$, $T_1 r - T_2 r = I\alpha$, $T_2 - m_2 g = m_2 a$, $\alpha = \frac{a}{r}$, and $P + Mg = Mg + T_1 + T_2$ yields the required result for I . The only difference in the second part is that the second equation becomes $T_1 r - T_2 r - C = I\alpha$, and so the same elimination yields $I = \frac{((m_1+m_2)P - 4m_1 m_2 g)r^2 - (m_1 - m_2)Cr}{(m_1+m_2)g - P}$, which can be seen to be smaller than that in part (i). The first four equations in this second part give $m_1 g - m_2 g - \frac{C}{r} = \left(m_1 + m_2 + \frac{I}{r^2}\right) a$, and so as $a > 0$, the required result follows.

10. After motion commences, the next at rest position has the string at $\frac{\pi}{3}$ to the vertical. Conserving energy between the two at rest positions gives $\lambda = 3mg$. Conserving energy for the general position and resolving radially, bearing in mind that the angle of the radius to the vertical is twice the angle of the string to the vertical, and using a double angle formula gives the required result. The discriminant being negative or completing the square demonstrates that the reaction force is always positive.

11. Various approaches can be used to find the energy terms. If potential energy zero level is taken to be at P, then the initial potential energy is $-2Mg\frac{L}{2}$. When the particle has fallen a distance *x*, the kinetic energy of the particle is $\frac{1}{2}mv^2$, the potential energy of the particle is $-mgx$, the potential energy of the part of the stationary piece of string of length *x* is $-\frac{x}{2L}2Mg\frac{x}{2}$, the potential energy of the remaining piece of (doubled up) string is $-\left(1 - \frac{x}{2L}\right) 2Mg \left(x + \frac{1}{2}\left(L - \frac{1}{2}x\right)\right)$, and the kinetic energy of the shorter moving piece is $\frac{1}{2}\frac{L - \frac{1}{2}x}{2L} 2M\, v^2$. This yields the first result and differentiation of it yields the second. As $0 < x \le 2L$, $ML - \frac{Mx}{4} \ge \frac{ML}{2}$, as $Mgx > 0$ and the denominator is twice a square $\frac{Mgx\left(mL + ML - \frac{Mx}{4}\right)}{2\left(mL + ML - \frac{xM}{2}\right)^2} > 0$, the final result follows.

12. The sketched region contained by AB, and the two line segments connecting A and B to the centre of the triangle. A simple approach for the pdf is via $P(X > x) \propto \left(\frac{1}{3} - x\right)^2$, finding the constant of proportionality and then differentiating to give the required result. $E(X) = \frac{1}{9}$.

For the second part, $g(x) = 192\left(\frac{1}{4} - x\right)^2$, and so $E(X) = \frac{1}{16}$.

13. That $P(z < Z < z + \delta z | a < Z < b) = \frac{P(z<Z<z+\delta Z)}{P(a<Z<b)}$ yields the part (i) result. Similarly, that $E(X|X > 0) = E(\sigma Z + \mu | \sigma Z + \mu > 0)$ yields the first result of part (ii), and that

$m = E\,(|X|) = E(X|X > 0)\,P(X > 0) + E(-X|X < 0)\,P(X < 0)$ yields the second.

Using $Var(X) = E(X^2) - \mu^2 = \sigma^2$, and so

$Var(|X|) = E(|X|^2) - (E|X|)^2 = E(X^2) - m^2 = \mu^2 + \sigma^2 - m^2$