CAMBRIDGE ASSESSMENT

STEP Solutions 2008

Mathematics

STEP 9465, 9470, 9475

UNIVERSITY of CAMBRIDGE
Local Examinations Syndicate

Contents

Step Mathematics (9465, 9470, 9475)

STEP I, Solutions
June 2008

Question 1

What does it mean to say that a number x is irrational?

It means that we cannot write $x = m/n$ where m and n are integers with $n \neq 0$.

Prove by contradiction statements A and B below, where p and q are real numbers.

A: *If pq is irrational, then at least one of p and q is irrational.*

B: *If $p+q$ is irrational, then at least one of p and q is irrational.*

We first prove statement A.

Assume that pq is irrational, but neither p nor q is irrational, so that both p and q are rational. But then pq is the product of two rational numbers, so is rational. This contradicts that assumption that pq is irrational. So statement A is true.

Now for statement B we argue similarly.

Assume that $p+q$ is irrational, but neither p nor q is irrational, so that both p and q are rational. But then $p+q$ is the sum of two rational numbers, so is rational. This contradicts the assumption that $p+q$ is irrational. So statement B is true.

Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: *If p and q are irrational, then $p+q$ is irrational.*

One example is $p = \sqrt{2}$, $q = -\sqrt{2}$.

If the numbers e, π, π^2, e^2 and $\mathrm{e}\pi$ are irrational, prove that at most one of the numbers $\pi + \mathrm{e}$, $\pi - \mathrm{e}$, $\pi^2 - e^2$, $\pi^2 + \mathrm{e}^2$ is rational.

We assume that the five given numbers are, indeed, irrational.

We have $(\pi + \mathrm{e}) + (\pi - \mathrm{e}) = 2\pi$, which is irrational (if p is irrational, then so is $2p$). So by statement B, at least one of $\pi + \mathrm{e}$ and $\pi - \mathrm{e}$ is irrational.

Similarly, $(\pi^2 + \mathrm{e}^2) + (\pi^2 - \mathrm{e}^2) = 2\pi^2$, which is irrational. So by statement B again, at least one of $\pi^2 + \mathrm{e}^2$ and $\pi^2 - \mathrm{e}^2$ is irrational.

Assume that both $\pi + \mathrm{e}$ and $\pi^2 - \mathrm{e}^2$ are rational. Then

$$\pi - \mathrm{e} = \frac{\pi^2 - \mathrm{e}^2}{\pi + \mathrm{e}}$$

would also be rational. But we know that at least one of $\pi + \mathrm{e}$ and $\pi - \mathrm{e}$ is irrational, so $\pi + \mathrm{e}$ and $\pi^2 - \mathrm{e}^2$ cannot both be rational. Similarly, we can't have both $\pi - \mathrm{e}$ and $\pi^2 - \mathrm{e}^2$ rational.

Thus if two of the four numbers are rational, they must be $\pi^2 + \mathrm{e}^2$ and one of $\pi \pm \mathrm{e}$.

Assume that $\pi^2 + \mathrm{e}^2$ and $\pi + \mathrm{e}$ are rational. Then $(\pi + \mathrm{e})^2 = (\pi^2 + \mathrm{e}^2) + 2\mathrm{e}\pi$ is the square of a rational number, so is rational. But then $2\mathrm{e}\pi = (\pi + \mathrm{e})^2 - (\pi^2 + \mathrm{e}^2)$ would be rational, contradicting the irrationality of $\mathrm{e}\pi$. Thus we cannot have both $\pi^2 + \mathrm{e}^2$ and $\pi + \mathrm{e}$ rational.

Similarly, if $\pi^2 + \mathrm{e}^2$ and $\pi - \mathrm{e}$ are both rational, we would have $2\mathrm{e}\pi = (\pi^2 + \mathrm{e}^2) - (\pi - \mathrm{e})^2$ being rational, again a contradiction.

Thus at most one of these four numbers is rational.

Question 2

The variables t and x are related by $t = x + \sqrt{x^2 + 2bx + c}$, where b and c are constants and $b^2 < c$. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t - x}{t + b},$$

and hence integrate $\dfrac{1}{\sqrt{x^2 + 2bx + c}}$.

We have, differentiating the given expression with respect to x:

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}x} &= 1 + \frac{2x + 2b}{2\sqrt{x^2 + 2bx + c}} \\ &= 1 + \frac{2(x + b)}{2(t - x)} \\ &= \frac{(t - x) + (x + b)}{t - x} \\ &= \frac{t + b}{t - x}.\end{aligned}$$

The required result follows on taking the reciprocal of both sides:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\mathrm{d}t/\mathrm{d}x} = \frac{t - x}{t + b}.$$

To find the integral, we use the given substitution for x, yielding:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 + 2bx + c}}\,\mathrm{d}x &= \int \frac{1}{t - x}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int \frac{1}{t - x}\,\frac{t - x}{t + b}\,\mathrm{d}t \\ &= \int \frac{1}{t + b}\,\mathrm{d}t \\ &= \ln|t + b| + k \\ &= \ln\left|x + b + \sqrt{x^2 + 2bx + c}\right| + k \\ &= \ln\left(x + b + \sqrt{x^2 + 2bx + c}\right) + k\end{aligned}$$

with the last line following as $x^2 + 2bx + c > x^2 + 2bx + b^2 = (x + b)^2$, so the parenthesised expression is positive.

Verify by direct integration that your result holds also in the case $b^2 = c$ if $x + b > 0$ but that your result does not hold in the case $b^2 = c$ if $x + b < 0$.

With $b^2 = c$, we have the integral

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 + 2bx + b^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{(x + b)^2}}\,\mathrm{d}x \\ &= \int \frac{1}{|x + b|}\,\mathrm{d}x\end{aligned}$$

We now consider the two cases discussed in the question. Firstly, if $x + b > 0$, then we have

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 + 2bx + b^2}}\,\mathrm{d}x &= \int \frac{1}{|x+b|}\,\mathrm{d}x \\ &= \int \frac{1}{x+b}\,\mathrm{d}x \\ &= \ln(x+b) + k'\end{aligned}$$

(we don't need absolute value signs as $x + b$ is positive), whereas before we had

$$\begin{aligned}\ln\bigl(x + b + \sqrt{x^2 + 2bx + c}\bigr) + k &= \ln\bigl(x + b + \sqrt{(x+b)^2}\bigr) + k \\ &= \ln\bigl(x + b + |x+b|\bigr) + k \\ &= \ln\bigl(x + b + (x+b)\bigr) + k \\ &= \ln(2(x+b)) + k \\ &= \ln(x+b) + \ln 2 + k\end{aligned}$$

Thus our earlier formula works in the case that $b^2 = c$ and $x + b > 0$, where we take $k' = k + \ln 2$ (which we may do, as they are arbitrary constants).

Next, when $x + b < 0$, direct integration yields:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 + 2bx + b^2}}\,\mathrm{d}x &= \int \frac{1}{|x+b|}\,\mathrm{d}x \\ &= \int -\frac{1}{x+b}\,\mathrm{d}x \\ &= -\ln|x+b| + k' \\ &= -\ln(-(x+b)) + k'\end{aligned}$$

as $x + b < 0$. But now our earlier formula yields

$$\begin{aligned}\ln\bigl(x + b + \sqrt{x^2 + 2bx + c}\bigr) + k &= \ln\bigl(x + b + \sqrt{(x+b)^2}\bigr) + k \\ &= \ln\bigl(x + b + |x+b|\bigr) + k \\ &= \ln\bigl(x + b - (x+b)\bigr) + k \\ &= \ln 0 + k\end{aligned}$$

which is not even defined. So the earlier result fails to give any answer in the case $b^2 = c$ when $x + b < 0$.

Question 3

Prove that, if $c \geqslant a$ and $d \geqslant b$, then

$$ab + cd \geqslant bc + ad. \qquad (*)$$

We have

$$\begin{aligned} ab + cd - bc - ad &= (a - c)b + (c - a)d \\ &= (c - a)(d - b) \\ &\geqslant 0 \qquad \text{as } c \geqslant a \text{ and } d \geqslant b, \end{aligned}$$

from which $(*)$ follows immediately.

(i) *If $x \geqslant y$, use $(*)$ to show that $x^2 + y^2 \geqslant 2xy$.*

If, further, $x \geqslant z$ and $y \geqslant z$, use $(*)$ to show that $z^2 + xy \geqslant xz + yz$ and deduce that $x^2 + y^2 + z^2 \geqslant xy + yz + zx$.

Prove that the inequality $x^2 + y^2 + z^2 \geqslant xy + yz + zx$ holds for all x, y and z.

Letting $a = b = y$ and $c = d = x$ in $(*)$, which we can do as $x \geqslant y$, yields $y^2 + x^2 \geqslant yx + yx$, that is

$$x^2 + y^2 \geqslant 2xy. \qquad (1)$$

Next, letting $a = b = z$, $c = x$ and $d = y$ in $(*)$ gives

$$z^2 + xy \geqslant zx + zy \qquad (2)$$

as we wanted.

Now adding the inequalities (1) and (2) gives us

$$x^2 + y^2 + z^2 + xy \geqslant 2xy + yz + zx.$$

Subtracting xy from both sides yields our desired result:

$$x^2 + y^2 + z^2 \geqslant xy + yz + zx. \qquad (3)$$

Finally, we have now proved inequality (3) when $x \geqslant y \geqslant z$, but we need to show that it is true whatever the values of x, y and z. But the inequality is *symmetric* in x, y and z, meaning that rearranging (permuting) the variables in any way does not change the statement. For example, if we swap x and z, we get

$$z^2 + y^2 + x^2 \geqslant zy + yx + xz,$$

which is exactly the same inequality.

So we can assume that the the values of x, y and z we are given satisfy $x \geqslant y \geqslant z$, without changing the statement of the inequality, and we know that the inequality holds in this case.

> ***(ii)*** *Show similarly that the inequality* $\dfrac{s}{t}+\dfrac{t}{r}+\dfrac{r}{s}\geqslant 3$ *holds for all positive* r*,* s *and* t*.*

We begin by assuming that $r \geqslant s \geqslant t > 0$, and set $c = r$, $a = s$, $d = 1/s$ and $b = 1/r$. Then by our assumption, we see that $c \geqslant a$ and $d \geqslant b$, so $(*)$ gives

$$\frac{s}{r}+\frac{r}{s}\geqslant 1+1. \tag{4}$$

Now set $c = s$, $a = t$, $d = 1/t$ and $b = 1/r$, and note that $c \geqslant a$ and $d \geqslant b$, so that $(*)$ gives

$$\frac{t}{r}+\frac{s}{t}\geqslant 1+\frac{s}{r}. \tag{5}$$

Adding the inequalities (4) and (5) gives

$$\frac{s}{r}+\frac{r}{s}+\frac{t}{r}+\frac{s}{t}\geqslant 1+1+1+\frac{s}{r},$$

so that

$$\frac{r}{s}+\frac{t}{r}+\frac{s}{t}\geqslant 1+1+1, \tag{6}$$

as we want.

Now this inequality is true for $r \geqslant s \geqslant t$, and by exactly the same argument it will also hold if $s \geqslant t \geqslant r$ or $t \geqslant r \geqslant s$, by cycling the variables.

If $r \geqslant t \geqslant s$, then we have to start again, but the argument is almost identical.

We set $c = r$, $a = t$, $d = 1/t$ and $b = 1/r$. Then by our assumption, we see that $c \geqslant a$ and $d \geqslant b$, so $(*)$ gives

$$\frac{t}{r}+\frac{r}{t}\geqslant 1+1. \tag{7}$$

We now set $c = r$, $a = s$, $d = 1/s$ and $b = 1/t$, and note that $c \geqslant a$ and $d \geqslant b$, so that $(*)$ gives

$$\frac{t}{r}+\frac{s}{t}\geqslant 1+\frac{s}{r}. \tag{8}$$

Once again, adding these two inequalities gives inequality (6), and by cycling the variables, it is also true if $t \geqslant s \geqslant r$ or $s \geqslant r \geqslant t$.

We have now shown (6) to be true for all six possible orderings of r, s and t, so it is true for all possible (positive) values of r, s and t.

Question 4

A function $\mathrm{f}(x)$ *is said to be* convex *in the interval* $a < x < b$ *if* $\mathrm{f}''(x) \geqslant 0$ *for all* x *in this interval.*

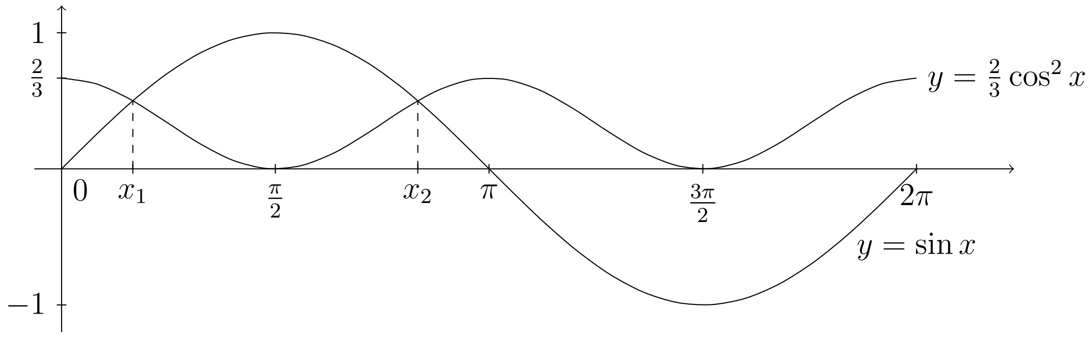
(i) *Sketch on the same axes the graphs of* $y = \frac{2}{3}\cos^2 x$ *and* $y = \sin x$ *in the interval* $0 \leqslant x \leqslant 2\pi$.

The function $\mathrm{f}(x)$ *is defined for* $0 < x < 2\pi$ *by*

$$\mathrm{f}(x) = \mathrm{e}^{\frac{2}{3}\sin x}.$$

Determine the intervals in which $\mathrm{f}(x)$ *is convex.*

We note that $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ from the double angle formula, so that the graph of $\frac{2}{3}\cos^2 x$ is a translated, stretched version of $y = \cos 2x$.



Now we have

$$\begin{aligned}\mathrm{f}(x) &= \mathrm{e}^{\frac{2}{3}\sin x}\\ \mathrm{f}'(x) &= (\tfrac{2}{3}\cos x)\mathrm{e}^{\frac{2}{3}\sin x}\\ \mathrm{f}''(x) &= (\tfrac{4}{9}\cos^2 x - \tfrac{2}{3}\sin x)\mathrm{e}^{\frac{2}{3}\sin x}\\ &= \tfrac{2}{3}(\tfrac{2}{3}\cos^2 x - \sin x)\mathrm{e}^{\frac{2}{3}\sin x}.\end{aligned}$$

As $\mathrm{e}^{\frac{2}{3}\sin x} > 0$ for all x, we have

$$\mathrm{f}''(x) \geqslant 0 \qquad \text{if and only if} \qquad \tfrac{2}{3}\cos^2 x - \sin x \geqslant 0.$$

But we have just drawn a graph of the two functions $y = \frac{2}{3}\cos^2 x$ and $y = \sin x$, so we see that $\mathrm{f}(x) \geqslant 0$ when $0 \leqslant x \leqslant x_1$ and when $x_2 \leqslant x \leqslant 2\pi$, where x_1 and x_2 are the x-coordinates of the points of intersection of the two graphs. So all we need to do is to solve the equation

$$\tfrac{2}{3}\cos^2 x - \sin x = 0$$

to determine the values of x_1 and x_2.

Using $\cos^2 x = 1 - \sin^2 x$ and then factorising gives

$$\begin{aligned} & \tfrac{2}{3}\cos^2 x = \sin x \\ \iff\quad & \tfrac{2}{3}(1-\sin^2 x) = \sin x \\ \iff\quad & 2 - 2\sin^2 x = 3\sin x \\ \iff\quad & 2\sin^2 x + 3\sin x - 2 = 0 \\ \iff\quad & (2\sin x - 1)(\sin x + 2) = 0 \\ \iff\quad & \sin x = \tfrac{1}{2} \end{aligned}$$

Therefore $x_1 = \frac{\pi}{6}$ and $x_2 = \frac{5\pi}{6}$, and the function $\mathrm{f}(x)$ is convex in the intervals $0 < x < \frac{\pi}{6}$ and $\frac{5\pi}{6} < x < 2\pi$.

(ii) *The function* $\mathrm{g}(x)$ *is defined for* $0 < x < \frac{1}{2}\pi$ *by*

$$\mathrm{g}(x) = \mathrm{e}^{-k\tan x}.$$

If $k = \sin 2\alpha$ *and* $0 < \alpha < \pi/4$, *show that* $\mathrm{g}(x)$ *is convex in the interval* $0 < x < \alpha$, *and give one other interval in which* $\mathrm{g}(x)$ *is convex.*

We have

$$\begin{aligned} \mathrm{g}(x) &= \mathrm{e}^{-k\tan x} \\ \mathrm{g}'(x) &= -k\sec^2 x.\mathrm{e}^{-k\tan x} \\ \mathrm{g}''(x) &= (k^2\sec^4 x - 2k\sec^2 x\tan x)\mathrm{e}^{-k\tan x} \\ &= k\sec^2 x(k\sec^2 x - 2\tan x)\mathrm{e}^{-k\tan x}. \end{aligned}$$

Therefore $g''(x) \geqslant 0$ when $k\sec^2 x - 2\tan x \geqslant 0$, that is, when $k\tan^2 x - 2\tan x + k \geqslant 0$. We can solve the equality $k\tan^2 x - 2\tan x + k = 0$ using the quadratic formula:

$$\begin{aligned} \tan x &= \frac{2 \pm \sqrt{4-4k^2}}{2k} \\ &= \frac{1 \pm \sqrt{1-k^2}}{k} \\ &= \frac{1 \pm \sqrt{1-\sin^2 2\alpha}}{\sin 2\alpha} && \text{substituting } k = \sin 2\alpha \\ &= \frac{1 \pm \cos 2\alpha}{\sin 2\alpha}. \end{aligned}$$

Thus we have two possibilities:

$$\tan x = \frac{1+\cos 2\alpha}{\sin 2\alpha} = \frac{2\cos^2\alpha}{2\sin\alpha\cos\alpha} = \cot\alpha = \tan(\tfrac{\pi}{2} - \alpha)$$

or

$$\tan x = \frac{1-\cos 2\alpha}{\sin 2\alpha} = \frac{2\sin^2\alpha}{2\sin\alpha\cos\alpha} = \tan\alpha.$$

Then, since $0 < \alpha < \frac{\pi}{4}$ and $0 < x < \frac{\pi}{2}$, we have $x = \alpha$ or $x = \frac{\pi}{2} - \alpha$.

It follows, since $k > 0$ and $\alpha < \frac{\pi}{2} - \alpha$, that $\mathrm{g}(x)$ is convex, that is, $\mathrm{g}''(x) > 0$, when $0 < x < \alpha$ or when $\frac{\pi}{2} - \alpha < x < \frac{\pi}{2}$. (In more detail, when $x \approx 0$, $\mathrm{g}''(x) \approx k > 0$, and when $x \approx \frac{\pi}{2}$, $\mathrm{g}''(x) \approx k\tan^2 x > 0$.)

Question 5

The polynomial $\mathrm{p}(x)$ *is given by*

$$x^n + \sum_{r=0}^{n-1} a_r x^r \,,$$

where $a_0, a_1, \ldots, a_{n-1}$ *are fixed real numbers and* $n \geqslant 1$. *Let* M *be the greatest value of* $|\mathrm{p}(x)|$ *for* $|x| \leqslant 1$. *Then* Chebyshev's theorem *states that* $M \geqslant 2^{1-n}$.

(i) *Prove Chebyshev's theorem in the case* $n = 1$ *and verify that Chebyshev's theorem holds in the following cases:*

(a) $\mathrm{p}(x) = x^2 - \frac{1}{2}$;

(b) $\mathrm{p}(x) = x^3 - x$.

In the case $n = 1$, Chebyshev's theorem states:

> Let $\mathrm{p}(x)$ be the polynomial $x + a_0$, and let M be the greatest value of $|\mathrm{p}(x)|$ for $|x| \leqslant 1$. Then $M \geqslant 1$.

If $a_0 > 0$, then when $x = 1$, $p(1) = 1 + a_0 > 1$, so $M > 1$.

If $a_0 < 0$, then when $x = -1$, $p(-1) = -1 + a_0 < -1$, so $|p(-1)| > 1$ and $M > 1$.

Finally, if $a_0 = 0$, then $p(x) = x$, so $|p(x)| = |x|$. It follows that $|p(x)| = |x| \leqslant 1$ when $|x| \leqslant 1$ and $p(1) = 1$, so $M = 1$.

Thus in all cases $M \geqslant 1$.

Now to verify the theorem in the specified cases. The obvious approach is to find the maximum absolute value of the function over the interval. It is important to verify that the function's maximum absolute value really is at least 2^{1-n} in both cases.

(a) $\mathrm{p}(x) = x^2 - \frac{1}{2}$ is a quadratic whose minimum value is at $x = 0$. So we only need to consider the value of $\mathrm{p}(0)$ and the values of $\mathrm{p}(x)$ at the endpoints of the interval: $\mathrm{p}(-1)$ and $\mathrm{p}(1)$. We have $\mathrm{p}(0) = -\frac{1}{2}$, $\mathrm{p}(-1) = \mathrm{p}(1) = \frac{1}{2}$, so $-\frac{1}{2} \leqslant \mathrm{p}(x) \leqslant \frac{1}{2}$, and hence $|\mathrm{p}(x)| \leqslant \frac{1}{2} = 2^{-1}$ with the maximum value taken on by $|p(x)|$ being $\frac{1}{2}$.

In this case, where $n = 2$, Chebyshev's theorem states that $M \geqslant 2^{-1}$. Since we have $M = 2^{-1}$ in this case, Chebyshev's theorem holds.

(b) Given $\mathrm{p}(x) = x^3 - x$ we first look for stationary points. We have $\mathrm{p}'(x) = 3x^2 - 1$ so there are stationary points at $x = \pm 1/\sqrt{3}$. We thus evaluate $\mathrm{p}(x)$ at these points and at the endpoints $x = \pm 1$. We have $\mathrm{p}(-1) = \mathrm{p}(1) = 0$, $\mathrm{p}(-1/\sqrt{3}) = 2/3\sqrt{3}$ and $\mathrm{p}(1/\sqrt{3}) = -2/3\sqrt{3}$. Hence $|\mathrm{p}(x)| \leqslant 2/3\sqrt{3}$, so that $M = 2/3\sqrt{3}$.

As $n = 3$, we wish to show that $M \geqslant \frac{1}{4}$. But $M^2 = \frac{4}{27} > \frac{4}{64} = \frac{1}{16}$, so $M > \frac{1}{4}$ as required.

A second approach, which is simpler and more direct, is to observe that all we need to do is to find *some* value of x in the interval $-1 \leqslant x \leqslant 1$ for which $|p(x)| \geqslant 2^{1-n}$, for then we know that the *maximum* value of $|p(x)|$ in this interval will be at least that. In (a), we have $|p(1)| = \frac{1}{2} \geqslant 2^{-1}$ and in (b), $|p(\frac{1}{2})| = |\frac{1}{8} - \frac{1}{2}| = \frac{3}{8} \geqslant 2^{-2}$. So we are done.

> ***(ii)*** *Use Chebyshev's theorem to show that the curve* $y = 64x^5+25x^4-66x^3-24x^2+3x+1$ *has at least one turning point in the interval* $-1 \leqslant x \leqslant 1$.

Let $\mathrm{p}(x) = x^5 + \frac{1}{64}(25x^4 - 66x^3 - 24x^2 + 3x + 1) = y/64$. The turning points of $\mathrm{p}(x)$ are the same as the turning points of y. Then if we let M be the greatest value of $|\mathrm{p}(x)|$ for $|x| \leqslant 1$, we have $M \geqslant 2^{-4} = \frac{1}{16}$. Let x_0 be the value of x in the interval $-1 \leqslant x \leqslant 1$ for which $|p(x_0)| = M$, i.e., where $|p(x)|$ takes its maximum value. (If there is more than one such point, choose any of them to be x_0.)

Now $p(-1) = \frac{1}{64}$ and $p(1) = \frac{3}{64}$, and so $|p(1)| < \frac{1}{16}$ and $|p(-1)| < \frac{1}{16}$. But since $|p(x_0)| \geqslant \frac{1}{16}$, we cannot have $x_0 = \pm 1$, so $-1 < x_0 < 1$.

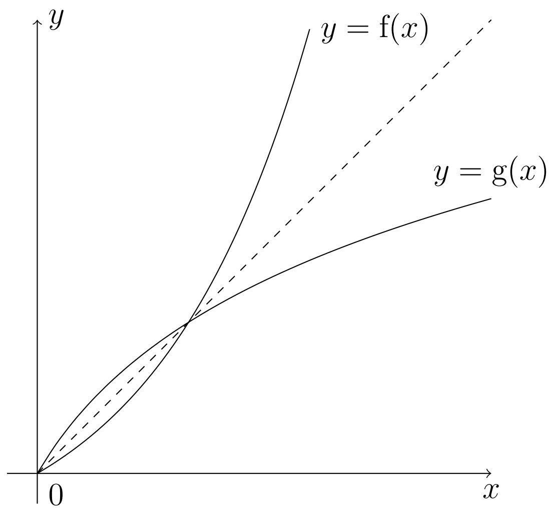
Then x_0 must be a local maximum or a local minimum: if $p(x_0) \geqslant \frac{1}{16}$ then it is at the the greatest value of $p(x)$ in the interval $-1 \leqslant x \leqslant 1$ (and is not at an endpoint); similarly, if $p(x_0) \leqslant -\frac{1}{16}$, then x_0 is at the least value. Either way, x_0 is a turning point as we wanted.

Question 6

The function f *is defined by*

$$\mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}, \qquad x \geqslant 0,$$

and the function g *is the inverse function to* f, *so that* $\mathrm{g}(\mathrm{f}(x)) = x$. *Sketch* $\mathrm{f}(x)$ *and* $\mathrm{g}(x)$ *on the same axes.*



Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_0^{k} \mathrm{g}(x)\,\mathrm{d}x = \frac{1}{2(\sqrt{\mathrm{e}}+1)}\,,$$

where $k = \dfrac{1}{\sqrt{\mathrm{e}}+1}$, *and explain this result by means of a diagram.*

We find $\mathrm{g}(x)$, the inverse of $\mathrm{f}(x)$, as follows, noting that $y = \mathrm{f}(x)$ if and only if $x = \mathrm{g}(y)$:

$$\begin{aligned} & y = \mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1} \\ \iff \quad & (\mathrm{e}-1)y = \mathrm{e}^x - 1 \\ \iff \quad & \mathrm{e}^x = (\mathrm{e}-1)y + 1 \\ \iff \quad & x = \ln\big((\mathrm{e}-1)y + 1\big) \end{aligned}$$

so that $\mathrm{g}(x) = \ln\big((\mathrm{e}-1)x + 1\big)$.

Now we can evaluate the integrals. We have

$$\begin{aligned}
\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}} \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}\,\mathrm{d}x \\
&= \frac{1}{\mathrm{e}-1}\Big[\mathrm{e}^x - x\Big]_0^{\frac{1}{2}} \\
&= \frac{1}{\mathrm{e}-1}\big((\mathrm{e}^{\frac{1}{2}} - \tfrac{1}{2}) - (1-0)\big) \\
&= \frac{1}{\mathrm{e}-1}\big(\mathrm{e}^{\frac{1}{2}} - \tfrac{3}{2}\big) \\
&= \frac{2\sqrt{\mathrm{e}} - 3}{2(\mathrm{e}-1)}.
\end{aligned}$$

For $\mathrm{g}(x)$, we can either use substitution and the standard result $\int \ln x\,\mathrm{d}x = x\ln x - x + c$ or integration by parts, effectively deriving the result. We demonstrate both methods.

Using substitution, we set $u = (\mathrm{e}-1)x+1$. When $x = 0$, $u = 1$, and when $x = k = 1/(\sqrt{\mathrm{e}}+1)$, we can easily calculate that $u = (\sqrt{\mathrm{e}} - 1) + 1 = \sqrt{\mathrm{e}}$. Finally, $\mathrm{d}u/\mathrm{d}x = \mathrm{e} - 1$. Thus

$$\begin{aligned}
\int_0^{k} \mathrm{g}(x)\,\mathrm{d}x &= \int_0^{k} \ln\big((\mathrm{e}-1)x+1\big)\,\mathrm{d}x \\
&= \int_1^{\sqrt{\mathrm{e}}} \ln u\,\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\
&= \int_1^{\sqrt{\mathrm{e}}} \frac{1}{\mathrm{e}-1}\,\ln u\,\mathrm{d}u \\
&= \frac{1}{\mathrm{e}-1}\Big[u\ln u - u\Big]_1^{\sqrt{\mathrm{e}}} \\
&= \frac{1}{\mathrm{e}-1}\big((\sqrt{\mathrm{e}}\ln\sqrt{\mathrm{e}} - \sqrt{\mathrm{e}}) - (\ln 1 - 1)\big) \\
&= \frac{1}{\mathrm{e}-1}\big(1 - \tfrac{1}{2}\sqrt{\mathrm{e}}\big) \\
&= \frac{2 - \sqrt{\mathrm{e}}}{2(\mathrm{e}-1)}.
\end{aligned}$$

Alternatively we can use integration by parts. We first note that

$$\begin{aligned}
\ln((\mathrm{e}-1)k+1) &= \ln\left(\frac{\mathrm{e}-1}{\sqrt{\mathrm{e}}+1} + 1\right) \\
&= \ln\big((\sqrt{\mathrm{e}}-1)+1\big) \\
&= \ln(\sqrt{\mathrm{e}}) \\
&= \tfrac{1}{2}.
\end{aligned}$$

Then we can evaluate our integral as follows:

$$\begin{aligned}
\int_0^k \mathrm{g}(x)\,\mathrm{d}x &= \int_0^k 1.\ln((\mathrm{e}-1)x+1)\,\mathrm{d}x \\
&= \big[x\ln((\mathrm{e}-1)x+1)\big]_0^k - \int_0^k x\left(\frac{\mathrm{e}-1}{(\mathrm{e}-1)x+1}\right)\mathrm{d}x \\
&= k\ln((\mathrm{e}-1)k+1) - \int_0^k \frac{(\mathrm{e}-1)x+1-1}{(\mathrm{e}-1)x+1}\,\mathrm{d}x \\
&= \tfrac{1}{2}k - \int_0^k 1 - \frac{1}{(\mathrm{e}-1)x+1}\,\mathrm{d}x \\
&= \tfrac{1}{2}k - \left[x - \frac{1}{\mathrm{e}-1}\ln((\mathrm{e}-1)x+1)\right]_0^k \\
&= \tfrac{1}{2}k - \left(\left(k - \frac{1}{\mathrm{e}-1}\ln((\mathrm{e}-1)k+1)\right) - (0-\ln 1)\right) \\
&= \tfrac{1}{2}k - \left(k - \frac{1}{2(\mathrm{e}-1)}\right) \\
&= \frac{1}{2(\mathrm{e}-1)} - \tfrac{1}{2}k \\
&= \frac{1}{2(\mathrm{e}-1)} - \frac{1}{2(\sqrt{\mathrm{e}}+1)} \\
&= \frac{1}{2(\mathrm{e}-1)} - \frac{\sqrt{\mathrm{e}}-1}{2(\mathrm{e}-1)} \\
&= \frac{2-\sqrt{\mathrm{e}}}{2(\mathrm{e}-1)}
\end{aligned}$$

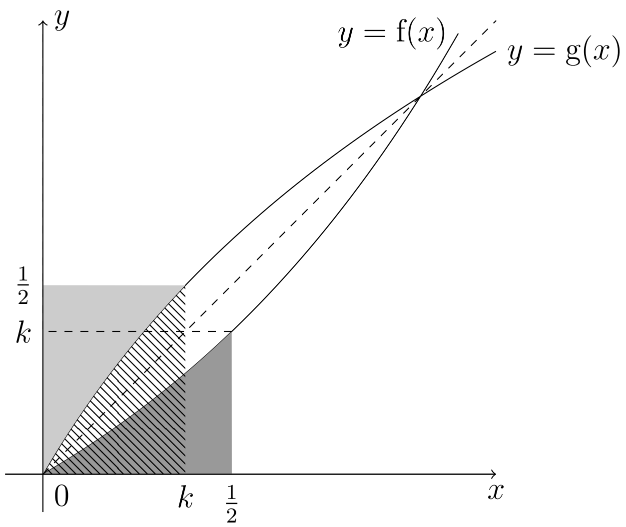
as we found using the substitution method.

We therefore have

$$\begin{aligned}
\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_0^k \mathrm{g}(x)\,\mathrm{d}x &= \frac{2\sqrt{\mathrm{e}}-3}{2(\mathrm{e}-1)} + \frac{2-\sqrt{\mathrm{e}}}{2(\mathrm{e}-1)} \\
&= \frac{\sqrt{\mathrm{e}}-1}{2(\mathrm{e}-1)} \\
&= \frac{1}{2(\sqrt{\mathrm{e}}+1)}
\end{aligned}$$

as we wanted.

Finally, to explain this result with the aid of a diagram, we want to fill in the two areas indicated by the integrals on our sketch of the functions above. It would be useful to know the value of $\mathrm{f}(\frac{1}{2})$ for this purpose: it is $(\mathrm{e}^{1/2}-1)/(\mathrm{e}-1) = 1/(\sqrt{\mathrm{e}}+1) = k$, which is very convenient. Since $\mathrm{g}(x)$ is the inverse of $\mathrm{f}(x)$, it follows likewise that $\mathrm{g}(k) = \frac{1}{2}$. We can now sketch the areas on our graph.

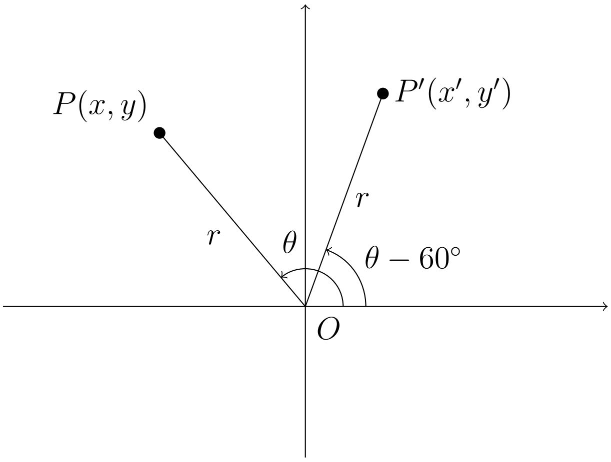


In this sketch, the dark shaded area is the integral $\int_0^{1/2} \mathrm{f}(x)\,\mathrm{d}x$ and the striped area is the integral $\int_0^k \mathrm{g}(x)\,\mathrm{d}x$. We have reflected the dark shaded area in the line $y = x$ to get the light shaded area also shown. It is now clear that the shaded and striped areas add to give a $\frac{1}{2} \times k$ rectangle, so the area is $k/2 = 1/2(\sqrt{\mathrm{e}} + 1)$, as we found.

Question 7

The point P has coordinates (x, y) with respect to the origin O. By writing $x = r\cos\theta$ and $y = r\sin\theta$, or otherwise, show that, if the line OP is rotated by $60°$ clockwise about O, the new y-coordinate of P is $\frac{1}{2}(y - \sqrt{3}\,x)$. What is the new y-coordinate in the case of an anti-clockwise rotation by $60°$?

The situation described is illustrated in the following diagram, where OP' is the image of OP under the specified rotation, P' having coordinates (x', y').



Then we have

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and

$$y' = r\sin(\theta - 60°)$$

We use the compound angle formula for sine to get

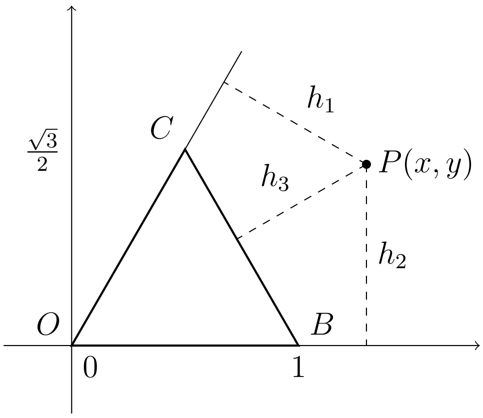
$$\begin{aligned} y' &= r(\sin\theta\cos 60° - \cos\theta\sin 60°) \\ &= \tfrac{1}{2}r\sin\theta - \tfrac{\sqrt{3}}{2}r\cos\theta \\ &= \tfrac{1}{2}y - \tfrac{\sqrt{3}}{2}x \\ &= \tfrac{1}{2}(y - \sqrt{3}\,x). \end{aligned}$$

Likewise, if the rotation is by $60°$ anticlockwise, we replace the $\theta - 60°$ by $\theta + 60°$ and repeat the above to get

$$y' = \tfrac{1}{2}(y + \sqrt{3}\,x).$$

An equilateral triangle OBC has vertices at O, $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, respectively. The point P has coordinates (x, y). The perpendicular distance from P to the line through C and O is h_1; the perpendicular distance from P to the line through O and B is h_2; and the perpendicular distance from P to the line through B and C is h_3.

Show that $h_1 = \frac{1}{2}\left|y - \sqrt{3}\,x\right|$ and find expressions for h_2 and h_3.



Clearly $h_2 = |y|$. (We need to take the absolute value as P might lie under the x-axis.)

For h_1, consider rotating the entire shape by 60° clockwise about O. This will rotate OC to the x-axis, and the perpendicular from P to OC will become vertical. The transformed y-coordinate of P is $\frac{1}{2}(y - \sqrt{3}\,x)$, as we deduced earlier, so $h_1 = \frac{1}{2}\big|y - \sqrt{3}\,x\big|$.

Finally, for h_3, we start by rotating anticlockwise by 60° around O. Then P ends up with y-coordinate $\frac{1}{2}(y + \sqrt{3}\,x)$ and the side BC ends up lying along the line $y = \frac{\sqrt{3}}{2}$; subtracting these then gives $h_3 = \frac{1}{2}\big|y + \sqrt{3}\,x - \sqrt{3}\big|$.

[An alternative argument is to translate the whole diagram by one unit to the left first, so that B moves to the origin and P moves to $(x-1, y)$. Then rotating around the new B gives P the new y-coordinate of $\frac{1}{2}(y + \sqrt{3}(x-1))$, which is the same as before.]

Show that $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}$ if and only if P lies on or in the triangle OBC.

We have

$$\begin{aligned} h_1 + h_2 + h_3 &= \tfrac{1}{2}\big(|2y| + \big|y - \sqrt{3}\,x\big| + \big|y + \sqrt{3}\,x - \sqrt{3}\big|\big) \\ &= \tfrac{1}{2}\big(|2y| + \big|\sqrt{3}\,x - y\big| + \big|\sqrt{3} - \sqrt{3}\,x - y\big|\big). \end{aligned}$$

Using the triangle inequality, we then have

$$\begin{aligned} h_1 + h_2 + h_3 &= \tfrac{1}{2}\big(|2y| + \big|\sqrt{3}\,x - y\big| + \big|\sqrt{3} - \sqrt{3}\,x - y\big|\big) \\ &\geqslant \tfrac{1}{2}\big|(2y) + (\sqrt{3}\,x - y) + (\sqrt{3} - \sqrt{3}\,x - y)\big| \\ &= \tfrac{1}{2}\sqrt{3}, \end{aligned}$$

with equality if and only if all of the bracketed terms are $\geqslant 0$ or all of the bracketed terms are $\leqslant 0$.

If all of the terms are negative or zero, then $2y \leqslant 0$, so $y \leqslant 0$. And $\sqrt{3}\,x - y \leqslant 0$ implies that $x \leqslant y/\sqrt{3} \leqslant 0$. But then we must have $\sqrt{3} - \sqrt{3}\,x - y > 0$, which is impossible. So we cannot have all three terms negative or zero.

Therefore, if we have equality, we must have all three terms positive or zero. But $2y \geqslant 0$ if and only if P lies on or above the x-axis, that is, on or above the line OB. Similarly, $\sqrt{3}\,x - y \geqslant 0$ if and only if $y \leqslant \sqrt{3}\,x$, which is true if and only if P lies on or below the line OC (which has equation $y = \sqrt{3}\,x$). Finally, $\sqrt{3} - \sqrt{3}\,x - y \geqslant 0$ if and only if

$y \leqslant \sqrt{3} - \sqrt{3}\,x$, which is true if and only if P lies on or below the line BC (which has equation $y = \sqrt{3} - \sqrt{3}\,x$).

Putting these together shows that $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}$ if and only if P lies on or inside the triangle OBC.

Question 8

> ***(i)*** *The gradient y' of a curve at a point (x, y) satisfies*
>
> $$(y')^2 - xy' + y = 0. \qquad (*)$$
>
> *By differentiating $(*)$ with respect to x, show that either $y'' = 0$ or $2y' = x$.*
>
> *Hence show that the curve is either a straight line of the form $y = mx + c$, where $c = -m^2$, or the parabola $4y = x^2$.*

Differentiating $(*)$ with respect to x, using the chain rule and product rule, gives

$$2y'y'' - y' - xy'' + y' = 0,$$

which, on cancelling terms and factorising, yields

$$(2y' - x)y'' = 0,$$

so either $y'' = 0$ or $2y' - x = 0$.

We now solve these two differential equations. Firstly, by integrating $y'' = 0$ twice with respect to x, we get $y' = a$, so $y = ax + b$ (where a and b are constants). Substituting this back into $(*)$ gives

$$a^2 - x.a + (ax + b) = 0,$$

so

$$a^2 + b = 0,$$

which gives us the straight line $y = mx + c$ with $m = a$ and $c = b = -a^2 = -m^2$.

In the other case, $y' = \frac{1}{2}x$, which on integrating gives $y = \frac{1}{4}x^2 + c$. Substituting this back into $(*)$ gives

$$(\tfrac{1}{2}x)^2 - x(\tfrac{1}{2}x) + (\tfrac{1}{4}x^2 + c) = 0,$$

so $c = 0$ and $y = \frac{1}{4}x^2$, or $4y = x^2$ as required.

> ***(ii)*** *The gradient y' of a curve at a point (x, y) satisfies*
>
> $$(x^2 - 1)(y')^2 - 2xyy' + y^2 - 1 = 0. \qquad (\dagger)$$
>
> *Show that the curve is either a straight line, the form of which you should specify, or a circle, the equation of which you should determine.*

Differentiating $(\dagger)$ with respect to x, using the chain rule and product rule, gives:

$$2x(y')^2 + (x^2 - 1).2y'y'' - 2yy' - 2xy'y' - 2xyy'' + 2yy' = 0,$$

which, on cancelling terms, yields

$$2(x^2 - 1)y'y'' - 2xyy'' = 0.$$

Finally, factorising brings us to our desired conclusion:

$$\big((x^2-1)y'-xy\big)y''=0,$$

so either $y''=0$ or $y'(x^2-1)-xy=0$.

We now solve these two equations. Again, integrating $y''=0$ twice gives $y'=a$, so $y=ax+b$. Substituting this back into (†) gives:

$$(x^2-1).a^2-2x(ax+b).a+(ax+b)^2-1=0,$$

which is equivalent to

$$a^2x^2-a^2-2a^2x^2-2abx+a^2x^2+2abx+b^2-1=0,$$

so

$$-a^2+b^2-1=0,$$

or $b^2=a^2+1$. Thus the equation is satisfied by straight lines $y=mx+c$ where $c^2=m^2+1$.

In the other case, $y'(x^2-1)-xy=0$.

It looks as though we could solve this by separating the variables to give:

$$\int \frac{1}{y}\,\mathrm{d}y=\int \frac{x}{x^2-1}\,\mathrm{d}x,$$

so $\ln y=\frac{1}{2}\ln|x^2-1|+c$. Doubling and exponentiating then gives

$$y^2=C|x^2-1|.$$

To determine C and if its value depends upon whether $|x|<1$ or $|x|>1$, we ought to try substituting back into (†). However, it is simpler to substitute in the result $y'(x^2-1)=xy$ directly into (†).

Substituting $y'=xy/(x^2-1)$ back into (†) gives

$$(x^2-1)\left(\frac{xy}{x^2-1}\right)^2-2xy\left(\frac{xy}{x^2-1}\right)+y^2-1=0.$$

Expanding brackets then gives

$$\frac{x^2y^2}{x^2-1}-\frac{2x^2y^2}{x^2-1}+y^2-1=0$$

so

$$-\frac{x^2y^2}{x^2-1}+y^2-1=0.$$

Multiplying by x^2-1 gives

$$-x^2y^2+(y^2-1)(x^2-1)=0$$

so

$$-x^2y^2+y^2x^2-y^2-x^2+1=0$$

or

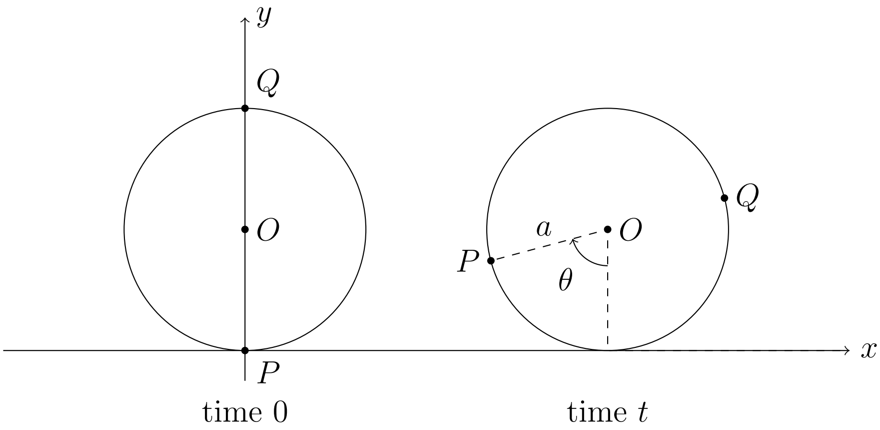
$$-x^2-y^2+1=0,$$

and we therefore deduce that the only other possible solution is the circle $x^2+y^2=1$.

We must finally check that the circle does, in fact, satisfy $y'=xy/(x^2-1)$. Differentiating $x^2+y^2=1$ with respect to x gives $2x+2yy'=0$, so $y'=-x/y=-xy/y^2=-xy/(1-x^2)$, as required.

Question 9

Two identical particles P and Q, each of mass m, are attached to the ends of a diameter of a light thin circular hoop of radius a. The hoop rolls without slipping along a straight line on a horizontal table with the plane of the hoop vertical. Initially, P is in contact with the table. At time t, the hoop has rotated through an angle θ. Write down the position at time t of P, relative to its starting point, in cartesian coordinates, and determine its speed in terms of a, θ and $\dot{\theta}$. Show that the total kinetic energy of the two particles is $2ma^2\dot{\theta}^2$.



The diagram shows the hoop rolling to the right, indicating the positions of the hoop at time 0 and time t.

Taking the origin to be at the initial position of P, the coordinates of the centre of the hoop at time t are $(a\theta, a)$, since the hoop has rolled a distance $a\theta$. Therefore P has coordinates $(a(\theta - \sin\theta), a(1 - \cos\theta))$ and position vector

$$\mathbf{r}_P = a(\theta - \sin\theta)\mathbf{i} + a(1 - \cos\theta)\mathbf{j}.$$

The velocity vector of P is then

$$\begin{aligned}
\dot{\mathbf{r}}_P &= \frac{\mathrm{d}}{\mathrm{d}t}\big(a(\theta - \sin\theta)\big)\mathbf{i} + \frac{\mathrm{d}}{\mathrm{d}t}\big(a(1 - \cos\theta)\big)\mathbf{j} \\
&= a\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\mathbf{i} - a\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{j} \\
&= a(\dot{\theta} - \cos\theta.\dot{\theta})\mathbf{i} + a\sin\theta.\dot{\theta}\,\mathbf{j} \\
&= a\dot{\theta}\big((1 - \cos\theta)\mathbf{i} + \sin\theta\,\mathbf{j}\big),
\end{aligned}$$

and hence P has speed v_P given by

$$\begin{aligned}
v_P^2 &= (a\dot{\theta})^2\big((1 - \cos\theta)^2 + (\sin\theta)^2\big) \\
&= (a\dot{\theta})^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta) \\
&= (a\dot{\theta})^2(2 - 2\cos\theta) \\
&= (a\dot{\theta})^2.4\sin^2\tfrac{1}{2}\theta \\
&= (2a\dot{\theta}\sin\tfrac{1}{2}\theta)^2,
\end{aligned}$$

so that $v_P = 2a|\dot\theta \sin\frac{1}{2}\theta|$.

Similarly, the coordinates of Q are $(a(\theta + \sin\theta), a(1 + \cos\theta))$, so Q has position vector

$$\mathbf{r}_Q = a(\theta + \sin\theta)\mathbf{i} + a(1 + \cos\theta)\mathbf{j}.$$

Arguing as before, the velocity vector of Q is then

$$\begin{aligned}\dot{\mathbf{r}}_Q &= a(\dot\theta + \cos\theta.\dot\theta)\mathbf{i} - a\sin\theta.\dot\theta\,\mathbf{j} \\ &= a\dot\theta\big((1+\cos\theta)\mathbf{i} - \sin\theta\,\mathbf{j}\big)\end{aligned}$$

so that Q has speed v_Q given by

$$\begin{aligned}v_Q^2 &= (a\dot\theta)^2\big((1+\cos\theta)^2 + (\sin\theta)^2\big) \\ &= (a\dot\theta)^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta) \\ &= (a\dot\theta)^2(2 + 2\cos\theta) \\ &= (a\dot\theta)^2.4\cos^2\tfrac{1}{2}\theta.\end{aligned}$$

Adding $v_P^2 = (a\dot\theta)^2.4\sin^2\frac{1}{2}\theta$ to this gives the total kinetic energy as

$$\begin{aligned}\tfrac{1}{2}mv_P^2 + \tfrac{1}{2}mv_Q^2 &= \tfrac{1}{2}m(v_P^2 + v_Q^2) \\ &= \tfrac{1}{2}m\big((a\dot\theta)^2.4\sin^2\tfrac{1}{2}\theta + (a\dot\theta)^2.4\cos^2\tfrac{1}{2}\theta\big) \\ &= \tfrac{1}{2}m(a\dot\theta)^2.4 \\ &= 2ma^2\dot\theta^2\end{aligned}$$

as required.

Given that the only external forces on the system are gravity and the vertical reaction of the table on the hoop, show that the hoop rolls with constant speed.

Consider the hoop as a single system. The only external forces on the hoop are gravity and the normal reaction. Both of these are vertical, while the hoop only moves in a horizontal direction. Therefore, no work is done on the hoop, so that GPE + KE is constant.

The gravitational potential energy of P and Q together, taking the centre of the hoop as potential energy zero, gives $mga(-\cos\theta) + mga(+\cos\theta) = 0$. So the GPE of the system is constant, meaning that the kinetic energy is also constant.

Since the total kinetic energy is $2ma^2\dot\theta^2$, it follows that $\dot\theta$ is constant, that is, the hoop rolls with the constant speed $a\dot\theta$.

Question 10

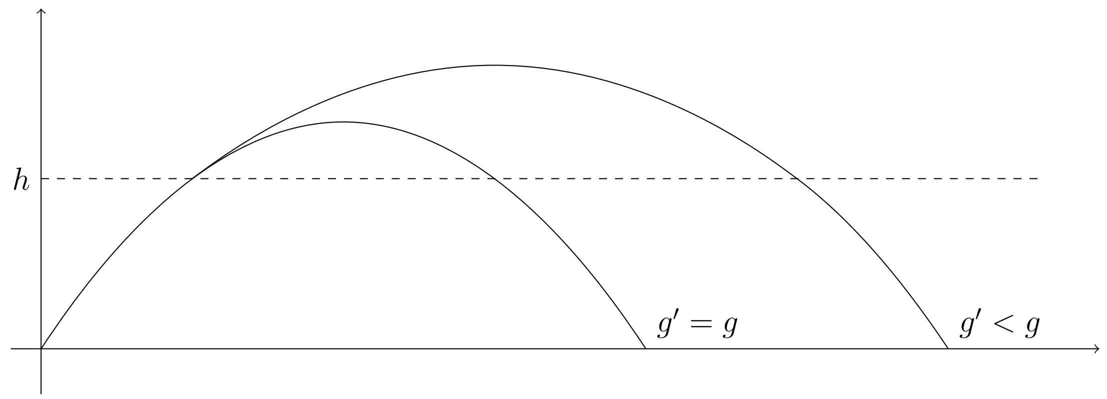
On the (flat) planet Zog, the acceleration due to gravity is g up to height h above the surface and g' at greater heights. A particle is projected from the surface at speed V and at an angle α to the surface, where $V^2\sin^2\alpha > 2gh$. Sketch, on the same axes, the trajectories in the cases $g' = g$ and $g' < g$.

We know that the path of a projectile is parabolic when the gravity is constant. When the gravity is less, the parabola will be "bigger", as the projectile will travel higher before returning to the ground, but the horizontal component of velocity will be unaffected.

So our sketch will consist of a parabola for the case $g = g'$ and a pair of parabolas joined at height h in the case $g' < g$. Note that the velocity does not change suddenly at height h, so the curve will be "smooth" at this point. (Technically, it has a continuous first derivative.)

We need to check that the particle does reach height h before we draw a sketch. The vertical component of velocity is initially $V\sin\alpha$. If the gravity were a constant g, then the maximum height reached would be s, where the formula $v^2 = u^2 + 2as$ gives us $0^2 = V^2\sin^2\alpha - 2gs$, so $s = V^2\sin^2\alpha/2g > h$, so the particle does reach height greater than h.

So here, then, is the sketch:



Show that the particle lands a distance d from the point of projection given by

$$d = \left(\frac{V - V'}{g} + \frac{V'}{g'}\right) V\sin 2\alpha,$$

where $V' = \sqrt{V^2 - 2gh\operatorname{cosec}^2\alpha}$.

We note that the horizontal speed is a constant $V\cos\alpha$, as there is no horizontal component of acceleration. Therefore the distance travelled is this times the time travelled. By symmetry, we find the time taken to reach the highest point on the trajectory, and then double it to find the total time.

First part: below height h.

We use the "suvat" equations to determine the time taken and the vertical speed at height h. Taking upwards as positive, we have $s = h$, $u = V\sin\alpha$, $a = -g$. Then $v^2 = u^2 + 2as$ gives

$v^2 = V^2 \sin^2 \alpha - 2gh$ and $v = u + at$ gives

$$\begin{aligned} t &= \frac{V \sin \alpha - \sqrt{V^2 \sin^2 \alpha - 2gh}}{g} \\ &= \left(\frac{V - \sqrt{V^2 - 2gh \operatorname{cosec}^2 \alpha}}{g} \right) \sin \alpha \\ &= \left(\frac{V - V'}{g} \right) \sin \alpha, \end{aligned}$$

writing $V' = \sqrt{V^2 - 2gh \operatorname{cosec}^2 \alpha}$.

Second part: above height h.

This time, $u = \sqrt{V^2 \sin^2 \alpha - 2gh} = V' \sin \alpha$, $v = 0$ and $a = -g'$, so $v = u + at$ gives

$$t = \frac{V' \sin \alpha}{g'}.$$

Therefore, the total time taken to reach the highest point is

$$\left(\frac{V - V'}{g} \right) \sin \alpha + \frac{V' \sin \alpha}{g'} = \left(\frac{V - V'}{g} + \frac{V'}{g'} \right) \sin \alpha.$$

Finally, we need to multiply this by 2 to get the total time taken and then by $V \cos \alpha$ to get the distance travelled, giving the distance

$$\begin{aligned} d &= 2 \left(\frac{V - V'}{g} + \frac{V'}{g'} \right) \sin \alpha . V \cos \alpha \\ &= \left(\frac{V - V'}{g} + \frac{V'}{g'} \right) V \sin 2\alpha, \end{aligned}$$

using $2 \sin \alpha \cos \alpha = \sin 2\alpha$.

Question 11

A straight uniform rod has mass m. Its ends P_1 and P_2 are attached to small light rings that are constrained to move on a rough circular wire with centre O fixed in a vertical plane, and the angle P_1OP_2 is a right angle. The rod rests with P_1 lower than P_2, and with both ends lower than O. The coefficient of friction between each of the rings and the wire is μ. Given that the rod is in limiting equilibrium (i.e., on the point of slipping at both ends), show that

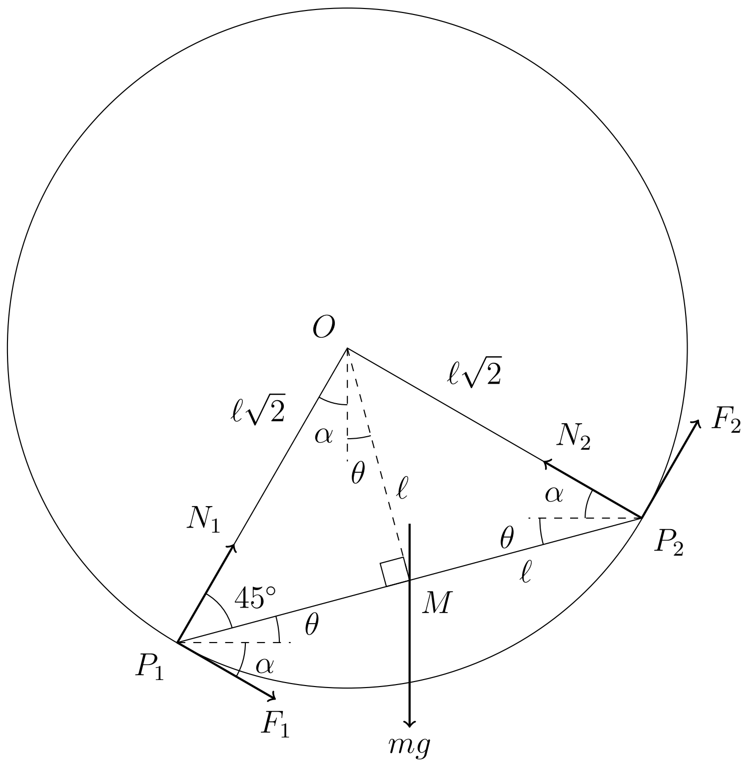
$$\tan\alpha = \frac{1-2\mu-\mu^2}{1+2\mu-\mu^2},$$

where α is the angle between P_1O and the vertical ($0 < \alpha < 45°$).
Let θ be the acute angle between the rod and the horizontal. Show that $\theta = 2\lambda$, where λ is defined by $\tan\lambda = \mu$ and $0 < \lambda < 22.5°$.

We present two methods for solving the problem. The first is a standard method using resolution of forces; the second is to use standard results about three forces acting on a large body. We specify that the length of the rod is 2ℓ, so that the radius of the circle is $\ell\sqrt{2}$.

Method 1: Resolving all the forces

We begin by drawing a clear sketch of the situation.



Note that as the rod is in limiting equilibrium, both of the frictional forces act to prevent it from slipping towards the horizontal, and $F_1 = \mu N_1$, $F_2 = \mu N_2$. Also, we see that $\theta = 45° - \alpha$, as the angle OP_1P_2 is 45° (the triangle being isosceles), so $0 < \theta < 45°$.

We now resolve the forces in two directions. We could resolve in any two directions, but so that we can exclude mg from at least one of the equations, we choose to resolve horizontally and vertically. Another sensible choice would have been to resolve along the directions of OP_1 and OP_2.

$$\mathscr{R}(\uparrow) \qquad N_1\cos\alpha - \mu N_1 \sin\alpha + N_2\sin\alpha + \mu N_2\cos\alpha - mg = 0 \qquad (1)$$

$$\mathscr{R}(\rightarrow) \qquad N_1\sin\alpha + \mu N_1\cos\alpha - N_2\cos\alpha + \mu N_2\sin\alpha = 0 \qquad (2)$$

We also need to take moments. There are four obvious places about which we can take moments: O, P_1, P_2 and M. For completeness, we show what happens if we calculate moments about all four points; clearly only one of these is necessary.

$$\mathscr{M}(\widehat{O}) \qquad mg.\ell\sin(45^\circ - \alpha) - F_1.\ell\sqrt{2} - F_2.\ell\sqrt{2} = 0 \qquad (3)$$

$$\mathscr{M}(\widehat{P_1}) \qquad mg.\ell\cos(45^\circ - \alpha) - N_2.\ell\sqrt{2} - F_2.\ell\sqrt{2} = 0 \qquad (4)$$

$$\mathscr{M}(\widehat{P_2}) \qquad N_1.\ell\sqrt{2} - F_1.\ell\sqrt{2} - mg.\ell\cos(45^\circ - \alpha) = 0 \qquad (5)$$

$$\mathscr{M}(\widehat{M}) \qquad N_1.\ell/\sqrt{2} - F_1.\ell/\sqrt{2} - N_2.\ell/\sqrt{2} - F_2.\ell/\sqrt{2} = 0 \qquad (6)$$

Our task is now to eliminate everything to find an expression for $\tan\alpha$ in terms of μ. We can use any one of the equations (3)–(6) to do this, but (6) appears to be the easiest to work with. (With the others, we would have to use a compound angle formula such as $\sin(45^\circ - \alpha) = \sin 45^\circ \cos\alpha - \cos 45^\circ \sin\alpha = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)$.)

Recalling that $F_1 = \mu N_1$ and $F_2 = \mu N_2$, equation (6) then gives us

$$N_1 - \mu N_1 = N_2 + \mu N_2$$

so

$$N_1(1-\mu) = N_2(1+\mu), \qquad (7)$$

or equivalently

$$\frac{N_1}{N_2} = \frac{1+\mu}{1-\mu}. \qquad (8)$$

$$(9)$$

We can also solve equations (1) and (2) simultaneously to get the results

$$N_1 = \frac{mg(\cos\alpha - \mu\sin\alpha)}{1+\mu^2}$$

and

$$N_2 = \frac{mg(\sin\alpha + \mu\cos\alpha)}{1+\mu^2},$$

so

$$\frac{N_1}{N_2} = \frac{\cos\alpha - \mu\sin\alpha}{\sin\alpha + \mu\cos\alpha}.$$

Equating this expression with equation (8) yields

$$\frac{\cos\alpha - \mu\sin\alpha}{\sin\alpha + \mu\cos\alpha} = \frac{1+\mu}{1-\mu}.$$

Dividing the numerator and denominator of the left hand side by $\cos\alpha$ then gives

$$\frac{1-\mu\tan\alpha}{\tan\alpha+\mu} = \frac{1+\mu}{1-\mu}.$$

A simple rearrangement of this then yields our desired result:

$$\tan\alpha = \frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}.$$

(Alternatively, one could use (7) to substitute for N_2 in equation (2), after writing $F_1 = \mu N_1$ and $F_2 = \mu N_2$. Factorising then gives

$$N_1(1+\mu)(\sin\alpha+\mu\cos\alpha) - N_1(1-\mu)(\cos\alpha-\mu\sin\alpha) = 0.$$

On dividing by $N_1\cos\alpha$ and expanding the brackets, we end up with an expression in $\tan\alpha$ which we can again rearrange to reach our desired conclusion.)

Now if $\tan\lambda = \mu$ with $0 < \lambda < 22.5°$, we have (recalling that $\theta = 45° - \alpha$)

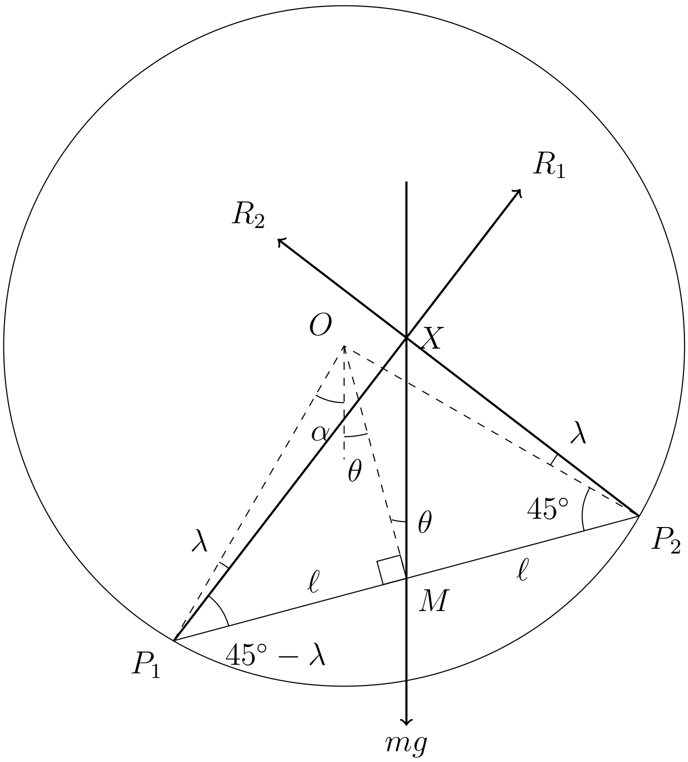
$$\begin{aligned}
\tan\theta &= \tan(45° - \alpha)\\
&= \frac{\tan 45° - \tan\alpha}{1+\tan 45°\tan\alpha}\\
&= \frac{1-\tan\alpha}{1+\tan\alpha} \qquad \text{as } \tan 45° = 1\\
&= \frac{1-\frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}}{1+\frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}}\\
&= \frac{(1+2\mu-\mu^2)-(1-2\mu-\mu^2)}{(1+2\mu-\mu^2)+(1-2\mu-\mu^2)}\\
&= \frac{4\mu}{2-2\mu^2}\\
&= \frac{2\mu}{1-\mu^2}\\
&= \frac{2\tan\lambda}{1-\tan^2\lambda}\\
&= \tan 2\lambda.
\end{aligned}$$

Then since $0 < \theta < 45°$, it follows that $\theta = 2\lambda$ as required.

Method 2: Three forces on a large body theorem

We recall the theorem that if three forces act on a large body in equilibrium, and they are not all parallel, then they are concurrent (i.e., they all pass through a single point). We combine the normal reaction, N, and friction, F, at each end of the rod into a single reaction force, R. This acts at an angle ϕ to the normal, where $\tan\phi = F/N$. In our case, $F = \mu N$ as the rod is on the point of slipping, so $\tan\phi = \mu$. But the question defines λ to be the acute angle such that $\tan\lambda = \mu$, from which it follows that $\phi = \mu$.

We now redraw the diagram showing only the total reaction forces, which we call R_1 and R_2 in this context, and we ensure that R_1, R_2 and the weight mg pass through a single point X.



The rest of the question is now pure trigonometry. We apply the sine rule to the triangles P_1MX and P_2MX, first noting that

$$\angle MXP_1 = 180° - (45° - \lambda) - (90° + \lambda) = 45° - \theta + \lambda$$

and

$$\angle MXP_2 = 180° - (90° - \theta) - (45° + \lambda) = 45° + \theta - \lambda$$

to get

$$\frac{\ell}{\sin(45° - \theta + \lambda)} = \frac{MX}{\sin(45° - \lambda)} \tag{10}$$

$$\frac{\ell}{\sin(45° + \theta - \lambda)} = \frac{MX}{\sin(45° + \lambda)}\,. \tag{11}$$

Then using the identity $\sin(90° - x) = \cos x$ twice, once with $x = 45° - \theta + \lambda$ and then with $x = 45° + \lambda$, we deduce from equation (11) that

$$\frac{\ell}{\cos(45° - \theta + \lambda)} = \frac{MX}{\cos(45° - \lambda)}\,. \tag{12}$$

We can now divide equation (12) by (10) to get

$$\tan(45° - \theta + \lambda) = \tan(45° - \lambda). \tag{13}$$

It follows immediately that $45° - \theta + \lambda = 45° - \lambda$ as both angles are acute, so $\theta = 2\lambda$, which answers the second part of the question.

This result then leads us to conclude that

$$\tan\theta = \tan 2\lambda = \frac{2\tan\lambda}{1 - \tan^2\lambda} = \frac{2\mu}{1 - \mu^2}.$$

Finally, as we know that $\alpha = 45^\circ - \theta$, we can deduce that

$$\begin{aligned}
\tan\alpha &= \tan(45^\circ - \theta) \\
&= \frac{\tan 45^\circ - \tan\theta}{1 + \tan 45^\circ \tan\theta} \\
&= \frac{1 - \tan\theta}{1 + \tan\theta} \qquad \text{as } \tan 45^\circ = 1 \\
&= \frac{1 - \frac{2\mu}{1-\mu^2}}{1 + \frac{2\mu}{1-\mu^2}} \\
&= \frac{(1-\mu^2) - 2\mu}{(1-\mu^2) + 2\mu} \\
&= \frac{1 - 2\mu - \mu^2}{1 + 2\mu - \mu^2}\,,
\end{aligned}$$

and we are done.

Question 12

In this question, you may use without proof the results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1) \qquad \textit{and} \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

The independent random variables X_1 and X_2 each take values 1, 2, …, N, each value being equally likely. The random variable X is defined by

$$X = \begin{cases} X_1 & \textit{if } X_1 \geqslant X_2 \\ X_2 & \textit{if } X_2 \geqslant X_1. \end{cases}$$

(i) *Show that* $\mathrm{P}(X = r) = \dfrac{2r-1}{N^2}$ *for* $r = 1, 2, \ldots, N$.

We have $X = r$ when either $X_1 = r$ and $X_2 < r$, or $X_2 = r$ and $X_1 < r$ or $X_1 = X_2 = r$. Therefore

$$\begin{aligned} \mathrm{P}(X = r) &= \mathrm{P}(X_1 = r \cap X_2 < r) + \mathrm{P}(X_2 = r \cap X_1 < r) + \mathrm{P}(X_1 = X_2 = r) \\ &= \frac{1}{N}.\frac{r-1}{N} + \frac{1}{N}.\frac{r-1}{N} + \frac{1}{N}.\frac{1}{N} \\ &= \frac{2r-1}{N^2}\,. \end{aligned}$$

Alternatively, one can argue as follows:

$$\begin{aligned} \mathrm{P}(X = r) &= \mathrm{P}(X_1 = r \cap X_2 \leqslant r) + \mathrm{P}(X_2 = r \cap X_1 \leqslant r) - \mathrm{P}(X_1 = X_2 = r) \\ &= \frac{1}{N}.\frac{r}{N} + \frac{1}{N}.\frac{r}{N} - \frac{1}{N}.\frac{1}{N} \\ &= \frac{2r-1}{N^2}\,. \end{aligned}$$

(ii) *Find an expression for the expectation, μ, of X and show that $\mu = 67.165$ in the case $N = 100$.*

By the definition of expectation, we have

$$\begin{aligned}\mu = \mathrm{E}(X) &= \sum_{r=1}^{N} r.\mathrm{P}(X=r)\\ &= \sum_{r=1}^{N} \frac{r(2r-1)}{N^2}\\ &= \frac{1}{N^2}\sum_{r=1}^{N}(2r^2-r)\\ &= \frac{1}{N^2}\big(\tfrac{2}{6}N(N+1)(2N+1)-\tfrac12 N(N+1)\big) \quad \text{using the given results}\\ &= \frac{\frac16 N(N+1)(4N+2-3)}{N^2}\\ &= \frac{(N+1)(4N-1)}{6N}\,.\end{aligned}$$

In the case $N=100$, this is $101\times 399/600 = 13\,433/200 = 67.165$ as required.

> ***(iii)*** *The median, m, of X is defined to be the integer such that $\mathrm{P}(X \geqslant m) \geqslant \frac12$ and $\mathrm{P}(X \leqslant m) \geqslant \frac12$. Find an expression for m in terms of N and give an explicit value for m in the case $N=100$.*

We have

$$\begin{aligned}\mathrm{P}(X\leqslant k) &= \sum_{r=1}^{k}\frac{2r-1}{N^2}\\ &= \frac{1}{N^2}\big(2.\tfrac12 k(k+1)-k\big)\\ &= \frac{k^2}{N^2}\end{aligned}$$

so that

$$\mathrm{P}(X\geqslant k) = 1-\mathrm{P}(X\leqslant k-1) = 1-\frac{(k-1)^2}{N^2}\,.$$

We are looking for the value of m which makes $\mathrm{P}(X\geqslant m)\geqslant\frac12$ and $\mathrm{P}(X\leqslant m)\geqslant\frac12$. The first condition gives

$$\begin{aligned} & 1-\frac{(m-1)^2}{N^2} \geqslant \tfrac12\\ \iff\quad & \frac{(m-1)^2}{N^2}\leqslant\tfrac12\\ \iff\quad & (m-1)^2\leqslant\tfrac12 N^2\\ \iff\quad & m-1\leqslant\tfrac{N}{\sqrt2}\\ \iff\quad & m\leqslant\tfrac{N}{\sqrt2}+1.\end{aligned}$$

The second condition, $\mathrm{P}(X \leqslant m) \geqslant \frac{1}{2}$, yields

$$\begin{aligned} & \frac{m^2}{N^2} \geqslant \tfrac{1}{2} \\ \iff \quad & m^2 \geqslant \tfrac{1}{2}N^2 \\ \iff \quad & m \geqslant \tfrac{N}{\sqrt{2}}\,. \end{aligned}$$

So we have $N/\sqrt{2} \leqslant m \leqslant (N/\sqrt{2}) + 1$, thus m is the smallest integer greater than $N/\sqrt{2}$ (which is not itself an integer as $\sqrt{2}$ is irrational). The smallest integer greater than or equal to a number x is called the *ceiling* of x, and is denoted by $\lceil x \rceil$, so we can state our result as $m = \lceil N/\sqrt{2} \rceil$.

In the case $N = 100$, $m = \lceil 100/\sqrt{2} \rceil = \lceil 70.7\ldots \rceil = 71$.

(iv) *Show that when N is very large,*

$$\frac{\mu}{m} \approx \frac{2\sqrt{2}}{3}\,.$$

We have formulæ for μ from part (ii) and for m from part (iii). Therefore we have, for large N,

$$\begin{aligned} \frac{\mu}{m} &= \frac{(N+1)(4N-1)}{6N} \Bigg/ \lceil N/\sqrt{2} \rceil \\ &\approx \frac{(N+1)(4N-1)}{6N(N/\sqrt{2})} \\ &= \frac{4N^2 + 3N - 1}{6N^2/\sqrt{2}} \\ &= \frac{4 + \frac{3}{N} - \frac{1}{N^2}}{6/\sqrt{2}} \\ &\approx \frac{4}{6/\sqrt{2}} \\ &= \frac{2\sqrt{2}}{3}\,. \end{aligned}$$

Question 13

Three married couples sit down at a round table at which there are six chairs. All of the possible seating arrangements of the six people are equally likely.

(i) *Show that the probability that each husband sits next to his wife is $\frac{2}{15}$.*

We call the couples H_1 and W_1, H_2 and W_2, H_3 and W_3. We seat H_1 arbitrarily, leaving $5! = 120$ ways of seating the remaining five people. If each husband sits next to his wife, then there are two seats in which W_1 can sit, each of which leaves four consecutive seats for the other two couples.

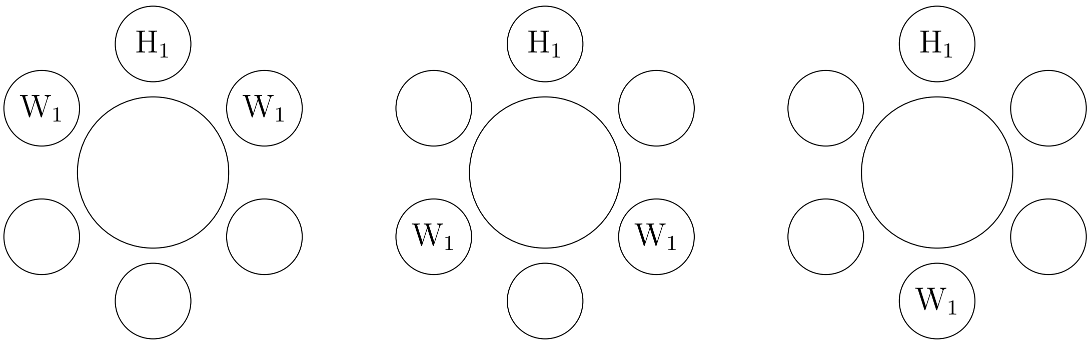
There are four choices for who to sit immediately next to W_1, and that forces the following seat as well (being the spouse of that person).

Next, there are two choices for who to seat the other side of H_1, and the final person must sit next to their spouse.

So there are $2 \times 4 \times 2 = 16$ ways to have all of the husbands sitting next to their wives, with a probability of $\frac{16}{120} = \frac{2}{15}$.

(ii) *Find the probability that exactly two husbands sit next to their wives.*

Assume to begin with that H_1 and W_1 are separated and the other two husbands are seated next to their wives. Once H_1 has been seated, there are five possible positions for W_1, as shown in the following diagrams (there are two possibilities shown in each of the first two):



Clearly the first two possibilities do not work: in the first, H_1 and W_1 are next to each other. In the second, whoever sits between H_1 and W_1 will be separated from their spouse. So W_1 must sit opposite H_1, one couple sits to the right of H_1 and the other couple to his left. There are two choices for which couple sits to the right of H_1, and two choices for whether the husband or wife sits next to H_1; similarly there are two choices for whether the husband or wife of the third couple sits next to H_1. So in all, there are $2 \times 2 \times 2 = 8$ ways to seat the couples with H_1 and W_1 separated and the other couples together.

Similarly, there are 8 ways with couple 2 separated and 8 ways with couple 3 separated, so there are $3 \times 8 = 24$ ways in total.

Thus the probability is $\frac{24}{120} = \frac{3}{15} = \frac{1}{5}$.

(iii) *Find the probability that no husband sits next to his wife.*

Method 1: First find the probability of exactly one husband sitting next to his wife.

Let us assume that H_3 and W_3 are the only pair next to each other. Then in the above diagrams, the left hand one fails as H_1 and W_1 are together. The right hand one also fails, as if H_3 and W_3 are together, H_2 and W_2 must also be. So the only valid configuration is the middle one, with either H_2 or W_2 between H_1 and W_1 and the other partner on the other side of either H_1 or W_1.

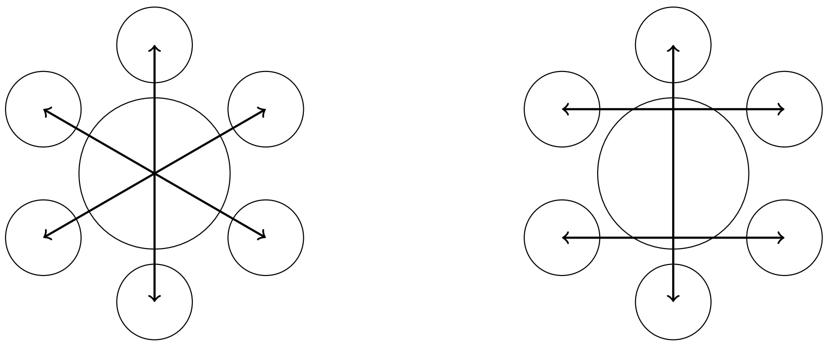
There are two choices for where W_1 will sit, two choices for which of H_2 or W_2 will sit between H_1 and W_1, two choices for where the other partner will sit, and two choices for which way round H_3 and W_3 will sit, giving $2 \times 2 \times 2 \times 2 = 16$ possibilities.

Finally, we need to multiply this by three, as any one of the three couples could be the adjacent one, giving $3 \times 16 = 48$ possibilities, and hence a probability of $\frac{48}{120} = \frac{2}{5} = \frac{6}{15}$.

Thus the probability that no husband sits next to his wife is $1 - \frac{2}{15} - \frac{3}{15} - \frac{6}{15} = \frac{4}{15}$.

Method 2: Find the probability directly.

If no husband sits next to his wife, there are two possible configurations as shown in these diagrams (the arrows join husbands with their wives):



The number of ways of arranging the first case (fixing H_1 at the top as usual) is $2 \times 2 \times 2 = 8$, as their are two ways of choosing which couple sits in which of the diagonal pairs of seats, two ways of couple 2 sitting and two ways of couple 3 sitting.

For the second case, which is no longer totally symmetrical between the three couples, if H_1 sits in the top seat, there are again 8 ways of seating the other two couples. As there are three choices for which couple sits opposite each other, there are $3 \times 8 = 24$ ways in all.

Thus in total there are $8 + 24 = 32$ ways, giving a total probability of $\frac{32}{120} = \frac{4}{15}$.

STEP II, Solutions
June 2008

STEP Mathematics II 2008: Solutions

1 (i) Given $(x_{n+1}, y_{n+1}) = (x_n^2 - y_n^2 + 1,\ 2x_n y_n + 1)$, it is easier to remove the subscripts and set $x^2 - y^2 + 1 = x$ and $2xy + 1 = y$. Then, identifying the y's (or x's) in each case, gives $y^2 = x^2 - x + 1$ and $y = \dfrac{1}{1-2x}$. Eliminating the y's leads to a polynomial equation in x; namely, $4x^4 - 8x^3 + 9x^2 - 5x = 0$.

Noting the obvious factor of x, and then finding a second linear factor (e.g. by the factor theorem) leads to $x(x-1)(4x^2 - 4x + 5) = 0$. Here, the quadratic factor has no real roots, since the discriminant, $\Delta = 4^2 - 4.4.5 = -64 < 0$. [Alternatively, one could note that $4x^2 - 4x + 5 \equiv (2x-1)^2 + 4 > 0\ \forall x$.]

The two values of x, and the corresponding values of y, gained by substituting these x's into $y = \dfrac{1}{1-2x}$, are then $(x, y) = (0, 1)$ and $(1, -1)$

(ii) Now $(x_1, y_1) = (-1, 1) \Rightarrow (x_2, y_2) = (a, b)$ and $(x_3, y_3) = (a^2 - b^2 + a,\ 2ab + b + 2)$. Setting both $a^2 - b^2 + a = -1$ and $2ab + b + 2 = 1$, so that the third term is equal to the first, and identifying the b's in each case, gives $b^2 = a^2 + a + 1$ and $b = \dfrac{-1}{1+2a}$.

One could go about this the long way, as before. However, it can be noted that the algebra is the same as in (i), but with $a = -x$ and $b = -y$. Either way, we obtain the two possible solution-pairs: $(a, b) = (0, -1)$ and $(-1, 1)$.

However, upon checking, the solution $(-1, 1)$ actually gives rise to a constant sequence (and remember that the working only required the third term to be the same as the first, which doesn't preclude the possibility that it is also the same as the second term!), so we find that there is in fact just the one solution: $(a, b) = (0, -1)$.

2 The correct partial fraction form for the given algebraic fraction is

$$\frac{1+x}{(1-x)^2(1+x^2)} \equiv \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{Cx+D}{1+x^2},$$

although these *can* also be put together in other correct ways that don't materially hinder the progress of the solution. The standard procedure now is to multiply throughout by the denominator of the LHS and compare coefficients or substitute in suitable values: which leads to $A = \frac{1}{2}$, $B = 1$, $C = \frac{1}{2}$ and $D = -\frac{1}{2}$.

In order to apply the binomial theorem to these separate fractions, we now use index notation to turn

$$\frac{1+x}{(1-x)^2(1+x^2)} \equiv A(1-x)^{-1} + B(1-x)^{-2} + Cx(1+x^2)^{-1} + D(1+x^2)^{-1}$$

into the infinite series

$$\tfrac{1}{2}\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(n+1)x^n + \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n+1} - \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

It should be clear at this point that the last two of these series have odd/even powers only, with alternating signs playing an extra part. The consequence of all this is that we need to examine cases for n modulo 4; i.e. depending upon whether n leaves a remainder of 0, 1, 2 or 3 when divided by 4.

For $n \equiv 0 \pmod 4$, the coefft. of x^n is $\frac{1}{2} + n + 1 + 0 - \frac{1}{2} = n + 1$;

A1 for $n \equiv 1 \pmod 4$, coefft. of x^n is $\frac{1}{2} + n + 1 + \frac{1}{2} - 0 = n + 2$;

A1 for $n \equiv 2 \pmod 4$, coefft. of x^n is $\frac{1}{2} + n + 1 + 0 + \frac{1}{2} = n + 2$;

A1 for $n \equiv 3 \pmod 4$, coefft. of x^n is $\frac{1}{2} + n + 1 - \frac{1}{2} + 0 = n + 1$.

For the very final part of the question, we note that $\dfrac{11000}{8181} = \dfrac{1.1}{0.9^2 \times 1.01}$, is a cancelled form of our original expression, with $x = 0.1$. (N.B. $|x| < 1$ assures the convergence of the infinite series forms). Substituting this value of x into

$$1 + 3x + 4x^2 + 4x^3 + 5x^4 + 7x^5 + 8x^6 + 8x^7 + 9x^8 + \ldots$$

then gives 1.344 578 90 to 8dp.

3 (i) Setting $\dfrac{dy}{dx} = 81x^2 - 54x = 0$ for TPs gives $(0 , 4)$ and $\left(\frac{2}{3}, 0\right)$. You really ought to know the shape of such a ("positive") cubic, and it is customary to find the crossing-points on the axes: $x = 0$ gives $y = 4$, and $y = 0$ leads to $x = -1$ and $x = \frac{2}{3}$ (twice). [If you have been paying attention, this latter zero for y should come as no surprise!] The graph now shows that, for all $x \geq 0$, $y \geq 0$; which leads to the required result – $x^2(1 - x) \leq \frac{4}{27}$ – with just a little bit of re-arrangement.

In order to prove the result by contradiction (*reduction ad absurdum*), we first assume that all three numbers exceed $\frac{4}{27}$. Then their product

$$bc(1 - a)ca(1 - b)ab(1 - c) > (\tfrac{4}{27})^3.$$

However, this product can be re-written in the form

$$a^2(1 - a).\ b^2(1 - b.)\ c^2(1 - c),$$

and the previous result guarantees that $x^2(1 - x) \leq \frac{4}{27}$ for each of a , b , c, from which it follows that

$$a^2(1 - a).\ b^2(1 - b.)\ c^2(1 - c) \leq (\tfrac{4}{27})^3,$$

which is the required contradiction. Hence, at least one of the three numbers $bc(1 - a)$, $ca(1 - b)$, $ab(1 - c)$ is less than, or equal to, $\frac{4}{27}$.

(ii) Drawing the graph of $y = x - x^2$ (there are, of course, other suitable choices, such as $y = (2x - 1)^2$ for example) and showing that it has a maximum at $(\frac{1}{2}, \frac{1}{4})$ gives

$$x(1 - x) \leq \tfrac{1}{4} \text{ for all } x.$$

The assumption that $p(1 - q)$, $q(1 - p) > \frac{1}{4} \Rightarrow p(1 - p).q(1 - q) > (\frac{1}{4})^2$.

However, we know that $x(1 - x) \leq \frac{1}{4}$ for each of p and q, and this gives us that

$$p(1 - p).q(1 - q) \leq (\tfrac{1}{4})^2.$$

Hence, by contradiction, at least one of $p(1 - q)$, $q(1 - p) \leq \frac{1}{4}$.

4 Differentiating implicitly gives $2\left(x + y\frac{\mathrm{d}y}{\mathrm{d}x} + ax\frac{\mathrm{d}y}{\mathrm{d}x} + ay\right) = 0$, from which it follows that

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x+ay}{ax+y}$ and hence the gradient of the normal is $\frac{ax+y}{x+ay}$.

Using $\tan(A - B)$ on this and $\frac{y}{x}$ gives $\tan\theta = \left|\frac{\frac{y}{x} - \frac{ax+y}{x+ay}}{1 + \frac{y}{x}\times\frac{ax+y}{x+ay}}\right| = \left|\frac{xy + ay^2 - ax^2 - xy}{x^2 + axy + axy + y^2}\right|$.

However, we know that $x^2 + y^2 + 2axy = 1$ from the curve's eqn., and so

$$\tan\theta = a\left|y^2 - x^2\right|.$$

(i) Differentiating this w.r.t. x then gives $\sec^2\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}x} = a\left(2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\right)$. Equating this to zero and using $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x+ay}{ax+y}$ from earlier then leads to $a(x^2 + y^2) + 2xy = 0$.

(ii) Adding $x^2 + y^2 + 2axy = 1$ and $a(x^2 + y^2) + 2xy = 0$ gives $(1 + a)(x + y)^2 = 1$.

(iii) However, subtracting these two eqns. instead gives $(1 - a)(y - x)^2 = 1$, and multiplying these two last results together yields $(1 - a^2)(y^2 - x^2)^2 = 1$.

Finally, using $\tan\theta = a\left|y^2 - x^2\right| \Rightarrow (y^2 - x^2)^2 = \frac{1}{a^2}\tan^2\theta$, and substituting this into the last result of (iii) then gives the required result: $\tan\theta = \frac{a}{\sqrt{1-a^2}}$. All that remains is to justify taking the positive square root, since $\tan\theta$ is | something |, which is necessarily non-negative.

5 Using a well-known double-angle formula gives $\int_0^{\pi/2} \frac{\sin 2x}{1+\sin^2 x}\,\mathrm{d}x = \int_0^{\pi/2} \frac{2\sin x\cos x}{1+\sin^2 x}\,\mathrm{d}x$, and this should suggest an obvious substitution: letting $s = \sin x$ turns this into the integral

$$\int_0^1 \frac{2s}{1+s^2}\,\mathrm{d}s.$$

This is just a standard log. integral (the numerator being the derivative of the denominator), leading to the answer $\ln 2$.

Alternatively, one could use the identity $\sin^2 x \equiv \frac{1}{2} - \frac{1}{2}\cos 2x$ to end up with

$$\int_0^{\pi/2} \frac{2\sin 2x}{3 - \cos 2x}\,\mathrm{d}x.$$

This, again, gives a log. integral, but without the substitution.

A suitable substitution for the second integral is $c = \cos x$, which leads to $\int_0^1 \frac{1}{2-c^2}\,\mathrm{d}c$.

Now you can either attack this using partial fractions, or you could look up what is a fairly standard result in your formula booklet. In each case, you get (after a bit of careful log and surd work) $\frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right)$.

Now $(1 + \sqrt{2})^5 = 1 + 5\sqrt{2} + 20 + 20\sqrt{2} + 20 + 4\sqrt{2} = 41 + 29\sqrt{2}$ (using the binomial theorem, for instance), and

$$41 + 29\sqrt{2} < 99 \Leftrightarrow 29\sqrt{2} < 58 \Leftrightarrow \sqrt{2} < 2 ,$$

which is obviously the case. Also, $1.96 < 2 \Rightarrow 1.4 < \sqrt{2}$. Thereafter, an argument such as

$$2^{1.4} > 1 + \sqrt{2} \Leftrightarrow 2^7 > (1 + \sqrt{2})^5 \Leftrightarrow 128 > 41 + 29\sqrt{2}$$

$$\Leftrightarrow 87 > 29\sqrt{2} \Leftrightarrow 3 > \sqrt{2}$$

from which it follows that $2^{\sqrt{2}} > 2^{7/5} > 1+\sqrt{2}$.

Taking logs in this result then gives $\sqrt{2}\ \ln 2 > \ln(1 + \sqrt{2}) \Rightarrow \ln 2 > \frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right)$; and

$$\int_0^{\pi/2} \frac{\sin 2x}{1+\sin^2 x}\,dx > \int_0^{\pi/2} \frac{\sin x}{1+\sin^2 x}\,dx .$$

6 (i) Firstly, $\cos x$ has period $2\pi \Rightarrow \cos(2x)$ has period π;

and $\sin x$ has period $2\pi \Rightarrow \sin\left(\frac{3x}{2}\right)$ has period $\frac{4}{3}\pi$.

Then $f(x) = \cos\left(2x+\frac{\pi}{3}\right) + \sin\left(\frac{3x}{2}-\frac{\pi}{4}\right)$ has period $4\pi = \text{lcm}(\pi, \frac{4}{3}\pi)$.

(ii) Any approach here is going to require the use of some trig. identity work. The most straightforward is to note that $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$ so that $f(x) = 0$ reduces to

$\cos\left(2x+\frac{\pi}{3}\right) = \cos\left(\frac{3x}{2}+\frac{\pi}{4}\right)$, from which it follows that $2x+\frac{\pi}{3} = 2n\pi \pm \left(\frac{3x}{2}+\frac{\pi}{4}\right)$

where n is an integer, using the symmetric and periodic properties of the cosine curve. Taking suitable values of n, so that x is in the required interval, leads to the answers

$x = -\frac{31\pi}{42}$ (from $n = -1$, with the – sign), $x = -\frac{\pi}{6}$ ($n = 0$, with both + and – signs),

$x = \frac{17\pi}{42}$ ($n = 1$, – sign) and $x = \frac{41\pi}{42}$ ($n = 2$, – sign).

Since $x = -\frac{\pi}{6}$ is a repeated root (occurring twice in the above list), the curve of $y = f(x)$ touches the x-axis at this point.

For those who are aware of the results that appear in all the formula books, but which seem to be on the edge of the various syllabuses, that I know by the title of the *Sum-and-Product Formulae*, such as $\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$, there is a second straightforward approach available here. For example, noting that

$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ gives $\cos\left(2x+\frac{\pi}{3}\right) + \cos\left(\frac{3\pi}{4}-\frac{3x}{2}\right) = 0$ which (from the above identity) then gives $2\cos\left(\frac{x}{4}+\frac{13\pi}{24}\right)\cos\left(\frac{7x}{4}-\frac{5\pi}{24}\right)$ = 0, and setting each of these two cosine terms equal to zero, in turn, yields the same values of x as before, including the repeat.

(iii) The key observation here is that $y = 2$ if and only if *both* $\cos\left(2x+\frac{\pi}{3}\right) = 1$ *and* $\sin\left(\frac{3x}{2}-\frac{\pi}{4}\right) = 1$, simultaneously. So we must solve

$\cos\left(2x+\frac{\pi}{3}\right) = 1 \Rightarrow 2x+\frac{\pi}{3} = 0, 2\pi, 4\pi, \ldots$, giving $x = \frac{5\pi}{6}, \frac{11\pi}{6}, \ldots$; and

$\sin\left(\frac{3x}{2}-\frac{\pi}{4}\right) = 1 \Rightarrow \frac{3x}{2}-\frac{\pi}{4} = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$, giving $x = \frac{\pi}{2}, \frac{11\pi}{6}, \ldots$.

Both equations are satisfied when $x = \frac{11\pi}{6}$, and this is the required answer.

7 (i) Differentiating $y = u\sqrt{1+x^2}$ gives $\frac{dy}{dx} = u.\frac{x}{\sqrt{1+x^2}} + \sqrt{1+x^2}.\frac{du}{dx}$; so that

$\frac{1}{y}\frac{dy}{dx} = xy + \frac{x}{1+x^2}$ becomes $\frac{1}{u\sqrt{1+x^2}}\left\{\frac{ux}{\sqrt{1+x^2}} + \sqrt{1+x^2}.\frac{du}{dx}\right\} = xu\sqrt{1+x^2} + \frac{x}{1+x^2}$.

Simplifying and cancelling the common term on both sides leads to

$$\frac{1}{u}.\frac{du}{dx} = xu\sqrt{1+x^2}.$$

This is a standard form for a first-order differential equation, involving the separation of variables and integration:

$$\int \frac{1}{u^2}.du = \int x\sqrt{1+x^2}\,dx \Rightarrow -\frac{1}{u} = \frac{1}{3}\left(1+x^2\right)^{3/2} \ (+C).$$

Using $x = 0$, $y = 1$ ($u = 1$) to find C leads to the final answer, $y = \frac{3\sqrt{1+x^2}}{4-\left(1+x^2\right)^{3/2}}$.

(ii) The key here is to choose the appropriate function of x. If you have really got a feel for what has happened in the previous bit of the question, then this isn't too demanding. If you haven't really grasped fully what's going on then you may well need to try one or two possibilities first. The product that needs to be identified here is $y = u\left(1+x^3\right)^{1/3}$. Once you have found this, the process of (i) pretty much repeats itself.

$\frac{dy}{dx} = u.x^2\left(1+x^3\right)^{-2/3} + \left(1+x^3\right)^{1/3}\frac{du}{dx}$ means that $\frac{1}{y}\frac{dy}{dx} = x^2y + \frac{x^2}{1+x^3}$ becomes

$$\frac{1}{u}.\frac{du}{dx} = x^2u\left(1+x^3\right)^{1/3}.$$

Separating variables and integrating:

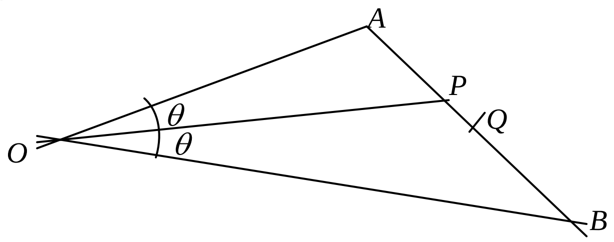
$$\int \frac{1}{u^2}.du = \int x^2\left(1+x^3\right)^{1/3}dx = -\frac{1}{u} = \frac{1}{4}\left(1+x^3\right)^{4/3} \ (+C);$$

and $x = 0$, $y = 1$ ($u = 1$) gives C and the answer $y = \dfrac{4\left(1+x^3\right)^{1/3}}{5-\left(1+x^3\right)^{4/3}}$.

(iii) Note that the question didn't actually require you to simplify the two answers in (i) and (ii), but doing so certainly enables you to have a better idea as to how to generalise the results:

$$y = \frac{(n+1)\left(1+x^n\right)^{1/n}}{(n+2)-\left(1+x^n\right)^{1+1/n}} .$$

8 It is never a bad idea to start this sort of question with a reasonably accurate diagram … something along the lines of



The first result is an example of what is known as the *Ratio Theorem*:

$$AP : PB = 1 - \lambda : \lambda \Rightarrow \mathbf{p} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b} .$$

Alternatively, it can be deduced from the standard approach to the vector equation of a straight line, via $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$.

Using the scalar product twice then gives

$$\mathbf{a} \bullet \mathbf{p} = \lambda a^2 + (1-\lambda)(\mathbf{a} \bullet \mathbf{b}) \text{ and } \mathbf{b} \bullet \mathbf{p} = \lambda(\mathbf{a} \bullet \mathbf{b}) + (1-\lambda)\, b^2 .$$

Equating these two expressions for $\cos\theta$, $\dfrac{\mathbf{a} \bullet \mathbf{p}}{ap} = \dfrac{\mathbf{b} \bullet \mathbf{p}}{bp}$, re-arranging and collecting up like terms, then gives $ab\{\lambda(a+b) - b\} = \mathbf{a} \bullet \mathbf{b}\ \{\lambda(a+b) - b\}$. There are two possible consequences to this statement, and *both* of them should be considered. Either $ab = \mathbf{a} \bullet \mathbf{b}$, which gives $\cos 2\theta = 1$, $\theta = 0$, $A = B$ and violates the non-collinearity of O, A & B; or the bracketed factor on each side is zero, which gives

$$\lambda = \frac{b}{a+b} .$$

However, if you know the *Angle Bisector Theorem*, the working is short-circuited quite dramatically:

$$\frac{AP}{PB} = \frac{OA}{OB} \Rightarrow \frac{(1-\lambda)(AB)}{\lambda(AB)} = \frac{a}{b} \Rightarrow b - b\lambda = a\lambda \Rightarrow \lambda = \frac{b}{a+b}.$$

Next, $AQ : QB = \lambda : 1 - \lambda \Rightarrow \mathbf{q} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$.

Then

$$OQ^2 = \mathbf{q} \bullet \mathbf{q} = (1-\lambda)^2 a^2 + \lambda^2 b^2 + 2\lambda(1-\lambda)\ \mathbf{a} \bullet \mathbf{b}$$

and $OP^2 = \mathbf{p} \bullet \mathbf{p} = (1-\lambda)^2 b^2 + \lambda^2 a^2 + 2\lambda(1-\lambda)\ \mathbf{a} \bullet \mathbf{b}$.

[N.B. This working can also be done by the *Cosine Rule*.]

Subtracting:

$$OQ^2 - OP^2 = (b^2 - a^2)\,[\lambda^2 - (1-\lambda)^2] = (b^2 - a^2)\,(2\lambda - 1)$$

and, substituting λ in terms of a and b into this expression, gives the required answer

$$= (b-a)(b+a) \times \frac{b-a}{b+a} = (b-a)^2.$$

9 (i) Using a modified version of the trajectory equation (which you are encouraged to have learnt), $y = h + x\tan\alpha - \dfrac{gx^2}{2u^2}\sec^2\alpha$, and substituting in $g = 10$ and $u = 40$ gives

$$y = h + x\tan\alpha - \frac{gx^2}{320}\sec^2\alpha\ .$$

Setting $x = 20$ and $y = 0$ into this trajectory equation and using one of the well-known *Pythagorean* trig. identities ($\sec^2\alpha = 1 + \tan^2\alpha$) leads to the quadratic equation

$$5t^2 - 80t - (4h - 5) = 0$$

in $t = \tan\alpha$.
[Note that you could have substituted $x = 20$ and $y = -h$ into the unmodified trajectory equation and still got the same result here.]
Solving, using the quadratic formula, and simplifying then gives

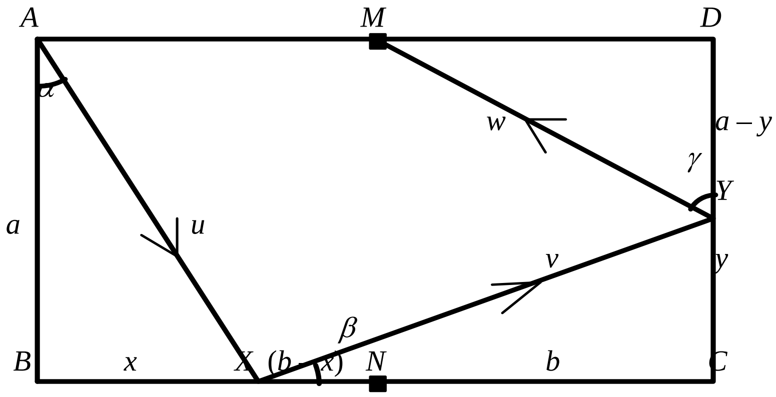
$$\tan\alpha = 8 \pm \sqrt{63 + \tfrac{4}{5}h}\ .$$

We reject $\tan\alpha = 8 + \sqrt{63 + \tfrac{4}{5}h}$, since this gives a very high angle of projection and hence a much greater time for the ball to arrive at the stumps. Now, since α is small, $\cos\alpha \approx 1$, and the time of flight $= \dfrac{x}{u\cos\alpha} = \dfrac{1}{2\cos\alpha} \approx \dfrac{1}{2}$.

(ii) $h > \tfrac{5}{4}$ for $\tan\alpha = 8 - \sqrt{64 + \varepsilon}\ ) < 0$.

(iii) Now $h = 2.5 \Rightarrow \tan\alpha = 8 - \sqrt{64 + 1} = 8 - 8\left(1 + \tfrac{1}{64}\right)^{1/2}$. The *Binomial Theorem* then allows us to expand the bracket, and it seems reasonable to take just the first term past the 1: $\tan\alpha = 8 - 8\left(1 + \tfrac{1}{128} + ...\right)$, so that $\tan\alpha \approx -\tfrac{1}{16}$. [We can ignore the minus sign, since this just tells us that the projection is *below* the horizontal.]
Using $\tan\alpha \approx \alpha$ for small-angles, and converting from radians into degrees using the conversion factor $180/\pi \approx 57.3$ then gives $\alpha \approx 3.6^\circ$.

10 On this sort of question, a good, clear diagram is almost essential, even when it is not asked-for.



(i) The two fundamental principles involved in collisions are the *Conservation of Linear Momentum* (*CLM*) and *Newton's Experimental Law of Restitution* (*NEL* or *NLR*).
For the collision at *X*, applying $CLM \parallel BC \quad \Rightarrow \quad mu\sin\alpha = mv\cos\beta$
.. $NEL \quad \Rightarrow \quad eu\cos\alpha = v\sin\beta$

Dividing these two gives $\tan\beta = e\cot\alpha$ or $\tan\alpha\tan\beta = e$.
At Y, "similarly", we have $\tan\beta\tan\gamma = e$. Hence $\alpha = \gamma$ (since all angles are acute).

(ii) A good approach here is to use similar Δs, and a bit of sensible labelling is in order (see the diagram). Let $BX = x$ ($XN = b - x$) and $CY = y$ ($DY = a - y$). Then

$$\tan\alpha = \frac{x}{a}, \quad \tan\beta = \frac{y}{2b-x}, \quad \tan\gamma = \frac{b}{a-y}.$$

Using $\alpha = \gamma$ to find (e.g) y in terms of a, b, x $\Rightarrow$ $ax - xy = ab$ $\Rightarrow$ $y = \dfrac{a(x-b)}{x}$.

Next, we use the result $\tan\alpha\tan\beta = e$ from earlier to get x in terms of a and b:

$$\frac{x}{a}\times\frac{a(x-b)/x}{2b-x} = e \;\Rightarrow\; x - b = 2be - ex \;\Rightarrow\; x = \frac{b(1+2e)}{1+e},$$

from which it follows that $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)}$.

At this stage, some sort of inequality argument needs to be considered, and a couple of obvious approaches might occur to you.

I $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)} = \dfrac{b}{a} + \dfrac{be}{a(1+e)} > \dfrac{b}{a}$ and $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)} = \dfrac{2b}{a} - \dfrac{be}{a(1+e)} < \dfrac{2b}{a}$

give $\dfrac{b}{a} < \tan\alpha < \dfrac{2b}{a}$; and the shot is possible, with the ball striking BC between N and C, whatever the value of e .

II As $e \to 0$, $\tan\alpha \to \dfrac{b}{a}+$ and as $e \to 1$, $\tan\alpha \to \dfrac{3b}{2a}-$, so that

$\dfrac{b}{a} < \tan\alpha < \dfrac{3b}{2a}$; and the shot is possible, with the ball striking BC between N and the midpoint of NC, whatever the value of e.

(iii) There are two possible approaches to this final part. The first, much longer version, involves squaring and adding the eqns. for the collision at X, and then again at Y, to get

$$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha) \quad\text{and}\quad w^2 = v^2(\sin^2\beta + e^2\cos^2\beta).$$

Now, noting that the initial KE = $\frac{1}{2}mu^2$ and the final KE = $\frac{1}{2}mw^2$, the fraction of KE lost is $\dfrac{\frac{1}{2}mu^2 - \frac{1}{2}mw^2}{\frac{1}{2}mu^2} = 1 - \dfrac{w^2}{u^2} = 1 - (\sin^2\alpha + e^2\cos^2\alpha)(\sin^2\beta + e^2\cos^2\beta)$

$$= 1 - \frac{\tan^2\alpha + e^2}{\sec^2\alpha}\times\frac{\tan^2\beta + e^2}{\sec^2\beta}.$$

From here, we use $\tan\alpha\tan\beta = e$ and $\sec^2\alpha = 1 + \tan^2\alpha$ to get

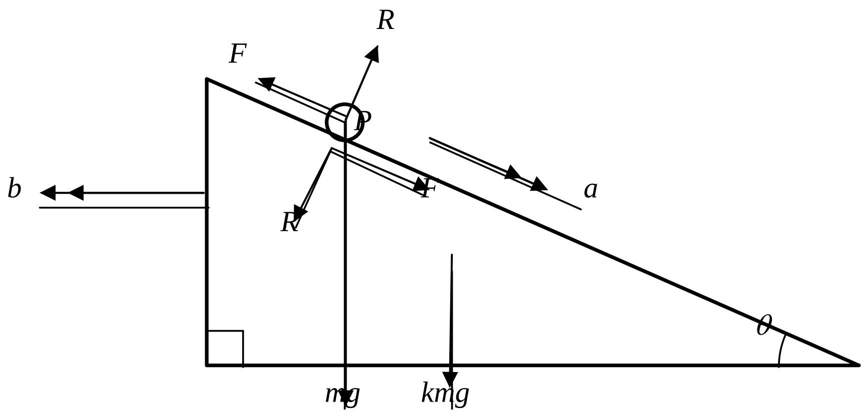
$$1 - \frac{t^2+e^2}{1+t^2}\times\frac{e^2/t^2+e^2}{1+e^2/t^2} = 1 - \frac{t^2+e^2}{1+t^2}\times\frac{e^2\left(1+t^2\right)/t^2}{\left(t^2+e^2\right)/t^2} = 1 - e^2, \text{ as required.}$$

However, it is very much quicker to note the following:

At X, the ↑-component of the ball's velocity becomes $e\times$ initial ↑-component ,
and
at Y, the →-component of the ball's velocity becomes $e\times$ initial →-component.
Hence its final velocity is eu and the fraction of the KE lost is then

$$\frac{\frac{1}{2}mu^2 - \frac{1}{2}me^2u^2}{\frac{1}{2}mu^2} = 1 - e^2.$$

11



Once again, a good, clear diagram is an important starting-point, and the above diagram shows the relevant forces – labelled using standard notations – along with the accelerations of P down the sloping surface of the wedge (a) and the wedge itself along the plane (b).

(i) Noting the acceleration components of P are $a\cos\theta - b$ ($\rightarrow$) and $a\sin\theta$ ($\downarrow$), we employ *Newton's Second Law* as follows:

N2L → for P $\quad m(a\cos\theta - b) = R\sin\theta - F\cos\theta$

N2L ↓ for P $\quad ma\sin\theta = mg - F\sin\theta - R\cos\theta$

N2L ← for wedge $\quad kmb = R\sin\theta - F\cos\theta$

From which it follows that $a\cos\theta - b = kb \Rightarrow b = \dfrac{a\cos\theta}{k+1}$.

Alternatively, one could use *N2L* to note *P*'s → accln. component and also the wedge's accln. ←, but instead use

CLM ↔ $\quad km\,bt = m(a\cos\theta - b)t$ (where t = time from release)

and this again leads to the above result for b.

Now, for P to move at 45° to the horizontal, $a\cos\theta - b = a\sin\theta$. Then

$$b = a(\cos\theta - \sin\theta) = \frac{a\cos\theta}{k+1}$$

$\Rightarrow (k+1)(\cos\theta - \sin\theta) = \cos\theta \Rightarrow k + 1 - (k+1)\tan\theta = 1$ and $\tan\theta = \dfrac{k}{k+1}$.

When $k = 3$, $\tan\theta = \frac{3}{4}$, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$ and $b = \frac{1}{5}a$.

Substituting these into the first two equations of motion from (i), along with the use of the *Friction Law (in motion)*, which assumes that $F = \mu R$, gives

$$m(\tfrac{4}{5}a - b) = \tfrac{3}{5}R - \tfrac{4}{5}F \quad\text{or}\quad 3R - 4F = m(4a - 5b) = 3ma \Rightarrow R(3 - 4\mu) = 3ma$$

and

$$\tfrac{3}{5}ma = mg - \tfrac{3}{5}F - \tfrac{4}{5}R \quad\text{or}\quad 4R + 3F = m(5g - 3a) \Rightarrow R(4 + 3\mu) = 5mg - 3ma.$$

Dividing, or equating for R:

$$\frac{4+3\mu}{3-4\mu} = \frac{5g-3a}{3a} \Rightarrow (12 + 9\mu)a = 5(3 - 4\mu)g - (9 - 12\mu)a \Rightarrow a = \frac{5(3-4\mu)g}{3(7-\mu)}.$$

(ii) Finally, if $\tan\theta \le \mu$, then both P and the wedge remain stationary. So, technically, the answer is "nothing".

12 Clearly, $X \in \{0, 1, 2, 3\}$ and working out the corresponding probabilities is a good thing to do at some point in this question (although it can, of course, be done numerically later when a value for p has been found).

$p(X = 0) = (1 - p)(1 - \tfrac{1}{3}p)(1 - p^2)$

$p(X = 1) = p(1 - \tfrac{1}{3}p)(1 - p^2) + (1 - p)\tfrac{1}{3}p(1 - p^2) + (1 - p)(1 - \tfrac{1}{3}p)p^2$

$\quad = p(1 - p)(\tfrac{4}{3} + \tfrac{5}{3}p - p^2)$

$p(X = 2) = p.\tfrac{1}{3}p(1 - p^2) + p(1 - \tfrac{1}{3}p)p^2 + (1 - p)\tfrac{1}{3}p.p^2$

$\quad = \tfrac{1}{3}p^2(1 + 4p - 3p^2)$

$p(X = 3) = \tfrac{1}{3}p^4$

[Of course, one of these could be deduced on a (1 – the sum of the rest) basis, but that can always be left as useful check on the correctness of your working, if you so wish.]

Then $E(X) = \sum x.p(x) = 0 + p(1 - p)(\tfrac{4}{3} + \tfrac{5}{3}p - p^2) + \tfrac{2}{3}p^2(1 + 4p - 3p^2) + p^4$

$\quad = \tfrac{4}{3}p + p^2$

Alternatively, if you have done a little bit of expectation algebra, it is clear that

$$E(X) = \sum E(X_i) = p + \tfrac{1}{3}p + p^2 = \tfrac{4}{3}p + p^2 .$$

Equating this to $\tfrac{4}{3} \Rightarrow 0 = 3p^2 + 4p - 4 \Rightarrow 0 = (3p - 2)(p + 2)$, and since $0 < p < 1$ it follows that $p = \tfrac{2}{3}$.

In the final part, you will need either (p_0 and p_1) or (p_2 and p_3):

$$p_0 = \tfrac{35}{243} \text{ and } p_1 = \tfrac{108}{243} \text{ or } \quad p_2 = \tfrac{84}{243} \text{ and } p_3 = \tfrac{16}{243}.$$

Next, a careful statement of cases is important (with, I hope, obvious notation):

p(correct pronouncement) = p(G and ≥ 2 judges say G) + p(NG and ≤ 1 judges say G)

$$= t . \frac{100}{243} + (1 - t) . \frac{143}{243} = \frac{143 - 43t}{243}$$

Equating this to $\tfrac{1}{2}$ and solving for $t \Rightarrow 243 = 286 - 86t \Rightarrow 86t = 43 \Rightarrow t = \tfrac{1}{2}$.

Alternatively, let p(King pronounces guilty) = q.

Then "King correct" = "King pronounces guilty and defendant *is* guilty"

or "King pronounces not guilty and defendant *is* not guilty"

so that p(King correct) = $qt + (1 - q)(1 - t)$.

Setting $qt + (1 - q)(1 - t) = \tfrac{1}{2} \Leftrightarrow (2q - 1)(2t - 1) = 0$, and since q is not identically equal to $\tfrac{1}{2}$, $t = \tfrac{1}{2}$.

13 (i) p(B in bag P) = p(B not chosen draw 1) + p(B chosen draw 1 and draw 2)

$$= \left(1 - \frac{k}{n}\right) + \frac{k}{n} \times \frac{k}{n + k}$$

$$= \frac{1}{n(n + k)}\left((n - k)(n + k) + k^2\right)$$

$$= \frac{n}{n + k}$$

This has its maximum value of 1 for $k = 0$, and for no other values of k. Since

$p = 1 - \dfrac{k}{n + k} \le 1$ and for $k = 0, p = 1$ but $k > 0$ for all $p < 1$).

(ii) p(Bs in same bag) = p(B_1 chosen on D_1 and neither chosen on D_2)
+ p(B_1 chosen on D_1 and both chosen on D_2)
+ p(B_1 not chosen on D_1 and B_2 chosen on D_2)

$$= \frac{k}{n} \times \frac{{}^{n+k-2}C_k}{{}^{n+k}C_k} + \frac{k}{n} \times \frac{{}^{n+k-2}C_{k-2}}{{}^{n+k}C_k} + \left(1 - \frac{k}{n}\right) \times \frac{k}{n+k}$$

Notice that, although the ${}^nC_r$ terms *look* very clumsy, they are actually quite simple once all the cancelling of common factors has been undertaken.

$$= \frac{k}{n} \times \frac{n(n-1)}{(n+k)(n+k-1)} + \frac{k}{n} \times \frac{k(k-1)}{(n+k)(n+k-1)} + \frac{k(n-k)}{n(n+k)}$$

$$= \frac{k}{n} \left\{ \frac{n^2 - n + k^2 - k + (n^2 + nk - n - nk - k^2 + k)}{(n+k)(n+k-1)} \right\}$$

$$= \frac{2k(n-1)}{(n+k)(n+k-1)}$$

Differentiating this expression gives

$$\frac{dp}{dk} = \frac{(n^2 + 2nk + k^2 - n - k) \times 2(n-1) - 2k(n-1) \times (2n + 2k - 1)}{[(n+k)(n+k-1)]^2}$$

$= 0$ when $n^2 + 2nk + k^2 - n - k = 2nk + 2k^2 - k$ since $n > 2$, $n - 1 \neq 0$

$\Rightarrow k^2 = n(n-1)$.

Now there is nothing that guarantees that k is going to be an integer (quite the contrary, in fact), so we should look to the integers either side of the (positive) square root of $n(n-1)$:

$$k = \left\lfloor \sqrt{n(n-1)} \right\rfloor \quad \text{and} \quad k = \left\lfloor \sqrt{n(n-1)} \right\rfloor + 1 .$$

In fact, since $n^2 - n = (n - ½)^2 - ¼$, $\left\lfloor \sqrt{n^2 - n)} \right\rfloor = n - 1$ and we find that,

when $k = n - 1$, $p = \dfrac{2(n-1)^2}{(2n-1)2(n-1)} = \dfrac{n-1}{2n-1}$

and when $k = n$, $p = \dfrac{2n(n-1)}{(2n)(2n-1)} = \dfrac{n-1}{2n-1}$ also,

and $k = n - 1$, n are the two values required.

STEP III, Solutions
June 2008

STEP Mathematics III 2008: Solutions

1. Following the hint yields

$$ax^2 + by^2 + (a+b)xy = \frac{1}{3}(x+y)$$

which is $\frac{1}{5} + xy = \frac{1}{3}(x+y)$

The same trick applied to the third equation gives $\frac{1}{7} + \frac{1}{3}xy = \frac{1}{5}(x+y)$.

The two equations can be solved simultaneously for *xy* and (*x*+*y*), giving

$$xy = \frac{3}{35} \text{ and } (x+y) = \frac{6}{7}$$

Thus *x* and *y* are the roots of the quadratic equation $35z^2 - 30z + 3 = 0$
(*x* and *y* are interchangeable).
a and *b* are then found by substituting back into two of the original equations and the full solution is

$$x = \frac{3}{7} \pm \frac{2}{35}\sqrt{30} = \frac{3}{7} \pm \frac{2}{7}\sqrt{\frac{6}{5}}$$

$$y = \frac{3}{7} \mp \frac{2}{35}\sqrt{30} = \frac{3}{7} \mp \frac{2}{7}\sqrt{\frac{6}{5}}$$

$$a = \frac{1}{2} \mp \frac{\sqrt{30}}{36} = \frac{1}{2} \mp \frac{1}{6}\sqrt{\frac{5}{6}}$$

$$b = \frac{1}{2} \pm \frac{\sqrt{30}}{36} = \frac{1}{2} \pm \frac{1}{6}\sqrt{\frac{5}{6}}$$

2. (i) On the one hand

$\sum_{r=0}^{n}\left[(r+1)^k - r^k\right] = \sum_{r=0}^{n}(r+1)^k - \sum_{r=0}^{n} r^k = \sum_{r=1}^{n+1} r^k - \sum_{r=0}^{n} r^k = (n+1)^k$ whilst expanding binomially yields

$$k\sum_{r=0}^{n} r^{k-1} + \binom{k}{2}\sum_{r=0}^{n} r^{k-2} + \binom{k}{3}\sum_{r=0}^{n} r^{k-3} + \ldots + \binom{k}{k-1}\sum_{r=0}^{n} r^{k-1} + \sum_{r=0}^{n} 1$$

$$= kS_{k-1}(n) + \binom{k}{2}S_{k-2}(n) + \binom{k}{3}S_{k-3}(n) + \ldots + \binom{k}{k-1}S_1(n) + (n+1)$$

and hence the required result.

Applying this in the case $k = 4$ gives

$$4S_3(n) = (n+1)^4 - (n+1) - \binom{4}{2}S_2(n) - \binom{4}{3}S_1(n)$$

which, after substitution of the two given results and factorization, yields the familiar

$S_3(n) = \frac{1}{4} n^2 (n+1)^2$

The identical process with $k = 5$ results in

$S_4(n) = \frac{1}{30} n(n+1)(6n^3 + 9n^2 + n - 1) = \frac{1}{30} n(n+1)(2n+1)(3n^2 + 3n - 1)$

(ii) Applying induction, with the assumption that $S_t(n)$ is a polynomial of degree $t+1$ in n for $t < r$ for some r, and then considering (*),

$(n+1)^{r+1} - (n+1)$ is a polynomial of degree $r+1$ in n ,

and each of the terms $-\binom{r+1}{j} S_{r+1-j}(n)$ is a polynomial of degree $r+2-j$ in n where $j \geq 2$, i.e. the degree is $\leq r$. A sum of polynomials of degree $\leq r+1$ in n , is a polynomial of degree $\leq r+1$ in n , and there is a single non-zero term in n^{r+1} from just $(n+1)^{r+1}$ so the degree of the polynomial is not reduced to $< r+1$, i.e. it is $r+1$. (The initial case is true to complete the proof.)

If, $S_k(n) = \sum_{i=0}^{k+1} a_i n^i = \sum_{r=0}^{n} r^k$ then $S_k(0) = a_0 + \sum_{i=1}^{k+1} a_i 0^i = \sum_{r=0}^{0} r^k = 0$ and so $a_0 = 0$

$S_k(1) = \sum_{i=0}^{k+1} a_i 1^i = \sum_{r=0}^{1} r^k = 1$ and so $\sum_{i=0}^{k+1} a_i = 1$ as required.

3. $\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta}$

So the line ON is $y = \frac{a\sin\theta}{b\cos\theta} x$

SP is $y = \frac{b\sin\theta}{a(\cos\theta + e)}(x + ae)$

Solving simultaneously by substituting for *x* to find the *y* coordinate of *T*,

$y = \frac{b\sin\theta}{a(\cos\theta + e)}\left(\frac{b\cos\theta}{a\sin\theta} y + ae\right)$

and using $b^2 = a^2(1-e^2)$ to eliminate a^2 gives the required result.

Then the *x* coordinate of *T* is $\frac{b^2\cos\theta}{a(1 + e\cos\theta)}$.

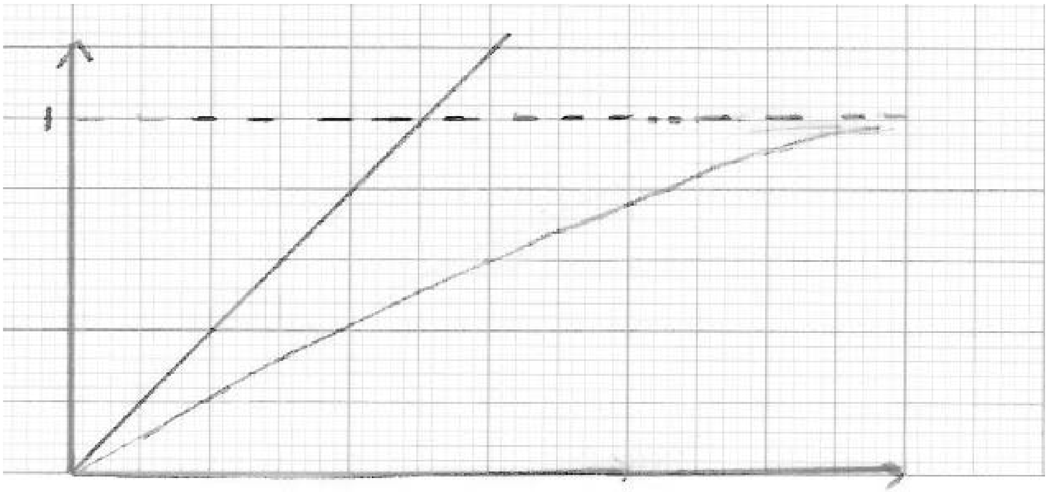
Eliminating θ using $\sec\theta + e = \frac{b^2}{ax}$ and $\tan\theta = \frac{by}{ax}$,

(x, y) satisfies $\left(\frac{b^2}{ax} - e\right)^2 = 1 + \left(\frac{by}{ax}\right)^2$

and again using $b^2 = a^2(1-e^2)$, this time to eliminate b^2, gives, following simplifying algebra

$(x + ae)^2 + y^2 = a^2$, as required.

4. (i)



The graph of $z = y$ has gradient 1 and passes through the origin.

The graph of $z = \tanh\left(\frac{y}{2}\right)$ which has gradient $\frac{1}{2}\operatorname{sec}h^2\left(\frac{y}{2}\right) \le \frac{1}{2}$ for $y \ge 0$ also passes though the origin and is asymptotic to $z = 1$.

Thus $y \ge \tanh\left(\frac{y}{2}\right)$ for $y \ge 0$.

If $x = \cosh y$, then $\sqrt{\frac{x-1}{x+1}} = \sqrt{\frac{\cosh y - 1}{\cosh y + 1}} = \sqrt{\frac{2\sinh^2\left(\frac{y}{2}\right)}{2\cosh^2\left(\frac{y}{2}\right)}} = \tanh\left(\frac{y}{2}\right)$

and as $y \ge \tanh\left(\frac{y}{2}\right)$ for $y \ge 0$, $ar\cosh x \ge \sqrt{\frac{x-1}{x+1}}$ for $x \ge 1$.

$\sqrt{\frac{x-1}{x+1}} = \sqrt{\frac{x-1}{x+1}}\sqrt{\frac{x-1}{x-1}} = \frac{x-1}{\sqrt{x^2-1}}$ for $x > 1$, and (*) is obtained.

(ii) By parts $\int ar\cosh x dx = x ar\cosh x - \sqrt{x^2-1} + c$

and $\int \frac{x-1}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} - ar\cosh x + c'$

Thus $\int_1^x ar\cosh x dx \ge \int_1^x \frac{x-1}{\sqrt{x^2-1}} dx$ for $x > 1$ gives

$x ar\cosh x - \sqrt{x^2-1} \ge \sqrt{x^2-1} - ar\cosh x$ for $x > 1$, which rearranges to give result

(iii) Integrating (ii) similarly gives $x ar\cosh x - \sqrt{x^2-1} \ge 2\left(\sqrt{x^2-1} - ar\cosh x\right)$ for $x > 1$, which also can be rearranged as desired.

5. There are a number of correct routes to proving the induction, though the simplest is to consider $\left(\left(T_{k+1}(x)\right)^2 - T_k(x)T_{k+2}(x)\right) - \left(\left(T_k(x)\right)^2 - T_{k-1}(x)T_{k+1}(x)\right)$

For $f(x) = 0$, $\left(T_n(x)\right)^2 - T_{n-1}(x)T_{n+1}(x) = 0$

and so $\dfrac{T_{n+1}(x)}{T_n(x)} = \dfrac{T_n(x)}{T_{n-1}(x)}$ provided that neither denominator is zero, leading to
$\dfrac{T_n(x)}{T_{n-1}(x)} = \dfrac{T_1(x)}{T_0(x)} = r(x)$,
and so $\dfrac{T_n(x)}{T_{n-1}(x)} \times \dfrac{T_{n-1}(x)}{T_{n-2}(x)} \times \ldots \times \dfrac{T_1(x)}{T_0(x)} = \left(r(x)\right)^n$
Thus $T_n(x) = \left(r(x)\right)^n T_0(x)$

Substituting this result into (*) for $n = 1$,
$\left(\left(r(x)\right)^2 - 2xr(x) + 1\right)T_0(x) = 0$, and as $T_0(x) \neq 0$, solving the quadratic gives
$r(x) = x \pm \sqrt{x^2 - 1}$

6. (i) Differentiating $y = p^2 + 2xp$ with respect to x gives

$p = 2p\dfrac{dp}{dx} + 2x\dfrac{dp}{dx} + 2p$ which can be rearranged suitably.

The differential equation $\dfrac{dx}{dp} + \dfrac{2}{p}x = -2$ has an integrating factor p^2 and integrating will give the required general solution.

Substituting $x = 2, p = -3$, leads to $A = 0$, i.e. $p = -\dfrac{3}{2}x$ which can be substituted in the original equation and so $y = -\dfrac{3}{4}x^2$.

(ii) The same approach as in part (i) generates $\dfrac{dx}{dp} + \dfrac{2}{p}x = -\dfrac{(\ln p + 1)}{p}$,

which with the same integrating factor has general solution

$x = -\dfrac{1}{4} - \dfrac{1}{2}\ln p + Bp^{-2}$

and particular solution

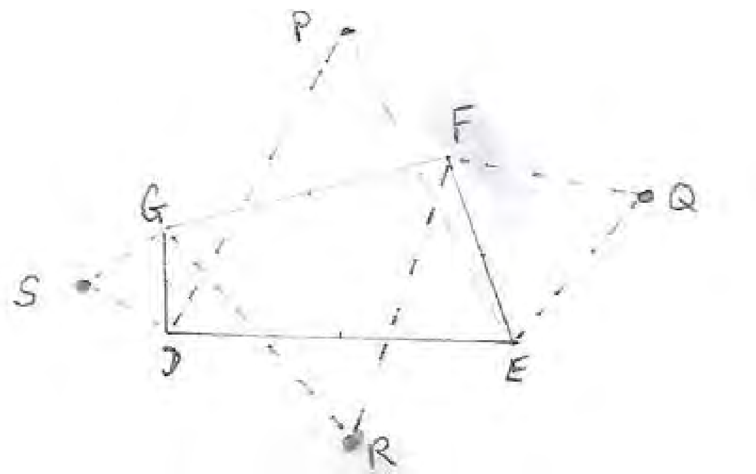
$x = -\dfrac{1}{2}\ln p - \dfrac{1}{4}$

Again, substitution of ln p (and p) in the original equation leads to the solution which is $y = -\dfrac{1}{2}e^{-2x-\frac{1}{2}}$

7. The starting point $c - a = \frac{1}{2}\left(1 + i\sqrt{3}\right)(b - a)$ leads to the given result.

Interchanging a and b gives $2c = (a + b) + i\sqrt{3}(a - b)$ if A, B, C are described clockwise.

(i) The clue to this is the phrase “can be chosen” and a sketch demonstrates that a pair of the equilateral triangles need to be clockwise, and the other pair anti-clockwise



Applying the results in the stem of the question to this configuration,

$$2p = (d + e) + i\sqrt{3}(e - d)$$
$$2q = (e + f) + i\sqrt{3}(e - f)$$
$$2r = (f + g) + i\sqrt{3}(g - f)$$
$$2s = (g + d) + i\sqrt{3}(g - d)$$

and so $2PS = (g - e) + i\sqrt{3}(g - e) = -2RQ$, PSQR is a parallelogram.
(The pairs could have been chosen with opposite parity leading to very similar working.)

(ii) Supposing LMN is clockwise, U is the centroid of equilateral triangle LMH, V of MNJ, and W of NLK, then
$3u = l + m + h$ where $2h = (l + m) + i\sqrt{3}(m - l)$ with similar results for *v* and *w*.

Both $6w$, and $3\left[(u + v) + i\sqrt{3}(u - v)\right]$ can be shown to equal $3(n + l) + i\sqrt{3}(l - n)$ and so UVW is a clockwise equilateral triangle.

8. (i) $p = -\frac{1}{2}$

$\left(1 + px\right)S = \frac{1}{3}x$ with all other terms cancelling and so $S = \frac{1}{3}x \Big/ \left(1 - \frac{1}{2}x\right) = \frac{2x}{3(2 - x)}$

Using the sum of a GP

$$S_{n+1} = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{1}{12}x^3 + \ldots + \frac{1}{3\times 2^n}x^{n+1} = \frac{\frac{1}{3}x\left(1-\frac{x^{n+1}}{2^{n+1}}\right)}{\left(1-\frac{x}{2}\right)}$$

Alternatively $S_{n+1} = S - (a_{n+2}x^{n+2} + \ldots) = S - \frac{1}{2^{n+1}}x^{n+1}S$

$$= \left(1-\frac{x^{n+1}}{2^{n+1}}\right)\frac{2x}{3(2-x)}$$

(ii) Using similar working to part (i)

$18 + 8p + 2q = 0$

$37 + 18p + 8q = 0$

so $p = -\frac{5}{2}, q = 1$

and so $\left(1 + px + qx^2\right)T = 2 + 3x$

giving $T = (2+3x)\Big/\left(1-\frac{5}{2}x + x^2\right) = \frac{4+6x}{2-5x+2x^2} = \frac{4+6x}{(2-x)(1-2x)}$

By partial fractions $T = \frac{14}{3}(1-2x)^{-1} - \frac{8}{3}\left(1-\frac{x}{2}\right)^{-1}$

and so $T_{n+1} = \frac{14}{3}\left(1 + 2x + (2x)^2 + \ldots + (2x)^n\right) - \frac{8}{3}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \ldots + \left(\frac{x}{2}\right)^n\right)$

$$= \frac{14}{3}\frac{\left(1-(2x)^{n+1}\right)}{1-2x} - \frac{8}{3}\frac{\left(1-\left(\frac{x}{2}\right)^{n+1}\right)}{1-\frac{x}{2}}$$

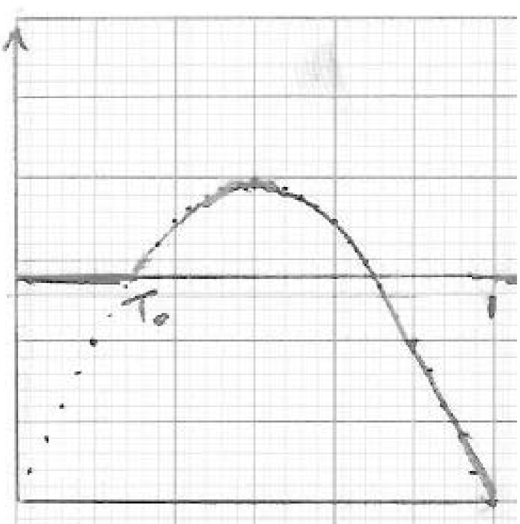
Section B: Mechanics

9. When the particle starts to move, friction is limiting and so

$mg\sin\pi T_0 - \mu mg = 0$

i.e. $\mu = \sin\pi T_0$

(i)

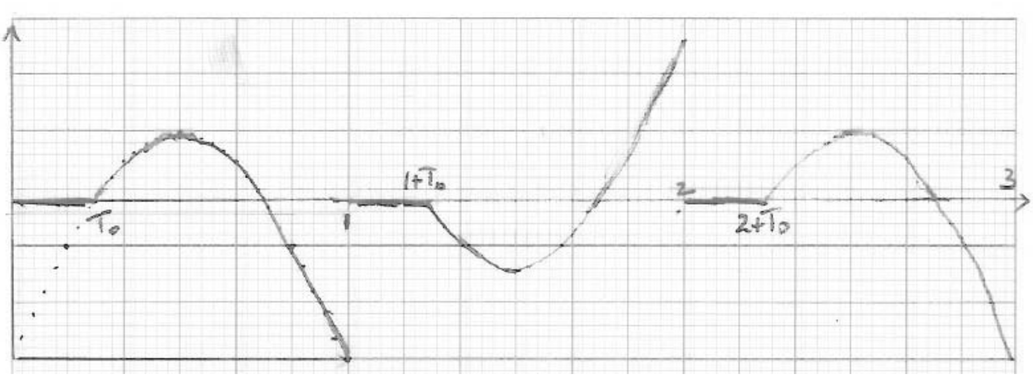


When the particle comes to rest, the area under the acceleration-time graph is zero

i.e. $\int_{T_0}^{1} g \sin \pi t - \mu_o g dt = 0$

Completing the manipulation and eliminating T_0 using the relation at the start of the question renders the required result.

(ii)



In the case $\mu = \mu_0$, the motion is periodic with period 2, the particle is stationary in intervals $(0, T_0), (1, 1 + T_0), (2, 2 + T_0) \ldots$, reversing its direction of motion after times 1, 2, 3, … , and returning to its starting point at time 2 (and 4,6….)

In the case $\mu = 0$, the motion is simple harmonic motion (period 2) superimposed on uniform motion , the particle instantaneously comes to rest at time 2, 4, …but otherwise always moves in the positive x direction.

($x = \frac{g}{\pi^2}(\pi t - \sin \pi t)$)

10. Considering the *r*th short string $T_r = mg + T_{r-1}$

Also we have $T_r = \frac{\lambda x_r}{l}$, and $T_1 = mg$

Thus $T_r = rmg$ and so the total length is given by $\sum_1^n (l + x_r) = \sum_1^n \left(l + \frac{rmgl}{\lambda} \right)$

$$= nl + \frac{mgl}{\lambda}\frac{n(n+1)}{2}$$

The elastic energy stored is $\sum_1^n \frac{\lambda x_r^{\ 2}}{2l} = \sum_1^n \lambda \left(\frac{lmg}{\lambda} \right)^2 \frac{r^2}{2l} = \frac{m^2 g^2 l}{12\lambda} n(n+1)(2n+1)$

For the uniform heavy rope, we let $M = nm$, $L_0 = nl$, and consider the limit as $n \to \infty$

$$L = \lim \left(L_0 + \frac{M}{n}\frac{g}{\lambda}\frac{L_0}{n}\frac{n(n+1)}{2} \right) = L_0 \left(1 + \frac{Mg}{2\lambda} \right)$$

and the elastic energy stored is

$$\lim \left(\frac{m^2 g^2 l}{12\lambda} n(n+1)(2n+1) \right) = \lim \left(\frac{M^2 g^2 L_0}{12\lambda} \frac{n(n+1)(2n+1)}{n^3} \right) = \frac{M^2 g^2 L_0}{6\lambda}$$

and eliminating *M* using the result just found for *L* we obtain $\frac{2\lambda (L - L_0)^2}{3L_0}$

11. If the resistance couple (constant) is L, then using $L = I\alpha$ for the second phase of the motion, $L = \frac{I\omega_0}{T}$ and rotational kinetic energy used up doing work against the couple in the second phase gives

$$\frac{1}{2} I\omega_0^{\ 2} = L \times n_2 \times 2\pi$$

Hence, eliminating *L* and simplifying gives the first result.

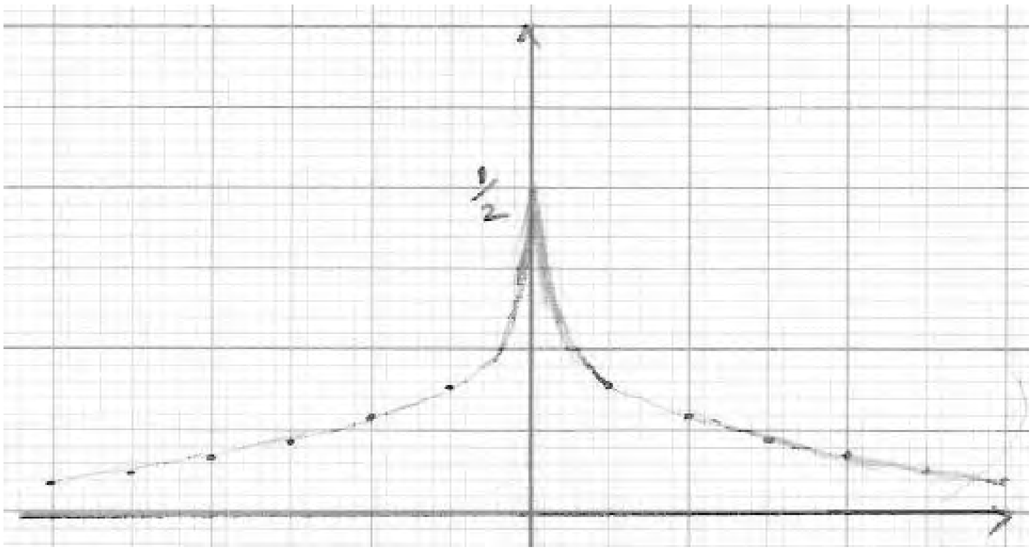
If the particle descends a distance *h* in the first phase of motion, then $h = 2\pi r n_1$. If the particle has speed *v* at the end of the first phase, then $v = r\omega_0$
and using the work-energy principle,

$$mgh - L \times n_1 \times 2\pi = \frac{1}{2} I\omega_0^{\ 2} + \frac{1}{2} mv^2$$

Hence, eliminating *h*, *v* and $\omega_0^{\ 2}$ obtains the second result.

Section C: Probability and Statistics

12.



$M_x(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f(x)dx = \int_{-\infty}^{0} \frac{1}{2} e^{x(1+\theta)}dx + \int_{0}^{\infty} \frac{1}{2} e^{-x(1-\theta)}dx$

$= \frac{1}{2(1+\theta)}\left[e^{x(1+\theta)}\right]_{-\infty}^{0} - \frac{1}{2(1-\theta)}\left[e^{-x(1-\theta)}\right]_{0}^{\infty} \quad = \frac{1}{2(1+\theta)} + \frac{1}{2(1-\theta)}$ (requiring $|\theta| < 1$)

$= \left(1-\theta^2\right)^{-1}$

$Var(X) = M_x{''}(0) - \left(M_x{'}(0)\right)^2$

$M_x{'}(\theta) = 2\theta\left(1-\theta^2\right)^{-2}$, $M_x{''}(\theta) = 2\left(1-\theta^2\right)^{-2} + 8\theta\left(1-\theta^2\right)^{-3}$

and so $M_x{'}(0) = 0$, $M_x{''}(0) = 2$, $Var(X) = 2$

Or alternatively, $M_x(\theta) = E\left(e^{\theta X}\right) = E\left(1 + \theta X + \frac{1}{2}\theta^2 X^2 + ...\right)$

$= \left(1-\theta^2\right)^{-1} = 1 + \theta^2 + \theta^4 + ...$

and so $E(X) = 0$, $E\left(\frac{1}{2}X^2\right) = 1$, $Var(X) = E\left(X^2\right) - \left(E(X)\right)^2 = 2$

If $T = Y/\sqrt{2n}$, then $M_T(\theta) = E\left(e^{\theta T}\right) = E\left(e^{\theta \sum \left(X_i/\sqrt{2n}\right)}\right) = \prod_{i=1}^{n} E\left(e^{\frac{\theta}{\sqrt{2n}}X_i}\right) = \left(1 - \frac{\theta^2}{2n}\right)^{-n}$

$$\log\left(M_T(\theta)\right) = -n\log\left(1-\frac{\theta^2}{2n}\right) = -n\left[-\frac{\theta^2}{2n}-\frac{\theta^4}{8n^2}-\frac{\theta^6}{24n^3}-.....\right] = \frac{\theta^2}{2}+\frac{\theta^4}{8n}+\frac{\theta^6}{24n^2}+.....$$

Thus as $n \to \infty$, $\log\left(M_T(\theta)\right) \to \frac{\theta^2}{2}$, and so $M_T(\theta) \to \exp\left(\frac{\theta^2}{2}\right)$

$P(|Y| \geq 25) = 0\cdot 05$ and $P\left(\left|Y/\sqrt{2n}\right| \geq 1\cdot 96\right) = 0\cdot 05$ and so

$25 = 1\cdot 96\sqrt{2n}$

$2n = \frac{25^2}{1\cdot 96^2} \approx \frac{625}{4}$

$n \approx \frac{625}{8} \approx 78$

13. P(1 ring created at first step) = $\frac{1}{2n-1}$,

P(0 rings created at first step) = $\frac{2n-2}{2n-1}$

E(number of rings created at first step) = $\frac{1}{2n-1}\times 1+\frac{2n-2}{2n-1}\times 0 = \frac{1}{2n-1}$

Regardless of what happens at first step, after the first step there $2n-2$ free ends.
Similarly after second step $2n-4$ free ends regardless, etc.

E(number of rings at end of process) = $\frac{1}{2n-1}+\frac{1}{2n-3}+\frac{1}{2n-5}+\frac{1}{2n-7}+......+\frac{1}{1}$

Var(number of rings at end of process) =

$$\frac{1}{2n-1}-\left(\frac{1}{2n-1}\right)^2+\frac{1}{2n-3}-\left(\frac{1}{2n-3}\right)^2+\frac{1}{2n-5}-\left(\frac{1}{2n-5}\right)^2+\frac{1}{2n-7}-\left(\frac{1}{2n-7}\right)^2+......+\frac{1}{1}-\left(\frac{1}{1}\right)^2$$

(as numbers of rings created at each step are independent)

$$=\frac{2(n-1)}{(2n-1)^2}+\frac{2(n-2)}{(2n-3)^2}+\frac{2(n-3)}{(2n-5)^2}+......+\frac{2}{3^2}$$

For $n = 40000$, E(number of rings created) = $1+\frac{1}{3}+\frac{1}{5}+.....+\frac{1}{79999}$

$$= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+.....+\frac{1}{80000}-\left(\frac{1}{2}+\frac{1}{4}+.....+\frac{1}{80000}\right)$$

$\approx \ln 80000 - \frac{1}{2}\ln 40000$

$= 2\ln 20$

≈ 6

This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by the Examiners. It does not indicate the details of the discussions which took place at an Examiners' meeting before marking commenced.

All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the Report on the Examination.

Cambridge Assessment will not enter into any discussion or correspondence in connection with this mark scheme.

More information about STEP can be found at:
http://www.admissionstests.cambridgeassessment.org.uk/adt/