STEP

Mathematics

STEP **9465, 9470, 9475**

STEP Hints and Answers

June 2006

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Section A: Pure Mathematics

1 Since $182^2 = 33124$ and $183^2 = 33489$, let $n = 182$.

Since $184^2 - 33127 = 729 = 27^2$, let $m = 2$.

Therefore $184^2 - 27^2 = 33127$, so $33127 = (184 - 27) \times (184 + 27) \Rightarrow 33127 = 157 \times 211$.

It is crucial to realise that 157 and 211 are both prime numbers, hence the only other factorization of 33127 is $33127 = 1 \times 33127$.

Therefore $33127 = (16564 - 16563) \times (16564 + 16563) = 16564^2 - 16563^2$, so $m = 16382$.

This question is based on the method of Fermat factorization, which can sometimes be used effectively to factorize integers. Notice how we have factorized a number without ever dividing.

2 A good diagram is crucial here. Notice carefully that you are required to **prove** that the maximum area grazed is $14\pi a^2$, and therefore that **assuming** that this occurs when the goat is tethered to a corner will achieve few marks. Unjustified assumptions are of little value in Mathematics.

Let the goat be tethered a distance x from a corner. Therefore, the goat can graze an area

$$A = \frac{16a^2\pi}{2} + \frac{(4a-x)^2\pi}{4} + \frac{(2a-x)^2\pi}{4} + \frac{(2a+x)^2\pi}{4} + \frac{(x)^2\pi}{4} = \frac{\pi}{4}\left(56a^2 + 4x^2 - 8ax\right)$$

So the area grazed $A = \pi\left[13a^2 + (x-a)^2\right]$. This is minimised when $x = a$, and maximised when $x = 0$ or $2a$ (since $0 \leqslant x \leqslant 2a$), hence $13\pi a^2 \leqslant A \leqslant 14\pi a^2$.

Notice that completing the square is an efficient, easy way of maximising or minimising a quadratic expression. Calculus could also be used.

3 Recall that in this question b, c, p and q are real numbers.

(i) A picture of the graph $y = x^2 + bx + c$ shows that $c < 0$ is a sufficient condition for the roots of $x^2 + bx + c = 0$ to be real and unequal, since it is U-shaped with negative y-intercept. However, $c < 0$ is not a necessary condition; consider for example the equation $x^2 + 5x + 6 = 0$.

(ii) For the equation to have two distinct positive real roots, there must be two distinct real roots ($b^2 - 4c > 0$), they must be of the same sign ($c > 0$), and they must be positive ($b < 0$, consider the turning point of the graph). It is much easier to see this graphically than to try to manipulate the quadratic formula.

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(iii) The first two parts of the question suggest that the nature of the roots of an equation can sometimes be ascertained by looking at an appropriate graph: in particular, the location of the turning points and the y-intercept. Therefore we consider the graph $y = x^3 + px + q$, its y-intercept at $(0\,,\,q)$, and the derivative $3x^2 + p$. The condition $q < 0$ ensures that the y-intercept is negative.

If $p > 0$ then $3x^2 + p = 0$ has no real solutions, so the cubic curve $y = x^3 + px + q$ has no turning points. Hence if $q < 0$ there will be one positive, real solution of $x^3 + px + q = 0$.

If $p < 0$ then the turning points are at $\left(\sqrt{\frac{-p}{3}}\,,\, \frac{2p}{3}\sqrt{\frac{-p}{3}} + q\right)$ and at $\left(-\sqrt{\frac{-p}{3}}\,,\, -\frac{2p}{3}\sqrt{\frac{-p}{3}} + q\right)$.

Notice that $4p^3 + 27q^2 \equiv 27\left(\frac{2p}{3}\sqrt{\frac{-p}{3}} + q\right)\left(-\frac{2p}{3}\sqrt{\frac{-p}{3}} + q\right)$, so that

a) if $4p^3 + 27q^2 < 0$ then the y coordinates of the turning points are of opposite signs, ensuring that there are three real roots, of which two are negative since $q < 0$;

b) if $4p^3 + 27q^2 = 0$ then one of the turning points has y coordinate 0, so the equation has exactly two real roots of opposite signs;

c) if $4p^3 + 27q^2 > 0$ then the y coordinates of the turning points are of the same sign, ensuring that there is one real root, which is positive since $q < 0$;

4 (i) When asked to sketch two graphs on the same axes, it is important to ensure that they look correct relative to each other; in this question, the gradient of each graph needs considering. The x-axis should be measured in radians. A good graph makes the argument obvious: the curves $y = \sin x$ and $y = x$ both pass through the origin, and both have gradient 1 there. Afterwards, $y = \sin x$ has gradient less than 1, and so is always "underneath" the line $y = x$.

(ii) Another graphical argument: the curves look very similar, hence $\dfrac{\sin x}{x} \approx 1$.

The polygon can be split into n isosceles triangle of base $\dfrac{P}{n}$ and vertical height $\dfrac{P}{2n} \div \tan\left(\dfrac{2\pi}{2n}\right)$.

So the area $= n \times \dfrac{1}{2} \times \dfrac{P}{n} \times \dfrac{P}{2n} \div \tan\left(\dfrac{2\pi}{2n}\right) = \dfrac{P^2}{4n\tan\left(\frac{\pi}{n}\right)}$

As instructed, we consider $\dfrac{\mathrm{d}}{\mathrm{d}n}\left[\dfrac{P^2}{4n\tan\left(\frac{\pi}{n}\right)}\right] = \dfrac{P^2}{4\left[n\tan\left(\frac{\pi}{n}\right)\right]^2} \times \left[\tan\left(\dfrac{\pi}{n}\right) - n\dfrac{\pi}{n^2}\sec^2\left(\dfrac{\pi}{n}\right)\right]$

$$= -1 \times \frac{P^2}{4\left[n\tan\left(\frac{\pi}{n}\right)\right]^2} \times \frac{\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right) - \left(\frac{\pi}{n}\right)}{\cos^2\left(\frac{\pi}{n}\right)}$$

$< -\dfrac{P^2}{4\left[n\tan\left(\frac{\pi}{n}\right)\right]^2} \times \dfrac{\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right) - \left(\frac{\pi}{n}\right)}{\cos^2\left(\frac{\pi}{n}\right)}$ by (i), which tells us that $\sin x < x$.

This derivative is positive, since $\left(\dfrac{\pi}{n}\right)\cos\left(\dfrac{\pi}{n}\right) - \left(\dfrac{\pi}{n}\right) < 0$, hence the area increases.

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n large $\Rightarrow \sin\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n}$ (from part (ii)), and clearly $\cos\left(\frac{\pi}{n}\right) \approx 1$. Therefore $\tan\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n}$, so the area of the polygon $\approx \frac{P^2}{4\pi}$. The radius of the circumcircle $= \frac{P}{2n} \div \sin\left(\frac{2\pi}{2n}\right) \approx \frac{P}{2n} \times \frac{n}{\pi}$ (since n is large), hence the circumcircle has area $\approx \pi \times \frac{P^2}{4\pi^2}$.

5 (i) $u^2 = 2x+1 \Rightarrow 2u\,\mathrm{d}u = 2\,\mathrm{d}x$ and $x-4 = \frac{1}{2}\left(u^2-9\right)$

$$\Rightarrow \int \frac{3}{(x-4)\sqrt{2x+1}}\,\mathrm{d}x = \int \frac{6}{(u^2-9)\,u}\,u\,\mathrm{d}u$$

$= \int \frac{1}{u-3} - \frac{1}{u+3}\,\mathrm{d}u$ (splitting the integrand into partial fractions)

$$= \ln(u-3) - \ln(u+3) + K = \ln\left(\frac{\sqrt{2x+1}-3}{\sqrt{2x+1}+3}\right) + K$$

(ii) The similarity of the two integrands suggests a similar substitution.

$u^2 = \mathrm{e}^x + 1 \Rightarrow 2u\,\mathrm{d}u = \mathrm{e}^x\,\mathrm{d}x$; also, $x = \ln 8 \Rightarrow u = 3$ and $x = \ln 3 \Rightarrow u = 2$.

$$\Rightarrow \int_{\ln 3}^{\ln 8} \frac{2}{\mathrm{e}^x\sqrt{\mathrm{e}^x+1}}\,\mathrm{d}x = \int_2^3 \frac{2}{(u^2-1)\,u}\,\frac{2u}{u^2-1}\,\mathrm{d}u = \int_2^3 \frac{4}{(u^2-1)^2}\,\mathrm{d}u$$

Let $\frac{4}{(u^2-1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2}$

$\Rightarrow 4 = A(u-1)(u+1)^2 + B(u+1)^2 + C(u+1)(u-1)^2 + D(u-1)^2$

Let $u = 1 \Rightarrow B = 1$. Let $u = -1 \Rightarrow D = 1$

Comparing the coefficients of $u^3 \Rightarrow 0 = A + C$.

Comparing the coefficients of $u^0 \Rightarrow 4 = -A + B + C + D$

$\Rightarrow A = -1,\ C = 1$

So the integral $= \left[\ln(u+1) - \ln(u-1) - \frac{1}{u-1} - \frac{1}{u+1}\right]_2^3 = \frac{7}{12} + \ln\frac{2}{3}$

Alternatively, the identity $\left[\frac{2}{u^2-1}\right]^2 \equiv \left[\frac{1}{u-1} - \frac{1}{u+1}\right]^2$ can be used to split the integrand into partial fractions.

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6 (i) The assertion that "$(a\,,\,b)$ lies on the curve $x^2 - 2y^2 = 1$" is equivalent to stating that $a^2 - 2b^2 = 1$. Since $(3a+4b)^2 - 2\,(2a+3b)^2 \equiv 9a^2 + 24ab + 16b^2 - 2\left(4a^2 + 12ab + 9b^2\right) \equiv a^2 - 2b^2$, the point $(3a+4b\,,\,2a+3b)$ lies on the curve $x^2 - 2y^2 = 1$ if $(a\,,\,b)$ does.

(ii) We are told that $Ma^2 - Nb^2 = 1$, and we want to find M and N so that $M\,(5a+6b)^2 - N\,(4a+5b)^2 = 1 = Ma^2 - Nb^2$.

Since $M\,(5a+6b)^2 - N\,(4a+5b)^2 \equiv a^2\,(25M - 16N) + ab\,(60M - 40N) + b^2\,(36M - 25N)$, we could let $25M - 16N = M$, $60M - 40N = 0$ and $36M - 25N = -N$.

These are all equivalent to $3M = 2N$, so let $M = 2$ and $N = 3$.

(iii) We require $(Pa + Qb)^2 - 3\,(Ra + Sb)^2 \equiv a^2 - 3b^2 = 1$.

$\Rightarrow P^2 - 3R^2 = 1$

and $2PQ = 6RS$

and $Q^2 - 3S^2 = -3$

The first of these equations suggests letting $P = 2$ and $R = 1$. Then the second equation reduces to $4Q = 6S$ which suggests letting $Q = 3$ and $S = 2$. These values are consistent with the third equation.

Therefore $(2a + 3b\,,\,a + 2b)$ is the simplest solution.

$(7a + 12b\,,\,4a + 7b)$ is another solution, since $7^2 - 3 \times 4^2 = 1, 7 \times 12 = 3 \times 7 \times 4$, and $12^2 - 3 \times 7^2 = -3$.

Equations of the form $x^2 - dy^2 = 1$ are called Pell equations; the techniques for solving them (which underlie this question) are explained in most undergraduate textbooks on Number Theory. It might be interesting to consider the equation $x^2 - 4y^2 = 1$: what happens when an argument similar to (iii) is pursued?

7 This question requires familiarity with the notation $|x|$, which can be defined as: $|x| \equiv x$ if $x \geqslant 0$, and $|x| \equiv -x$ if $x < 0$.

(i) The graph of $y = \operatorname{cosec} x$ has asymptotes $x = 0$ and $x = \pi$. Both this graph and the line $y = \frac{2}{\pi}x$ pass through the point $\left(\frac{\pi}{2}\,,\,1\right)$, so the equation $x \sin x = \frac{\pi}{2}$ (which is equivalent to $\frac{2}{\pi}x = \operatorname{cosec} x$) has two solutions for $0 < x < \pi$. The smaller of these is $\frac{\pi}{2}$ and the larger is defined in the question to be α.

The two graphs should make it clear that $\operatorname{cosec} x \leqslant \frac{2}{\pi}x$ for $\frac{\pi}{2} \leqslant x \leqslant \alpha$, hence in the same domain $\frac{\pi}{2} \leqslant x \sin x \Rightarrow x \sin x - \frac{\pi}{2} \geqslant 0$. By a similar argument, $x \sin x - \frac{\pi}{2} < 0$ for $\alpha < x \leqslant \pi$. This analysis enables us to remove correctly the modulus in the integrand:

$$\int_{\frac{\pi}{2}}^{\pi} \left| x \sin x - \frac{\pi}{2} \right| \,\mathrm{d}x = \int_{\frac{\pi}{2}}^{\alpha} x \sin x - \frac{\pi}{2}\,\mathrm{d}x + \int_{\alpha}^{\pi} \frac{\pi}{2} - x \sin x\,\mathrm{d}x$$

$$= \left[\sin x - x\cos x - \frac{\pi x}{2}\right]_{\frac{\pi}{2}}^{\alpha} + \left[-\sin x + x\cos x + \frac{\pi x}{2}\right]_{\alpha}^{\pi}$$

$$= \left(\sin\alpha - \alpha\cos\alpha - \frac{\pi\alpha}{2}\right) - \left(1 - \frac{\pi^2}{4}\right) + \left(-\pi + \frac{\pi^2}{2}\right) - \left(-\sin\alpha + \alpha\cos\alpha + \frac{\pi\alpha}{2}\right)$$

$$= 2\sin\alpha + \frac{3\pi^2}{4} - \alpha\pi - 2\alpha\cos\alpha - \pi - 1$$

(ii) As suggested by the first part of the question, a careful sketch of the graph of $y = \left||e^x - 1| - 1\right|$ is sensible. It should show that

if $x \leqslant 0$ then $\left||e^x - 1| - 1\right| \equiv e^x$

if $0 < x \leqslant \ln 2$ then $\left||e^x - 1| - 1\right| \equiv 2 - e^x$

if $x > \ln 2$ then $\left||e^x - 1| - 1\right| \equiv e^x - 2$.

Hence the area in the domain $0 \leqslant x \leqslant \ln 2$ is

$$\int_0^{\ln 2} 2 - e^x \, dx = (2\ln 2 - 2) - (0 - 1) = \ln 4 - 1$$

8 (i) The volume of $OABC = \frac{1}{3} \times$ the area of triangle $OAB \times OC = \frac{1}{6}abc$.

(ii) Using the scalar product with vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$,

$$\sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \cos\theta = \begin{pmatrix} a \\ 0 \\ -c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix} = c^2 \Rightarrow \cos\theta = \frac{c^2}{\sqrt{a^2+c^2}\sqrt{b^2+c^2}}$$

The cosine rule ($AB^2 = AC^2 + BC^2 - 2 \times AC \times BC \times \cos\theta$) will also yield this result.

The area of triangle ABC will be $\frac{1}{2} \times \sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \sin\theta$

$$= \frac{1}{2} \times \sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \sqrt{1 - \left(\frac{c^2}{\sqrt{a^2+c^2}\sqrt{b^2+c^2}}\right)^2} \quad \text{(because } \sin^2\theta \equiv 1 - \cos^2\theta\text{)}$$

$$= \frac{1}{2} \times \sqrt{(a^2+c^2)(b^2+c^2) - c^4}$$

$$= \frac{1}{2} \times \sqrt{a^2b^2 + b^2c^2 + c^2a^2}$$

So $\frac{1}{3} \times \left(\frac{1}{2} \times \sqrt{a^2b^2 + b^2c^2 + c^2a^2}\right) \times d = \frac{1}{6}abc \Rightarrow \frac{1}{d^2} = \frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2b^2c^2}$

which simplifies to the stated result.

A similar result is true for the right-angled triangle PQR, in which X is the foot of the perpendicular from the right-angle Q to the hypotenuse PR: $\frac{1}{PQ^2} + \frac{1}{QR^2} = \frac{1}{QX^2}$

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Section B: Mechanics

9 It is very difficult to answer this question without a clear diagram and the use of consistent notation. It is strongly recommended (though not essential) that the motion of the three objects be considered separately.

Before the string breaks, let the acceleration of the system be a, and let T and S be the tensions in the strings. Therefore (using Newton's Second Law on each object):

$$xg - T = xa \qquad T - S = 4a \qquad S - yg = ya.$$

Their sum $\Rightarrow a = \dfrac{g\,(x-y)}{x+y+4} = \dfrac{g\,(6-2y)}{10}$ since $x + y = 6$. The block takes time $t_1 = \sqrt{\dfrac{2d}{a}} = \sqrt{\dfrac{20d}{g\,(6-2y)}}$ to travel d. At this point it has velocity $v = \sqrt{2ad} = \sqrt{\dfrac{gd\,(6-2y)}{5}}$.

When the string over pulley P breaks, let the new acceleration be f and let the tension in the string over pulley Q be S_1.

Newton's Second Law tells us that $-S_1 = 4f$ and $S_1 - yg = yf \quad \Rightarrow f = \dfrac{-yg}{y+4}$

Therefore it takes $t_2 = -\sqrt{\dfrac{gd\,(6-2y)}{5}} \div \dfrac{-yg}{y+4} = \sqrt{\dfrac{d\,(6-2y)}{5g}}\left(\dfrac{y+4}{y}\right)$ to come to rest.

Hence the total time $T = t_1 + t_2 = \sqrt{\dfrac{d}{5g}}\ \mathrm{f}\,(y)$, where $\mathrm{f}\,(y)$ is as given.

To minimise T, set $\dfrac{\mathrm{d}f}{\mathrm{d}y} = 0 \Rightarrow \dfrac{10}{(6-2y)^{\frac{3}{2}}} + \left[\left(1+\dfrac{4}{y}\right)\dfrac{-1}{\sqrt{6-2y}} + \sqrt{6-2y}\left(\dfrac{-4}{y^2}\right)\right] = 0$

$$\Rightarrow 10 - \left(1+\frac{4}{y}\right)(6-2y) - (6-2y)^2\left(\frac{4}{y^2}\right) = 0$$

$$\Rightarrow (6-2y)\left(y^2+4y+4\,(6-2y)\right) = 10y^2$$

$$\Rightarrow -2y^3 + 4y^2 - 72y + 144 = 0$$

$$\Rightarrow y^3 - 2y^2 + 36y - 72 = 0$$

$$\Rightarrow (y-2)\left(y^2+36\right) = 0$$

$$\Rightarrow y = 2$$

Notice that the value of y does not depend on d: whenever the string is cut, a 2:1 division of the total mass of 6 kg will result in the shortest time taken.

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10 Since $x = Vt\cos 45^\circ$ and $y = Vt\sin 45^\circ - \frac{1}{2}gt^2$, we derive $y = x - \dfrac{gx^2}{V^2}$ as the cartesian equation of the trajectory. Next we consider the intersection of this parabola and the line $y = x\tan\alpha + b$. This occurs when $x\tan\alpha + b = x - \dfrac{gx^2}{V^2} \Rightarrow gx^2 + V^2x\left(-1+\tan\alpha\right) + bV^2 = 0$.

If this quadratic equation has only one solution then the line and the curve will touch. This will happen if the discriminant $(b^2 - 4ac) = 0$, i.e. $V^4\left(-1+\tan\alpha\right)^2 = 4gbV^2$

$\Rightarrow V\left(-1+\tan\alpha\right) = \pm 2\sqrt{gb}$

Since the particle cannot reach the roof if $\alpha \geqslant 45^\circ$ (consider the gradient of the parabolic path of projection at the origin), it can be seen that necessarily $\tan\alpha < 1$. Therefore $-1+\tan\alpha < 0$, so the negative square root must be chosen: $V\left(-1+\tan\alpha\right) = -2\sqrt{gb}$

If the condition for touching is satisfied, then this will occur where $x = \dfrac{-V^2\left(-1+\tan\alpha\right)}{2g}$ (using the quadratic formula with the discriminant equal to 0) at time $t = \dfrac{x\sqrt{2}}{V} = \dfrac{-V\left(-1+\tan\alpha\right)}{g\sqrt{2}}$.

To answer the last part of the question, a clear diagram of Q, and the forces acting on it, is recommended. The initial horizontal velocity of Q is $U\cos\alpha$, and its horizontal acceleration is $-g\cos\alpha\sin\alpha$ (caused by the horizontal component of the normal reaction).

Given that the particles touch at time $t = \dfrac{-V\left(-1+\tan\alpha\right)}{g\sqrt{2}}$, it is required that

$$U\cos\alpha \times \frac{-V\left(-1+\tan\alpha\right)}{g\sqrt{2}} - \frac{1}{2} \times -g\cos\alpha\sin\alpha \times V^2\left(\frac{-1+\tan\alpha}{g\sqrt{2}}\right)^2 = \frac{-V^2\left(-1+\tan\alpha\right)}{2g}$$

(this is because $s = ut + \frac{1}{2}at^2$ must be equal for both particles in the horizontal direction).

$$\Rightarrow U\cos\alpha + \frac{1}{2} \times \cos\alpha\sin\alpha \times V\left(\frac{-1+\tan\alpha}{\sqrt{2}}\right) = \frac{V}{\sqrt{2}}$$

$$\Rightarrow 2\sqrt{2}U\cos\alpha = 2V - V\cos\alpha\sin\alpha \times \left(-1+\tan\alpha\right)$$

$$\Rightarrow 2\sqrt{2}U\cos\alpha = V\left(2 + \cos\alpha\sin\alpha - \sin^2\alpha\right)$$

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11 It is essential to realise that we cannot consider each individual collision, since we do not know the masses of particles A_1 to A_{n-2}. We must consider the conservation of momentum and of kinetic energy overall.

(i) If only one particle were moving after all collisions have taken place, it would have to be A_n with velocity v.

This would require $mu = \lambda mv$ (conservation of momentum) and $mu^2 = \lambda mv^2$ (conservation of kinetic energy).

Hence $u^2 = \lambda^2 v^2$ and $u^2 = \lambda v^2 \Rightarrow \lambda^2 = \lambda \Rightarrow \lambda = 1$ or 0, neither of which is permitted.

(ii) Let the final speeds of A_{n-1} and A_n be v and w respectively, where both are positive.

This scenario requires $mu = mv + \lambda mw$ (conservation of momentum) and $mu^2 = mv^2 + \lambda mw^2$ (conservation of kinetic energy).

Therefore $(v + \lambda w)^2 = v^2 + \lambda w^2$

$\Rightarrow 2v\lambda w + \lambda^2 w^2 = \lambda w^2$

$\Rightarrow 2v = w(1 - \lambda)$ which implies that $v < 0$ since $\lambda > 1$. This contradicts the supposition of the question: if A_{n-1} moves backwards then it will not be the only particle moving other than A_n.

(iii) Let the final velocities of A_{n-2}, A_{n-1} and A_n be p, q and r respectively, where all are positive, and $p < q < r$. Also, we must let the mass of A_{n-2} be km, since we do not know what it is.

This scenario requires $mu = kmp + mq + \lambda mr$ (conservation of momentum) and $mu^2 = kmp^2 + mq^2 + \lambda mr^2$ (conservation of kinetic energy).

$\Rightarrow (kp + q + \lambda r)^2 = kp^2 + q^2 + \lambda r^2$

$\Rightarrow k^2p^2 + q^2 + \lambda^2 r^2 + 2kpq + 2\lambda qr + 2\lambda kpr = kp^2 + q^2 + \lambda r^2$

$\Rightarrow 2kpq + 2\lambda kpr - kp^2 + k^2p^2 = \lambda r^2 - \lambda^2 r^2 - 2\lambda qr$

$\Rightarrow kp\,[2q + 2\lambda r - p(1 - k)] = r^2(\lambda - \lambda^2) - 2\lambda qr$

Since $\lambda > 1$, the RHS is negative.

But $q > p \Rightarrow 2q - p(1 - k) > 2p - p(1 - k) \equiv p + pk > 0$.

Hence the RHS < 0 but the LHS > 0: a contradiction.

(iv) The two particles must be A_0 and A_n, with velocities x and y respectively.

Therefore, $mu = mx + \lambda my$ (conservation of momentum) and $mu^2 = mx^2 + \lambda my^2$ (conservation of kinetic energy).

$\Rightarrow (u - \lambda y)^2 = u^2 - \lambda y^2$

$\Rightarrow \lambda^2 y^2 - 2u\lambda y = -\lambda y^2$

$\Rightarrow \lambda y - 2u = -y$

$\Rightarrow y = \dfrac{2u}{1+\lambda} \Rightarrow x = \lambda y - u = u\left(\dfrac{2}{1+\lambda} - 1\right) = u\left(\dfrac{1-\lambda}{1+\lambda}\right).$

Note that x is negative, as is required.

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Section C: Probability and Statistics

12 There are many arguments to derive the first result: the neatest is probably to argue that if there is no road from Oxtown to Camville then the third road must be blocked (with probability p), and also on **both** of the other two roads it is **not** the case that **both** of the sections are **un**blocked. Therefore,

$$P\,(\text{no road from Oxtown to Camville}) = p\left(1-(1-p)^2\right)^2 = p\left(2p-p^2\right)^2 = p^3\,(2-p)^2.$$

It is crucial to recognise that the second paragraph is asking for a conditional probability:

$P\,(\text{the chosen road isn't blocked and the others are, given that the chosen road isn't blocked})$

$$= \frac{\frac{1}{3}(1-p)\left(2p-p^2\right)^2 + \frac{2}{3}p\left(2p-p^2\right)(1-p)^2}{\frac{1}{3}(1-p)+\frac{2}{3}(1-p)^2} = \frac{\left(2p-p^2\right)(1-p)\left[\left(2p-p^2\right)+2p\,(1-p)\right]}{(1-p)\left[1+2\,(1-p)\right]}$$

$$= \frac{\left(2p-p^2\right)\left(4p-3p^2\right)}{(3-2p)} = \frac{p^2\,(2-p)\,(4-3p)}{(3-2p)}$$

Notice that when $p=1$ this probability equals 1; but if $p=1$ there will be no route from Oxtown to Camville at all! As p tends to 1 this probability tends to 1: as blocked roads become more and more certain, then if you do (just) manage to get to Camville it's very likely that the other roads would have been blocked. The resolution to this apparent paradox is that when $p=1$ you're conditioning on an event of zero probability: notice the cancelled factor of $1-p$, which requires $p \neq 1$. The violation of this explains the apparent contradiction.

13 (i) Given the information in the question, we can assume that the number of diamonds in $100N$ grams of chocolate is distributed Poisson $(0.1N)$. Therefore, the probability of there being no diamonds in $100N$ grams of chocolate is $\mathrm{e}^{-0.1N}$, and the expected number of diamonds in $100N$ grams of chocolate is $0.1N$

$$P\,(\text{I have no diamonds}) = \frac{1}{6}\left(\mathrm{e}^{-0.1}+\mathrm{e}^{-0.2}+\ldots+\mathrm{e}^{-0.6}\right) = \frac{\mathrm{e}^{-0.1}}{6}\left(\frac{1-\mathrm{e}^{-0.6}}{1-\mathrm{e}^{-0.1}}\right)$$

A tree diagram might make this calculation more clear: there are six alternative branches, representing the possible scores on a die, at the end of each of which are two branches, representing either finding no diamonds or finding some.

A similar "expectation" tree diagram explains the argument that I expect to find $\frac{1}{6}(0.1+0.2+\ldots+0.6) = 0.35$ diamonds. These are examples of conditional expectations: $0.2 = E\,(\text{the number of diamonds I find } \textbf{given that} \text{ I roll a 2})$, analogous to $\mathrm{e}^{-0.2} = P\,(\text{I find no diamonds } \textbf{given that} \text{ I roll a 2})$. The idea of conditional expectation is developed considerably in undergraduate mathematics.

Hints and Answers, STEP I June 2006

(ii) P (I have no diamonds)

$$= \left(\frac{1}{6}\right)\left(e^{-0.1}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)\left(e^{-0.2}\right) + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)\left(e^{-0.3}\right) + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)\left(e^{-0.4}\right) + \ldots$$

(because if I score my first 6 on the rth roll, then the previous $r-1$ rolls have all scored "not 6")

$$= \frac{\left(\frac{1}{6}\right)e^{-0.1}}{1 - \left(\frac{5}{6}\right)e^{-0.1}}$$

$$= \frac{e^{-0.1}}{6 - 5e^{-0.1}}$$ using the formula for the sum to infinity of a geometric progression.

The question suggests we recall that $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$, because a similar argument to that used in part (i) tells us that the number of diamonds I expect to find is

$$\left(\frac{1}{6}\right)0.1 + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)0.2 + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)0.3 + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)0.4 + \ldots$$

$$= \left(\frac{1}{6}\right)0.1\left[1 + 2\left(\frac{5}{6}\right) + 3\left(\frac{5}{6}\right)^2 + 4\left(\frac{5}{6}\right)^3 + \ldots\right]$$

$$= \left(\frac{1}{6}\right)0.1\left(1 - \frac{5}{6}\right)^{-2} = 0.6 \text{ diamonds.}$$

14 (i) $$P\,(\text{red sweet is first drawn as } r\text{th sweet}) = \left(\frac{n}{n+1}\right)^{r-1}\frac{1}{n+1}$$

$$P \text{ maximal} \Rightarrow \frac{dP}{dn} = 0 \Rightarrow \frac{(n+1)^r(r-1)n^{r-2} - n^{r-1}r(n+1)^{r-1}}{(n+1)^{2r}} = 0$$

$\Rightarrow (n+1)(r-1) - nr = 0$

$\Rightarrow n = r - 1$

Notice that we can use this result to estimate n: if the rth sweet is red for the first time, it is sensible (because it makes the observed event most likely) to estimate that there are $r-1$ blue sweets in the bag.

(ii) P (red sweet is first drawn as rth sweet) =

$$\left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\cdots\left(\frac{n-r+2}{n-r+3}\right)\left(\frac{1}{n-r+2}\right) = \frac{1}{n+1}$$

The value of this probability decreases as n increases. Therefore, to maximise this probability, n needs to be as small as possible. Since r sweets have been chosen from the bag, this implies that $n+1 \geqslant r$: there must have been more sweets in the bag initially than are chosen during this procedure. Hence the minimum n is $r-1$.

Step II, Hints and Answers
June 2006

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HINTS AND ANSWERS

Q1 If you read through at least part (i) of the question, you will see that it is necessary to work with u_1, u_2, u_3 and u_5 (and hence, presumably – as the sequence is defined recursively – with u_4 also). Although it is not the only way to go about the problem, it makes sense to work each of these terms out first. Each will be an expression involving k and should ideally be simplified as you go. Thus, $u_1 = 2$ gives $u_2 = k - 18$, $u_3 = k - \frac{36}{k-18} = \frac{k^2 - 18k - 36}{k-18}$, etc. Then, for (a), $u_2 = 2$; for (b), $u_3 = 2$; and, for (c), $u_5 = 2$. Each result leads to a polynomial equation (of increasing orders) to be solved. Finally, you need to remember that, in the case of (c) for instance, of the four solutions given by the resulting equation, two of them must have arisen already in parts (a) and (b) – you'll see why if you think about it for a moment. Ideally, you would see this beforehand, and then this fact will help you factorise the quartic polynomial by the *factor theorem*.

A simple line of reasoning can be employed to establish the first result in (ii) without the need for a formal inductive proof. If $u_n \geq 2$, then $u_{n+1} = 37 - \frac{36}{u_n} \geq 37 - \frac{36}{2} = 19 > 2$. Since $u_1 = 2$, it follows that all terms of the sequence are ≥ 2. In fact, most of them are much bigger than this. Then, for the final part of the question, the informal observation that, eventually, all terms effectively become equal is all that is required. Setting $u_{n+1} = u_n = l$ (say) leads to a quadratic, with two roots, one of which is obviously less than 2 and can therefore be rejected.

Answers: (i) $k =$ (a) 20; (b) 0; (c) $\pm 6\sqrt{2}$. (ii) 36.

Q2 The formula books give a series for e^x. Setting $x = 1$ then gives you e as the limit of an infinite sum of positive terms, and the sum of the first four of these will then provide a lower bound to its value.

In the next part, you (again) can provide a perfectly sound argument for the required result without having to resort to a formally inductive one (although one would be perfectly valid, of course). Noting firstly that $4! = 24 > 16 = 2^4$, $(n + 4)!$ consists of the product of $4!$ and n positive integers, each greater than 2; while 2^{n+4} consists of 16 and a further n factors of 2. Since each term in the first number is greater than the corresponding term in the second, the result follows. [Alternatively, $4! > 2^4$ and $n! > 2^n \Rightarrow (n + 1)! = (n + 1) \times n! > 2 \times n!$ (since $n > 4$) $> 2 \times 2^n$ (by hypothesis) $= 2^{n+1}$, and proof follows by induction.] Now, adding the terms in the expansion for e **beyond** the cubed one, and noting that each is less than a corresponding power of $\frac{1}{2}$ using the result just established, gives $e < \frac{8}{3}$ + the sum-to-infinity of a convergent GP.

There are two common methods for showing that a stationary value of a curve is a max. or a min. One involves the second derivative evaluated at the point in question.

There are several drawbacks involved with this approach. One is that you have to differentiate twice (which is ok with simple functions). A second is that you need to know the exact value(s) of the variable being substituted (which isn't the case here). Another is that the sign of $\frac{d^2y}{dx^2}$ doesn't necessarily tell you what is happening to the curve. (Think of the graph of $y = x^4$, which has $\frac{d^2y}{dx^2} = 0$ at the origin, yet the stationary point here ***is*** a minimum!)

Thus, it is the other approach that you are clearly intended to use on this occasion. This examines the sign of $\frac{dy}{dx}$ slightly to each side of the point in question. When $x = \frac{1}{2}$, using $e < \frac{67}{24}$ shows …..; at $x = 1$, using $e > \frac{8}{3}$ shows …..; and at $x = \frac{5}{4}$, we can use any suitable bound for e, such as $e < 3$ for instance, to show that …..

Finally, since the answers are given in the question, it is important to state carefully the reasoning that supports these answers.

Q3 If you fail to notice that $\frac{1}{5+\sqrt{24}} = 5 - \sqrt{24}$, then this question is going to be a bit of a non-starter for you. The idea of conjugates, from the use of the *difference of two squares*, should be a familiar one. As is the *binomial theorem*, which you can now use to expand both $(5 + \sqrt{24})^4$ and $(5 - \sqrt{24})^4$. When you do this, you will see that all the $\sqrt{24}$ bits cancel out, to leave you with an integer. For the next part, some fairly simple inequality observations, such as

$$20.25 < 24 < 25 \Rightarrow 4.5 < \sqrt{24} < 5 \text{ and } 2 \times 100 = 200 < 208 = 11 \times 19 \Rightarrow \frac{2}{19} < \frac{11}{100}$$

help to establish the required results. It follows that $0.1^4 < (5 - \sqrt{24})^4 < 0.11^4$ and the difference between the integer and $(5 + \sqrt{24})^4$ is this small number, which lies between …..

For part (ii), it is simply necessary to mimic the work of part (i) but in a general setting, again starting with the key observations that $\frac{1}{N+\sqrt{N^2-1}} = N - \sqrt{N^2-1}$ and that the binomial expansions for $\left(N+\sqrt{N^2-1}\right)^k + \left(N-\sqrt{N^2-1}\right)^k$ will lead to the cancelling of all surd terms, to give an integer, M say. Now $\left(N-\sqrt{N^2-1}\right)^k$ is positive, and the reciprocal of a number > 1, so $\left(N-\sqrt{N^2-1}\right)^k \to 0+$ as $k \to \infty$. Also,

$$2N - \tfrac{1}{2} < N + \sqrt{N^2-1} < 2N \Rightarrow \frac{1}{2N-\frac{1}{2}} > N - \sqrt{N^2-1} > \frac{1}{2N}.$$

Thus $\left(N+\sqrt{N^2-1}\right)^k = M - \left(N-\sqrt{N^2-1}\right)^k$ differs from an integer (M) by less than $\left(\frac{1}{2N-\frac{1}{2}}\right)^k = (2N - \tfrac{1}{2})^{-k}$.

Answers: (i) 9601.9999

Q4 Using the given substitution, the initial result is established by splitting the integral into its two parts, and then making the simple observation that $\int_0^{\pi} x\, \mathrm{f}(\sin x)\ \mathrm{d}x = \int_0^{\pi} t\, \mathrm{f}(\sin t)\, \mathrm{d}t$.

This result is now used directly in (i), along with a substitution (such as $c = \cos x$). The resulting integration can be avoided by referring to your formula book, or done by using partial fractions. In (ii), the integral can be split into two; one from 0 to π, the second from π to 2π. The first of these is just (i)'s integral, and the second can be determined by using a substitution such as $y = x - \pi$ (the key here is that the limits will then match those of the initial result, which you should be looking to make use of as much as possible). In part (iii), the use of the double-angle formula for $\sin 2x$ gives an integral involving sines and cosines, but this must also count as a function of $\sin x$, since $\cos x = \sqrt{1-\sin^2 x}$. Thus the initial result may be applied here also. Once again, the substitution $c = \cos x$ reduces the integration to a standard one.

Answers: (i) $\frac{1}{4}\pi \ln 3$; (ii) $-\frac{1}{2}\pi \ln 3$; (iii) $\pi \ln \frac{4}{3}$.

Q5 The crucial observation here is that the integer-part (or *INT* or "floor") function is a whole number. Thus, when drawing the graphs, the two curves must coincide at the left-hand (integer) endpoints of each unit range, with the second curve slowly falling behind in the first instance, and remaining at the integer level in the second. Note that the curves with the *INT* function-bits in them will jump at integer values, and you should not therefore join them up at the right-hand ends (to form a continuous curve).

The easiest approach in (i) is not to consider $\int y_1\, \mathrm{d}x - \int y_2\, \mathrm{d}x$ (i.e. separately), but rather $\int (y_1 - y_2)\mathrm{d}x$. This gives a multiple of $x - [x]$ to consider at each step, and this simply gives a series of "unit" right-angled triangles of area $\frac{1}{2}$ to be summed.

In (ii), several possible approaches can be used, depending upon how you approached (i). If you again focus on the difference in area across a representative integer range, then you end up having to sum $k + \frac{11}{6}$ from $k = 1$ to $k = n - 1$. Otherwise, there is some integration (for the continuous curves) and some summation (for the integer-part lines) to be done, which may require the use of standard summation results for $\sum k$ and $\sum k^2$.

Answers: (i) $\frac{3}{2}n(n-1)$.

Q6 The two vectors to be used are clearly $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$**.** The inequality arises when you note that $\cos^2\theta \le 1$. The statement is an equality (equation) when $\cos\theta = \pm 1$, in which case the two vectors must be parallel, so that one is a (non-zero) multiple of the other. [The question cites an example of a result widely known as the *Cauchy-Schwarz Inequality*.] The equality case of the inequality is then used in the two following parts; simply in (i) – since we must have $y = z = \ldots..$, from which it follows that $x = \frac{1}{2}$ this. In (ii), you should check that this is indeed an equality case of the inequality when the two vectors are ….. and ….. The parallel condition (one being a multiple of the other) now gives p, q and r in terms of some parameter (say λ), and you can substitute them into the linear equation (of the two given this is clearly the more straightforward one to use), find λ, and then deduce p, q and r; these values actually being unique.

Answers: $x = \lambda a$, $y = \lambda b$ and $z = \lambda c$; (i) $x = 7$; (ii) $p = 24$, $q = 6$, $r = 1$.

Q7 This is a reasonably routine question to begin with. The general gradient to the curve can be found by differentiating either implicitly or parametrically. Finding the gradient and equation of line *AP* is also standard enough; as is setting $y = b$ in order to find the coordinates of Q: $\left(\frac{(1-k)a}{(1+k)}, b\right)$. The equation of line *PQ* follows a similar line of working, to get $y = \left(\frac{-(1-k^2)b}{2ka}\right)x + \frac{b(1+k^2)}{2k}$. If you are not familiar with the $t = \tan\frac{1}{2}$-angle identities, the next part should still not prove too taxing, as you should be able to quote, or derive (from the formula for $\tan(A + B)$ in the formula books), the formula for $\tan 2A$ soon enough; and the widely known, "*Pythagorean*", identity $\text{cosec}^2 A = 1 + \cot^2 A$ will help you sort out the gradient and intercept of *PQ* to show that the two forms of this line are indeed the same when $k = \tan(\frac{1}{2}\alpha)$.

A sketch of the ellipse, though not explicitly asked-for, should be made (at least once) so that you can draw on the lines *PQ* in the cases $k = 0$ and $k = 1$.

Answers: Yes; *PQ* is the vertical tangent to the ellipse.
Yes; *PQ* is the horizontal tangent to the ellipse.

Q8 I'm afraid that this question involves but a single idea: namely, that of intersecting lines. The first two parts are simple "bookwork" tasks, requiring nothing more than an explanation of the vector form of a line equation as $\mathbf{r}$ = p.v. of any point on the line + some scalar multiple of any vector (such as $\mathbf{y} - \mathbf{x}$, in this case) parallel to the line; then the basic observation that $CB \parallel OA \Rightarrow \overrightarrow{CB} = \lambda\mathbf{a}$ to justify the second result.

Thereafter, it is simply a case, with (admittedly) increasingly complicated looking position vectors coming into play, of equating **a**'s and **c**'s in pairs of lines to find out the position vector of the point where they intersect. If the final part is to be answered numerically, then the parameter λ must cancel somewhere before the end.

Answers: (ii) $\mathbf{d} = \left(\frac{1}{1-\lambda}\right)\mathbf{c}$; $\mathbf{e} = \frac{1}{3}\mathbf{a}$ **;** $\mathbf{f} = \mathbf{c} + \frac{1}{2}\lambda\mathbf{a}$; $\mathbf{g} = \left(\frac{2\lambda}{2+3\lambda}\right)\mathbf{a} + \left(\frac{2}{2+3\lambda}\right)\mathbf{c}$; $\mathbf{h} = \frac{2}{5}\mathbf{a}$.

Thus $OH : HA = 2 : 3$ (as H lies two-fifths of the way along the line OA).

Q9 The most important thing you can do on a question like this, is to draw a good, decent-sized diagram first, marking on it all the relevant forces. In fact, since a lot of extra forces come into play in the second part of the question, a completely new diagram here is pretty much essential. It is also helpful to have the painter, P, in a general position on the ladder; say, a distance xa from its base up along it. [Note that xa is so much better than x, so that – since all distances are now multiples of a – these will cancel in the moments equation and make things *look* simpler.] Now resolve twice and take moments (easier about the base of the ladder), and use the *Friction Law* (in its inequality form, since we don't need to know when it attains its maximum). And then sort out the remaining algebra. On this occasion, it is not unreasonable to assume that P is at the top of the ladder when slipping is most likely, and go from there.

In (ii), the extra forces involved are the weight of the table, the reaction forces between its legs and the ground **and** the reaction of the ladder's base on the table (previously ignored when the ladder was on the ground). The standard approach now is to assume that the system is rotationally stable and see when slipping occurs; then to assume that the system is translationally stable and see when tilting occurs. Again, this involves resolving twice and taking moments; using the *Friction Law* – with equilibrium broken when one of the reactions between table and ground is zero – and deciding which, if any, happens first.

Answers: Table slips on ground when P is distance $5a$ up the ladder. Table turns about edge furthest from the wall when P is distance $\frac{11}{3}a$ up the ladder. Thus, tilting occurs first.

Q10 The first two collisions, between A and B and then between B and C, each require the application of the principles of *conservation of linear momentum* (CLM) and *Newton's experimental law of restitution* (NEL or NLR). This will give the intermediate and final velocities of B along with the final velocities of A and C (although the latter is not needed anywhere) in terms of u. [It is simplest to take all velocities in the same direction, along AB, so that "opposite" directions will then be accounted for entirely (and consistently) by signs alone.] For a second collision between A and B, $V_A > V_B$ (irrespective of their signs!) and this leads to a quadratic equation in k. Note that any negative solutions are inappropriate here.

Using $k = 1$ (which presumably MUST lie in the range found previously), the velocities of all particles can now be noted less algebraically. The time between contacts is in two parts: the time for B to reach C, and then the time for A to catch up with B (from its new position when B & C collide). After B leaves C, it is only the relative speed of A and B that matters, and this simplifies the working considerably.

Answers: (i) $0 < k < \frac{3}{2}$.

Q11 The equations of motion in the x- and y-directions can be found by integrating up from accelerations, or by using the *constant-acceleration formulae*. Setting $y = 0$ gives $t = 0$ or $t = \ldots..$ (as usual). Substituting this into the expression for x then gives the distance OA.

In (i), the time when $\dot{x} = 0$ must occur before the time found above. This gives an inequality involving sine and cosine, which can be simplified to give the tangent of the angle required.

In (ii), OB is just OA with $\theta = 45^\circ$. Then OA is maximised either by calculus (a little trickier here) or by using the double-angle formulae for sines and cosines and then working with an expression of the form $a \cos 2\theta + b \sin 2\theta + c$, for which there is a standard piece of work to yield the form $R \cos(2\theta - \emptyset) + c$, which has an obvious maximum of $R + c$ (with R here being in terms of f and g).

For the very last part, $f = g$ with $\theta = 45^\circ$ gives $x = y$ for B's motion, and the particle moves up, and then down, a straight line inclined at 45° to the horizontal, to land at its original point of projection.

Answers: (i) $\alpha = \arctan\left(\frac{g}{2f}\right)$; (ii) answer as above.

Q12 In (i), the probability that one wicket is taken is

$$p(A1 \cap B0 \cap C0) + p(A0 \cap B1 \cap C0) + p(A0 \cap B0 \cap C1),$$

each of which is a product of three terms from a binomial distribution. The probability that it was Arthur who took the wicket is then the conditional probability

$$\frac{p(1,0,0)}{p(1,0,0) + p(0,1,0) + p(0,0,1)}.$$

Although this looks a pretty ferocious creature with all its terms in it, in fact almost all of them cancel in the fraction, and you are left with a few products to deal with (most involving further cancellable terms).

Part (ii) is a "quickie" – $30 \times \left(\frac{1}{36} + \frac{1}{25} + \frac{1}{41}\right)$ – to point you towards the use of the simple value of 3 in the next part. In (iii), since n is large and p is small, the Binomial can be approximated by the Poisson; and $p(W \geq 5) = 1 - \{p_0 + p_1 + p_2 + p_3 + p_4\}$. From here, you can use either the approximation $e^3 = 20$ (as given) and work with Poisson terms directly, or just resort to the use of the Poisson tables in your formula books.

Answers: (i) $\frac{3}{10}$.

Q13 To be honest, this was more of a counting question than anything, at least to begin with. Although it is possible to attack (i) by multiplying and adding various probabilities, it is most easily approached by examining the 24 permutations of {1, 2, 3, 4} individually, and seeing what choice is made in each case. To make life easy for yourself, be systematic in listing these possibilities.

This example should point you in the right direction, but don't be tempted to just write down the answer that you've spotted without any justification for how it arises ***in the general case***. To begin with, deal with what happens when the largest cone is offered first; then the second-largest being first; then the third. By this stage it should be easy to justify the general case as to what happens when the r^{th} largest cone is the first to be offered – then the largest is chosen if it appears first of the remaining (r – 1) cones that are bigger than the r^{th}. With probability …..

Answers: (i) $P_4(2) = \dfrac{7}{24}$; $P_4(3) = \dfrac{4}{24}$ or $\dfrac{1}{6}$; $P_4(1) = \dfrac{2}{24}$ or $\dfrac{1}{12}$;

(ii) $\dfrac{1}{n}\left\{0 + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots.. + \dfrac{1}{n-1}\right\}$ or $\frac{1}{n}\sum_{r=1}^{n-1}\frac{1}{r}$

Q14 For $y = \dfrac{1}{x\ln x}$, $y \to -\infty$ as $x \to 0$ and $y \to 0$ (+ve) as $x \to \infty$ are the obvious asymptotic tendencies of the graph. Since ln 1 = 0, there is also a discontinuity at x = 1, and you must decide what happens to the graph either side of this point.

For the rest of the question, its is essential to be able to integrate $\dfrac{1}{x\ln x}$. This can be done either by the sneaky observation that it can be written in the form $\dfrac{1/x}{\ln x}$, so that the numerator is exactly the derivative of the denominator – a standard log. integral form – or by using a substitution such as $u = \ln x$.

In (i) and (ii), it is now just a case of substituting in the limits and sorting out the log. work. Having gained the answer for (ii), in log. form, the numerical approximation arises from using the first few terms of the series, given in the formula books, for $\ln(1 + x)$ with $x = \ldots..$

In the very final part, a range is given that turns out to be outside the non-zero part of the *pdf*. A little bit of work needs to be done to justify this, and then you can write down the answer.

Answers: (i) $\lambda = \dfrac{1}{\ln\frac{1}{2}}$ or $-\dfrac{1}{\ln 2}$; (iv) 0.

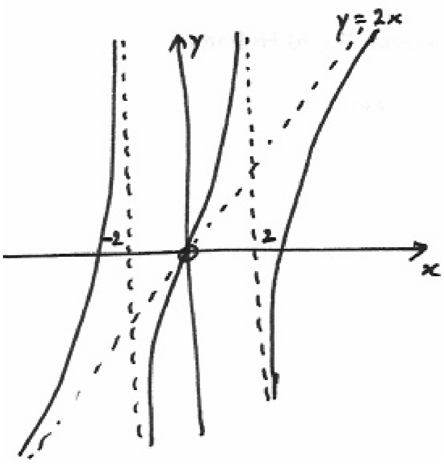
Step III, Hints and Answers
June 2006

<table>
<tr><td>1</td><td>

$y = \frac{2x(x^2-5)}{x^2-4}$

$= 2x - \frac{2x}{(x-2)(x+2)}$

Asymptotes are $y = 2x$, $x = \pm 2$.



$\frac{dy}{dx} = 2 - \frac{2(x-2)(x+2)-4x^2}{(x-2)^2(x+2)^2}$

(or equivalent).

Equation of the tangent at O is

$y = \frac{5x}{2}$.
</td></tr>
<tr><td>(i)</td><td>

$3x(x^2-5) = (x^2-4)(x+3)$

$\Leftrightarrow \frac{2x(x^2-5)}{x^2-4} = \frac{2x}{3} + 2 \quad (x \neq \pm 2)$

$y = \frac{2}{3}x + 2$ cuts the sketched curve in three points, so three roots.
</td></tr>
<tr><td>(ii)</td><td>

$4x(x^2-5) = (x^2-4)(5x-2)$

$\Leftrightarrow \frac{2x(x^2-5)}{x^2-4} = \frac{5x}{2} - 1 \quad (x \neq \pm 2)$.

$y = \frac{5x}{2} - 1$ passes through the intersection of $x = 2$ and $y = 2x$ and is parallel to $y = \frac{5x}{2}$ so just one root.
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<tr><td>(iii)</td><td>

$4x^2(x^2-5)^2 = (x^2-4)^2(x^2+1)$

$\Leftrightarrow \frac{2x(x^2-5)}{x^2-4} = \pm\sqrt{(x^2+1)} \quad (x \neq \pm 2)$

$y = \pm\sqrt{(x^2+1)}$ has two branches with asymptotes $y = \pm x$, so there are six roots.
</td></tr>
<tr><td>2 (i)</td><td>

First "show" by change of variable $\theta = -\phi$ (say).

Then
</td></tr>
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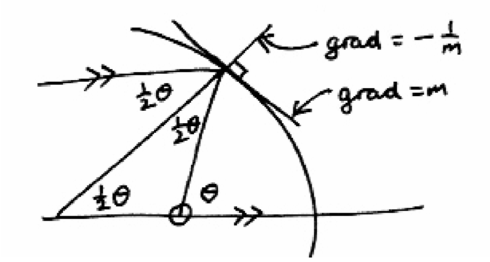
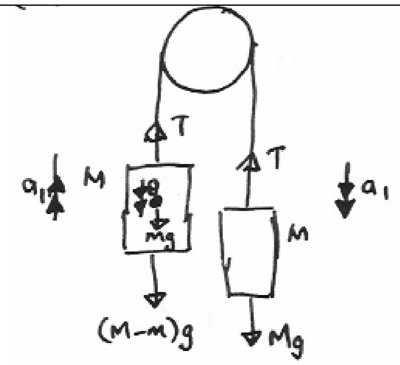
	$2I=\int_{-\pi/2}^{\pi/2}\frac{\cos^2\theta}{1+\sin\theta\sin 2\alpha}d\theta+\int_{-\pi/2}^{\pi/2}\frac{\cos^2\theta}{1-\sin\theta\sin 2\alpha}d\theta$ $=\int_{-\pi/2}^{\pi/2}\frac{2}{\sec^2\theta-\tan^2\theta\sin^2 2\alpha}d\theta$ and next "show" follows.
(ii)	$J=\sec 2\alpha\int_{-\pi/2}^{\pi/2}\frac{1}{1+(\cos 2\alpha\tan\theta)^2}\cos 2\alpha\sec^2\theta d\theta$ $=\sec 2\alpha\int_{-\pi/2}^{\pi/2}\frac{1}{1+u^2}du$ (since $\cos 2\alpha>0$) $=\pi\sec 2\alpha$
(iii)	$I\sin^2 2\alpha+J\cos^2 2\alpha=\pi$. Result follows after use of (ii).
(iv)	In this case, $\cos 2\alpha<0$, so $J=-\pi\sec 2\alpha$. Then $I=\frac{1}{2}\pi\operatorname{cosec}^2\alpha$
3 (i)	$\tan x$ is an odd function. Express both sides in terms of $\tan x$. From identity, substitute series and result follows by equating coefficients of powers of x.
(ii)	Show that $\cot x+\tan x=2\operatorname{cosec}2x$ and follow same method.
(iii)	Identity follows from $1+\cot^2 x=\operatorname{cosec}^2 x$. Equate coefficients to show that all coefficients for even n are zero, and $a_1=1, a_3=\frac{1}{3}$.
4	Let $x=y$ and deduce first result. $2f(x)=f(2x)$ $\Rightarrow 2f'(x)=2f'(2x)$ $\Rightarrow 2f''(x)=4f''(2x)$ then put $x=0$ to get $f(0)=0, f''(0)=0$. Similarly all higher order derivatives are zero, so by Maclaurin the most general function is cx, where c is a constant.
(i)	Use properties of logs to show that $G(x)+G(y)=G(x+y)$. Deduce that $g(x)=e^{cx}$.
(ii)	Show that $H(u)+H(v)=H(u+v)$ so $h(x)=c\ln x$.
(iii)	Let $T(x)=t(\tan x)$. Deduce that $t(x)=c\arctan x$.
5	There are essentially two different configurations, corresponding to clockwise and anticlockwise arrangements of α,β,γ taken in order. In what follows, $\omega=\frac{-1+\sqrt{3}}{2}$, the cube root of unity with modulus 1 and argument $\frac{2\pi}{3}$; $1+\omega+\omega^2=0$ (*) is assumed.

	Then either $\beta-\gamma=\omega(\gamma-\alpha)$ and $\beta-\gamma=\omega^2(\gamma-\alpha)$ expresses equality of adjacent sides and the correct angle between them for each of the two cases; by SAS this establishes an equilateral triangle. These two are equivalent to $[\beta-\gamma-\omega(\gamma-\alpha)][\beta-\gamma-\omega^2(\gamma-\alpha)]=0$. The required form is an expanded version of this, using (*). NB It is essential to be clear that the argument works both ways. If α,β,γ are the roots of the equation given, $-a=\alpha+\beta+\gamma, b=\alpha\beta+\beta\gamma+\gamma\alpha, c=-\alpha\beta\gamma$. Then $a^2-3b=\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha$ so $a^2-3b=0$ is equivalent to the expression in the first part. Result follows. $z\to pw$ is an enlargement combined with rotation, so object and image are similar. $pw\to pw+q$ is a translation so object and image are congruent. Hence under the composition $z\to pw+q$ object and image are similar. Result follows. Aliter. Substitute $z=pw+q$ in the first equation, and simplify. Compare coefficients to determine A and B in terms of a, b and c. Then $a^2-3b=0\Rightarrow A^2-3B=0$, so result follows.
6	$x=r\cos\theta, y=r\sin\theta, r=r(\theta)$ $\Rightarrow \dfrac{dy}{dx}=\dfrac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}$ and result follows.  Gradient of the normal is $\tan\dfrac{\theta}{2}=t$, say. Then we have $t=-\dfrac{\frac{dr}{d\theta}-r\tan\theta}{\frac{dr}{d\theta}\tan\theta+r}, \tan\theta=\dfrac{2t}{1-t^2}$ This reduces to

	$\frac{dr}{d\theta} = rt$ $\Rightarrow \ln r = \int \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} d\theta$ $= -2\ln\left[c\cos\frac{\theta}{2}\right]$ $\Rightarrow \frac{2}{c^2 r} = 1 + \cos\theta$ (using $1+\cos\theta = 2\cos^2\frac{\theta}{2}$) This corresponds to the standard equation of a parabola in polars.
7 (i)	Express sinhx in terms of exponentials, factorise and solve to get $u = -e^{x}$ or $u = e^{-x}$ (or $-\cosh x \pm \sinh x$). Use both of these as equal to $\frac{dy}{dx}$ and integration to get alternative solutions $y = -e^{\pm x} + c.$ From the given conditions the particular integral is $y = 1 - e^{-x}$.
(ii)	Solve the quadratic as before to get either $u = \frac{-1 \pm \cosh y}{\sinh y}$ (or equivalent) $\Rightarrow \frac{dx}{dy} = \frac{\sinh y}{-1 \pm \cosh y}$ $\Rightarrow x = \ln(\cosh y - 1) + c_1$ or $x = -\ln(\cosh y + 1) + c_2$ Only the first can satisfy the conditions $x = 0, y = 0$; then we have $x = \ln\frac{2}{1+\cosh y}$ $\Rightarrow \cosh y = 2e^{-x} - 1$ This is undefined for $x > 0$. For $x \to -\infty \Rightarrow \cosh y \to \infty$, and there will be two branches, corresponding to $y \to \pm\infty$, as cosh is an even function. So $x \to -\infty \Rightarrow \cosh y \to \infty \Rightarrow y \to \infty \Rightarrow e^{y} \approx 4e^{-x} \Rightarrow y = -x + \ln 4$ in one case, and similarly $y = x - \ln 4$ in the other.

8	Use (iv) with $f(x) \equiv 1, g(x) \equiv 1$ to show that $\Delta 1 = 0$. Use (iii) with $\lambda \equiv k, f(x) \equiv 1$ to show that $\Delta k = 0$. By (iv), (i) $\Delta x^2 = 2x$; ditto $\Delta x^3 = 3x^2$. Now show $\Delta kx^n = knx^{n-1}$ by induction. Initial step is $\Delta k = 0$; inductive hypothesis is that $\Delta kx^N = kNx^{N-1}$. Use (iii) and (iv) with hypothesis to show that $\Delta kx^{N+1} = k(N+1)x^N$. Now express any $P_k(x)$, a polynomial of degree k, as a sum of such powers, and so use (ii) to establish required result.
9	Take O as the zero level for potential energy. Then PE of bead at B is mgy; PE of particle at P is $mgr - mgl$. For perpetual equilibrium, the PE must have the same value in any position, in particular its value at H; result follows. Express equation shown in polar coordinates to get $r = \dfrac{2h}{1+\sin\theta}$ Differentiate and make $\dot{\theta}$ the subject so $\dot{\theta} = -\dfrac{\dot{r}(1+\sin\theta)^2}{2h\cos\theta}$. These two expressions give the desired result. By conservation of energy if PE is constant so is KE. Hence KE in a general position is equal to the initial value. That gives $V^2 = \left(r\dot{\theta}\right)^2 + 2\dot{r}^2$ Speed of the particle at P is $\left\|\dot{r}\right\|$. Use the expressions for $V^2, \dot{\theta}$ to derive the required result.
10	Use conservation of angular momentum for the first result. Use conservation of energy to derive $v^2 = \dfrac{k^2+a^2}{k^2}\Omega^2 - (k^2+r^2)\omega^2$ and so by use of the first result and $v = -\dfrac{dr}{dt}$ second result follows. Now use $\omega = \dfrac{d\theta}{dt}$ and $\dfrac{dr}{d\theta} = \dfrac{dr}{dt}\Big/\dfrac{d\theta}{dt}$ and the two displayed result to derive the third. The suggested substitution transforms the third displayed equation to $\dfrac{du}{d\theta} = \sqrt{1+u^2}$. Invert and integrate to get the desired result. Hence $r = \dfrac{k}{\sinh(\theta+\alpha)}$. As $\theta \to \infty, r \to 0+$, but $r = 0$ is impossible.

11



The equations of motion are

$$T-(M-m)g=(M-m)a_1$$

$$Mg-T=Ma_1$$

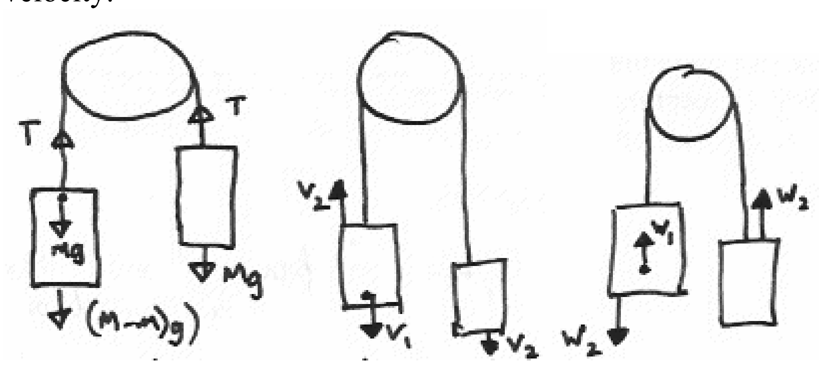
$$\Rightarrow a_1=\frac{mg}{2M-m}$$

Now consider relative motion of the tile with acceleration $(g+a_1)$.

If the time of the first stage is t_1, $s=ut+\frac{1}{2}at^2$ gives

$$t_1=\sqrt{\frac{(2M-m)h}{Mg}}$$

and then for the absolute motion of the tile $v=u+at$ gives the required final velocity.



The middle diagram shows the situation before the impact and the third after. The forces acting on the left-hand system (lift plus tile) are exactly the same as those on the right, so the changes in momentum must be equal in the first stage of the motion. Thus given that all is stationary initially

$$-(M-m)v_2+mv_1=Mv_2$$

$$\Rightarrow v_2=\frac{m}{2M-m}v_1=\alpha v_1 \text{ (*), say.}$$

In the collision, the equality of impulsive tensions given means that the change in momentum on one side equals change in momentum on the other. Hence we have

$$-Mw_2-Mv_2=-mw_1-mv_1+(M-m)(w_2+v_2)$$

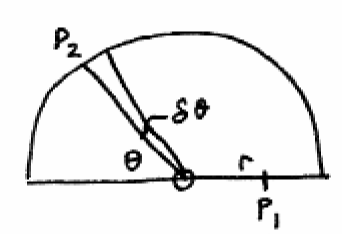
$$\Rightarrow w_2+v_2=\alpha(w_1+v_1)$$

Thus from the two last equations

$w_2=\alpha w_1$ (**).

Newton's experimental law and the two asterisked equations give

$w_1=ev_1$.

	Then the change of energy in the collision $\frac{1}{2}(2M-m)(v_2^2-w_2^2)+\frac{1}{2}m(v_1^2-w_1^2)$ simplifies to the required expression when the above relations are substituted. Loss of energy of a tile dropping to the floor of a fixed lift and bouncing would be just the same.
12	Model each tourist as trial with success probability ½. If X is the number of potential passengers $X \sim Bin(1024,\frac{1}{2})$, ie $N(512,16^2)$ approximately. Lost profit corresponds to $X>480$. Hence if L is the loss, we have $E[L]=\sum_{k=1}^{32} kpr(X=480+k)+32pr(X>512)$ $=\sum_{k=1}^{32} k.\frac{1}{16}.\phi\left(-2+\frac{k}{16}\right)+16$ $\approx \int_0^{32}\frac{x}{16}\phi\left(-2+\frac{x}{16}\right)dx+16$ $=\int_0^{32}\frac{x}{16}\cdot\frac{1}{\sqrt{2\pi}}\cdot\exp\frac{(x-32)^2}{512}dx+16$ Now use substitution to show that this evaluates to $\frac{16}{\sqrt{2\pi}}\left(e^{-2}-1\right)+32\Phi(2)$. In the course of year the expectation is 50 times that figure, so that is the maximum tolerable licence fee.
13	 There are three cases to consider: (i) both on the circumference, (ii) P_1 on the diameter and P_2 on the circumference, and (iii) vice versa. For case (i), if P_1 lies in the arc $(\alpha,\alpha+\delta\alpha)$ P_2 lies in the arc $(\theta,\theta+\delta\theta)$, with probability $\frac{\delta\theta}{\pi+2}$, the area is $\frac{1}{2}\lvert r\rvert\sin\theta$. The expected area given P_1 lies in the arc $(\alpha,\alpha+\delta\alpha)$ is by integration $\frac{1}{\pi+2}$. For case (ii), if P_1 lies in $(r,r+\delta r)$ and P_2 lies in the arc $(\theta,\theta+\delta\theta)$, with probability $\frac{\delta\theta}{\pi+2}$, the area is $\frac{1}{2}\lvert r\rvert\sin\theta$. The expected area given P_1 lies in $(r,r+\delta r)$ from O is by integration $\frac{\lvert r\rvert}{\pi+2}$. Case (iii) is essentially the same as case (ii). Thus the expected area is $\int_0^{\pi}\frac{1}{\pi+2}.\frac{1}{\pi+2}d\alpha+2\int_{-1}^{1}\frac{\lvert r\rvert}{\pi+2}.\frac{1}{\pi+2}dr$ where the first integral corresponds to case (i) and the second to (ii) and (iii).

	This evaluates to the answer given.
14	$E[aX_1+bX_2]=aE[X_1]+bE[X_2]$ $E[X_1X_2]=E[X_1]E[X_2]$ $E[P]=2\mu_1+2\mu_2$ $E[P^2]=4E[X_1^2]+8E[X_1]E[X_2]+4E[X_2^2]$ $E[X_1^2]=\mu_1^2+\sigma_1^2$ $\text{var}[P]=4(\sigma_1^2+\sigma_2^2)$ The standard deviation is the square root of that expression. $E[A]=\mu_1\mu_2$ $E[A^2]=\mu_1^2\mu_2^2$ $\text{var}[A]=\sigma_1^2\mu_2^2+\sigma_2^2\mu_1^2+\sigma_1^2\sigma_2^2$ Again the standard deviation is the square root. Now find $\text{cov}[P,A]=2\mu_2\sigma_1^2+2\mu_1\sigma_2^2$ This is not zero (as independence would imply) with given conditions. Similarly $\text{cov}[Z,A]=2\sigma_1^2\mu_2+2\sigma_2^2\mu_1-\alpha(\mu_1^2\sigma_2^2+\mu_2^2\sigma_1^2+\sigma_1^2\sigma_2^2)$ That too is non-zero when α is not the excluded value. We consider the exceptional case with the given information. We have $\mu_1=\mu_2=2, \sigma_1^2=\sigma_2^2=1, \alpha=\frac{8}{9}$. Only three values of *A* are possible - 1, 3 and 9 - and they correspond to unique values of *Z*. Dependence can be shown by considering, for example, $pr\left(Z=\frac{28}{9}\right)=\frac{1}{4},\ pr\left(Z=\frac{28}{9}\middle\|A=3\right)=0$.

Report on the Components
June 2006

REPORT ON THE COMPONENTS

9465 – Mathematics I

General comments

This paper was found to be more difficult than last year's; somewhat worryingly, this was perhaps because the paper placed a greater emphasis on algebraic and numerical manipulation than previously. At this level, the fluent, confident and correct handling of mathematical symbols (and numbers) is necessary and is expected; many good starts to questions soon became unstuck after a simple slip. The applied questions appeared to be beyond many candidates; it has been suggested that this reflects the reduction of the amount of applied mathematics in single maths A-level.

There were of course some excellent scripts, but the examiners were left with the overall feeling that some candidates were not ready to sit the examination. The use of past papers to ensure adequate preparation is strongly recommended. A student's first exposure to STEP questions can be a daunting, demanding experience; it is a shame if that takes place during a public examination on which so much rides.

Comments on individual questions

1 Some candidates could not square correctly a three digit number; of those who did, not all recognised that $184^2 - 33127 = 27^2$. This was intended to be a straightforward "warm-up" question, but it was not to be found to be so.

2 The key word in this question was "prove", but most candidates *assumed* that the goat would graze the maximum area if it were tethered to a corner of the barn, and the minimum if tethered to the middle of a side. This unjustified assumption severely reduced the awarded marks. Candidates are advised to ensure they understand what, at this level, is required by an instruction to *prove* a result.

3 Parts (i) and (ii) were reasonably well done, but very few successful attempts to (iii) were seen. Most candidates had not realised that they were supposed to be thinking about the graph of $y = x^3 + px + q$, in particular its stationary points and its y-intercept.

4 The two graphs were well drawn, although sometimes the horizontal scale was in degrees. The area formula was often derived correctly, but the subsequent differentiation often contained a major error, most commonly a failure to apply the chain rule when differentiating tan(□ / n). Most candidates found it hard to construct a coherent argument (using part (ii)) about the ratio of the polygon to the circumcircle.

5 This integration question was tackled much more successfully than last year's. It was particularly pleasing to see how many candidates were able to cope with the unusual partial fractions that arise in part (ii); some imaginative methods were seen.

6 This was a popular, straightforward question, which was often answered very well. However, algebraic errors still occurred – even when expanding
$(3a + 4b)^2$.

7 Only a few candidates saw the connection between the two halves of part (i), and therefore most evaluations of the definite integral failed to remove the modulus of the integrand correctly. Interestingly, part (ii) was often found more straightforward.

8 This was a well-answered question (using either vectors or the cosine rule), although some candidates tried to derive the result about d by using the vector equation of the plane ABC: the instruction "hence" required the use of the first two parts of the question. Candidates should ensure that they understand the distinction between "hence" and "hence or otherwise".

9 It was not thought that this would be a difficult question, but many candidates were unable to model correctly the motion of three connected particles. A common error was to consider $xg - yg$, the resultant force acting on the whole system, but to divide it by $x + y$ rather than $x + y + 4$ when calculating the acceleration.

10 Very few attempts at this question were seen, and those that did rarely progressed beyond the first paragraph.

11 Hardly any attempts at this question were seen, but those candidates who did tackle it were usually able to produce a mostly accurate solution. It was remarkable how few diagrams, not to mention labelled diagrams, were seen: it is always much easier for both the candidate and the examiner if symbols are clearly defined in a diagram.

12 Many different (and correct) arguments were seen to the first part: candidates' careful analysis of the different possibilities was encouraging. Unfortunately, hardly any candidates recognised that the second part of the question was asking for a conditional probability. This prompted the examiners' concern that candidates were too reliant on a verbal clue such as "given that", and found it very hard to identify the inherently conditional structure of an event such as was described in this question.

13 Very few attempts at this question were seen, although it was not expected to be popular since it was known that some candidates would not have studied the Poisson distribution. However, knowledge of it (and the Normal distribution) remains in the published specification, and so candidates may wish to ensure that they are familiar with both of these.

14 Part (i) was well answered by most of those who attempted it. Solutions to part (ii) often began with a correct product of fractions, but it was surprising how often factorials were employed in an (unsuccessful) attempt to simplify an expression that cancelled down very easily to $1/(n + 1)$. The implicit fact that $n + 1 \geq r$ was not often realised, leading to "$n = 0$" as the modal answer.

9470 – Mathematics II

General comments

This was an accessible paper, with up to half the marks on each question available to candidates of a suitable potential, The candidature represented the usual range of mathematical talents, with a goodly number of truly outstanding students, many more who were able to show insight and flair on some of the questions they attempted, and (sadly) a significant number of students for whom the experience was not to prove a particularly profitable one. Of the total entry of nearly 700, around 40% were awarded grade 1's (or better), while only around 20% received an unclassified grade.

Really able candidates generally produced solid attempts at six questions, while the weaker brethren were often to be found scratching around at bits and pieces of several questions, with little of substance being produced. In general, few candidates submitted serious attempts at more than six questions – a practice that is not to be encouraged, as it uses valuable examination time to little or no avail. It is, therefore, important for candidates to spend a few minutes at some stage of the examination deciding upon their optimal selection of questions to attempt.

As a rule, question 1 is intended to be accessible to all takers, with question 2 usually similarly constructed. In the event, at least one – and usually both – of these two questions were among candidates' chosen questions. Of the remaining selections, the majority of candidates supplied attempts at the questions in Section A (Pure Maths) only. There were relatively few attempts at the Applied Maths questions in Sections B & C, with Mechanics proving by far the more popular of the two options. Question 10, in particular, was relatively popular. Overall, there were remarkably few efforts submitted to the Statistics questions in Section C, although several of these were of exceptional quality.

On a more technical note, many solutions to those questions which were not already quite structured suffered a lamentable lack of clearly directed working. Large numbers of candidates would benefit considerably from the odd comment to indicate the direction that their working was taking. This was especially the case in questions 3, 5, 10 and 13, where it was often very difficult for examiners to decide what candidates were attempting to do, and where they had gone wrong, without any clear indication as to what they themselves thought they were doing.

Comments on individual questions

1 Almost all candidates attempted this question and most managed at least some measure of success; although the high level of algebra required to see matters through to a successful conclusion proved to be a decisive factor in whether attempts got much over half-marks. A minority of candidates worked with u_n and u_{n+r} (for the appropriate r's) and thereby made the algebra rather harder for themselves; whereas it had been intended that they should work with u_1 (with the given value of 2) and the appropriate u_r in order to determine periodicity. The other major problem arose when candidates worked backwards from (say) u_5 towards u_1, rather than forwards. This often generated nested sets of bracketed expressions of the form

$$u_5 = k - \cfrac{36}{k - \cfrac{36}{k - \frac{36}{k - \dots}}}$$

which only the hardiest were able to unravel successfully; while a forwards approach would have found each of $u_2, u_3, \ldots$ successively as much simpler (rational) terms.

Another common error arose when candidates failed to note that, if $k = 20$ gives a constant sequence, then, for a sequence of period 2, the answers "$k = 20$ and 0" can't both be correct. Similarly, for a sequence of period 4, the values 0 and 20 should appear as possible solutions when equating u_5 to u_1, but should be discounted. Whilst many candidates noted these points – and some shrewdly used their existence to help factorise the arising quartic equation in k – it is still clearly the case that a large proportion of A-level students, even the better ones, are happy to assume that any solution to an equation they end up having to solve is valid, irrespective of the context of the underlying problem or the logic of their work (viz. *necessary and/or sufficient conditions*).

Although only the most basic of arguments was required to establish that $u_n \geq 2$ at the beginning of part (ii), it was clear that most candidates were really not comfortable handling inequalities, and lacked practice in constructing reasonable mathematical arguments. Far too many failed to work generally at all, and simply showed that the first few terms were greater than or equal to 2, concluding with a waffle-y "etc., etc., etc." sort of argument. In the very last part, it was important to appreciate that a limit is approached when successive terms effectively become the same. No formal work beyond this simple idea was required, and the resulting quadratic gave two solutions, only one of which was greater than 2. Rather a lot of candidates were happy with this idea and rattled it through very quickly.

2 This question was the second most popular on the paper (in terms of the number of attempts) and really sorted out those who were comfortable with inequalities from those that weren't. Those who were generally scored very high marks on the question; even those who weren't generally managed several bits and pieces to get around half-marks on it.

Once again, there was an informal (possibly induction-type) proof required for the second bit of the question, although this was handled slightly more capably than the easier one in Q1, possibly because so many candidates seemed happier to effectively produce a formally inductive line of reasoning. Most candidates then picked up on the purpose of this bit in helping create a convergent GP to sum, which helped establish the next inequality for e.

The differentiation proved undemanding, and most candidates managed to realise that the minimum and maximum points referred to would be established by considering the sign of $\frac{dy}{dx}$ at $x = \frac{1}{2}$, 1 and $\frac{5}{4}$. Rather fewer were entirely happy to use the given bounds on e to help them do so, with many going off to lengthier (although often equally correct) workings-out. (In the final part, the use of $e < 3$ would have done the trick.) Those candidates who used approximations rather than inequalities were missing the point, as were those who tried to use $\frac{d^2y}{dx^2}$ without actually knowing the exact values of x which they could use in it.

3 A lot of candidates made a faltering start to this question before moving on to pastures greener. This was usually occasioned by a realisation that life was going to be very tough here – which it was if they failed to appreciate that $\frac{1}{5+\sqrt{24}} = 5 - \sqrt{24}$. Those who saw this early on generally made their way to at least the first 8 marks. Although there are other

ways to go about the first part, the use of the binomial theorem, with the $\sqrt{24}$-bits all cancelling out, establishes that the given expression is indeed an integer (without necessarily having to find out which). The three modest inequalities that followed were easily established with just a modicum of care. However, it was again the case that candidates' lack of comfort with inequalities once more prevented a convincing conclusion to (i) since most candidates resorted to approximation: showing that $N \approx 9601.9999$ is NOT the same as showing that, because N lies between … and … , it is actually equal to it (to four decimal places). Sadly, most candidates did not seem to understand such a difference in logical terms.

For part (ii), it was necessary only to mimic the work of part (i) but in a general setting. Most candidates attempting this question were happy to leave it at this point; of those who continued, many picked up two or three marks – only a handful actually polished it off properly.

4 Another difficult start again put most candidates off this question at the outset (if not before) and there were relatively few efforts at it. Most of these were pretty decent and scored well. The use of the initial result in (i) was straightforward, provided one is prepared to spot a decent substitution (such as $c = \cos x$). The formula books then helped bypass the integration required. In (ii), the given integral splits into the answer to (i) + a second integral, which must be considered separately. A simple linear substitution helped here, although quite a few candidates incorrectly assumed a result over the interval $(\pi, 2\pi)$ similar to the given one could just be assumed to hold. This was often the case in (iii) also, although fewer candidates tried such a move: the $\sin(2x)$ forcing them to consider more sensible approaches, such as (again) a linear substitution (after using the double angle formula for sine).

5 Despite the introduction of a non-standard function – often called the *floor* or the *INT* function – this was a popular question to attempt. As mentioned earlier, finding the areas required candidates to structure their working and, since there are several ways to break up the bits of the process, a teensy-weensy bit of explanation would have been greatly appreciated by the examiners. The easiest approach to the area in (i) is to work straightaway with the difference $(y_1 - y_2)$ which immediately gives a whole load of "unit triangles" to sum. Attempts varied from excellent-and-concise all the way down to scrambled-heap-of-integrations-and summations. Part (ii) was handled similarly, although it is strange to say that – despite the slightly greater degree of care needed with the various bits and pieces – there were slightly more correct answers arrived at here.

6 In hindsight, it might have been more generous to have included an "or otherwise" option to the very opening part of this question, as many candidates – particularly overseas ones – preferred an algebraic approach to obtaining the given result, rather than the vector one asked-for. It does, however, illustrate a pretty important examination point: namely, that if you don't actually answer the question that has been asked, you may not actually get any marks for your time and effort! These candidates reduced the given inequality to

$$(bx - ay)^2 + (cy - bz)^2 + (az - cx)^2 \geq 0,$$

and this represents some pretty decent mathematics. It is also very easy to deduce when equality holds in the result from this alternative statement. Such candidates were able to get the remaining sixteen marks on the question, however.

Part (i) didn't actually require candidates to use the given result to solve this quadratic equation, but those who did were guided towards the helpful notion of considering the equality case of the given result, which was intended to help them approach part (ii). [The question cites an example of a result widely known as the *Cauchy-Schwarz Inequality*.]

Overseas candidates apart, this was not a very popular question at all. Those who attempted it generally did quite well, and a surprisingly high proportion of them saw it through right to the end.

7 This proved to be a relatively popular choice of question, usually being pretty well-done, at least up to the point where trig. identities came into play, and often all the way through. It is suspected that the principal reasons for this were that the question had a fairly routine start, and then developed in a fairly straightforward A-level manner thereafter.

Most attempts established the opening result easily enough, and also managed to acquire Q's coordinates without much difficulty, and usually the equation of the line PQ also. A common shortfall at the next stage was not so much the introduction of the trig., which clearly put some candidates off, but rather the use of the trig. to show that the two lines were the same when these identities were used. A very surprising number of candidates seemed content to suggest that the two forms of the line were the same **on the basis of their gradients only**.

Those who got as far as the last part usually handled it very capably, showing that the two cases led to PQ being the vertical and horizontal tangents (respectively) to the ellipse.

8 Clearly vectors weren't a popular choice for candidates, as there were very few attempts made at this question. The first six marks, however, are gifts and almost all attemptees gained these. Thereafter, it is simply a case, with (admittedly) increasingly complicated looking position vectors coming into play, of equating **a**'s and **c**'s in pairs of lines to find out the position vector of the point where they intersect. Candidates' efforts tailed off fairly uniformly as the question progressed, and examiners cannot recall anyone actually getting to the end and finding **h** (the p.v. of H) correctly, although there were several attempts that gained all but the final two marks.

9 These leaning-ladder questions are actually pretty standard, and it was disappointing to see so few attempts made at this one. More disappointing still was the lack of a decent diagram from which candidates might have been able to extract some support for their working. Similar dismay was evoked by the widespread inability, on the part of almost all candidates, to be able to say what mechanical principles they were attempting to use at any stage of their working. Of the relatively small number of attempts seen, most suffered from at least one of these deficiencies. Consequently, although there were many partially or totally successful attempts at (i), the number of even half-decent attempts at (ii) were very few. The extra forces that needed to be considered in (ii) were either overlooked completely, or were missing from (i)'s diagram that candidates were trying to re-use.

The other painfully obvious shortfall here lay in candidates' dislike of using the *Friction Law* in its more general, inequality, statement rather than in the equality case given by limiting equilibrium. Such a shortfall was overlooked, even when it wasn't explained correctly (although it contributed substantially to problems in part (ii), when working was to be found). Those making a stab at (i) usually managed to make correct statements from resolving and taking moments, although arguments putting everything together and explaining why the ladder was stable were often less than entirely satisfactory.

10 The most popular of the three Mechanics questions, and generally the best done. Even so, marking was often made unnecessarily difficult by candidates' failure to explain what was going on and/or simplify their working at suitable stages in the proceedings. Setting up and finding the post-collision velocities of the various particles was relatively straightforward – although the algebra did prove too demanding for quite a few candidates – and most attempts correctly indicated the condition required to give a second collision between A and B. The number of unsuccessful attempts to solve the resulting quadratic was a surprise – most presumably faltering due to the lack of a unit x^2 term! – as was the number who preferred to use the quadratic formula rather than factorisation.

Problems generally arose here in part (ii), where a lack of explanation was a big problem. Those candidates who simply work out times and distances, without saying what they are supposed to be, do themselves no favours, as it is very difficult for the examiners to give credit to the working until a coherent strategy has emerged. Any error, no matter how small – and especially those made by candidates working "in their heads" – can render it almost impossible to spot such a strategy and reward it. On a more fundamental level, part (ii) should have opened up with the statement of the three relevant velocities, given in terms of u, using $k = 1$. Most efforts made mistakes because this simple task was left until much later on in the working, and some candidates even insisted on working with a general k throughout.

11 It was felt by examiners that this was the nicest (and easiest) of the three Mechanics questions, yet it drew very few serious attempts from the candidature. Most serious efforts coped very easily with the first two parts. Thereafter, it was often the case that maximising OA proved to be a greater difficulty than it should have done, despite the fact that the option to use calculus was available (although much less concise an approach than using a trigonometric one). There had been concerns that, for the final part, candidates might not grasp what was going on but, happily, this proved not to be the case and several candidates spotted the significance of having $f = g$ **and** described the resulting motion adequately.

12 This was the least popular of the Statistics questions, even amongst the relatively small number of attempts at any Section C questions. Of those seen, examiners can recall only two which got the answer of $\frac{3}{10}$ in (i). This was due to the almost total lack of appreciation that the result "1 wicket taken" required three probabilities.

The clear guidance towards the use of a Poisson distribution in (ii) and (iii) was, however, picked up by candidates. The calculation of the ensuing probabilities, either directly or via tables, was actually very straightforward, and candidates coped very easily when they ventured this far.

13 To be honest, this was more of a counting question than anything, at least to begin with, and several candidates picked up relatively large amounts of marks for very little working. Whilst several attacked (i) by multiplying and adding various probabilities, it was possibly most easily approached by looking at the 24 permutations of {1, 2, 3, 4} individually. Those candidates who adopted a *mix-'n'-match* approach without explanation often got themselves into a bit of a muddle, but still picked up several of the marks available here.

The example provided by (i) was intended to help direct candidates' thinking in (ii) as well as give them with a non-trivial case to use as a check. Of the attempts received, many explained things very poorly, even when they arrived at the correct expression. Sadly, rather too many seemed to deduce the (correct) answer on the basis of (i)'s example alone, and seemed unable to grasp that anything needed to be explained or justified.

14 This was a relatively popular choice of question, perhaps partially because it started off with a couple of bits of Pure Maths: namely, curve-sketching and integration. Strangely, though, very few sketches were fully correct, even when followed-through by "reciprocating" a correct sketch of $y = x \ln x$.

Further progress was going to be impossible without integrating $\frac{1}{x \ln x}$, and some attempts fell at this hurdle. Pleasingly, several candidates spotted the log. form immediately, while many others correctly used the substitution $u = \ln x$, or equivalent.

Thereafter, it was a routine statistical exercise in some respects. However, the log. work required to simplify matters in (i) proved beyond rather too many candidates – whereas it proved much less of a difficulty in (ii). Only a few candidates realised that there was a standard series expansion ready to hand for $\ln(\frac{4}{3})$, and those that did generally only went up to the cubed term, which was a shame as the given answer arose from using the next one as well.

The final twist, in part (iv), of giving a range that turned out to be outside the non-zero part of the *pdf*, was twigged by slightly more than half of the candidates that got this far.

9475 - Mathematics III

General comments

Candidates found the paper in general hard. Nevertheless they seemed aware of the importance of depth and quality of answers rather than disposed to do lots of bits. There were disproportionately small numbers of attempts on applied questions, particularly as regards the statistics section. Questions 1 and 7 were very popular. It is pleasing to see that there is a core of candidates who are well up to the challenge and who produce pleasing solutions, and many others who make sustained attempts to rise to it.

Comments of individual questions

1 Many candidates attempted algebraic approaches to the later parts instead of relating the problems to the graph.

2 This question required sustained and confident technical ability. Level of completion reflected the levels of those abilities. The last part required a more subtle appreciation of how integration by substitution works than is usual.

3 This question required a repeated and systematic use of trigonometric identities and their derivation together with the technique of equating coefficients. It tested the ability of candidates to work in a systematic and accurate fashion.

4 This question started with the use of function notation together with an appreciation of the chain rule. This would be unfamiliar to many candidates but at the same time accessible to those who had understood the chain rule well. The later parts required confident use of function notation in a creative way.

5 There were many different approaches to this question; the key to their acceptability is the extent to which they can be justified as sufficiently general. It is clearly not acceptable to illustrate the result with one or more special cases only. It was also important to note the "if and only if" phrase in the question. Those who managed to reach beyond the first proof produced different approaches to the second part, particularly with a preference for the algebraic approach.

6 The lack of structure in this question meant that there were again varied approaches. The solution required some degree of confidence at interrelating elementary geometry with calculus in the less familiar context of polar coordinates. Note that the question requires a solution which starts with a geometrical property and finishes with a parabola, and not the other way.

7 This was a very popular question, with many examples of sustained and correct work. The solution was ambiguous at every stage with a heavy premium on balancing the options with the appropriate correct choice at each stage. It tested candidates robustness very well.

8 This question was only accessible to those who were able to take seriously the required justification in terms of the rules given at each stage of the solution. The last part can be approached in several different variations of the same idea, and did require a careful

construction of an inductive proof from the several elements of the first part. Again it was a test of sustained thought.

9 The first part depends on the intuition that potential energy must be constant if the system is in equilibrium in all positions. The remainder of the question can be attempted without that by assuming the established result, but it makes its return in the complementary intuition that kinetic energy must also be constant by the conservation law.

10 This question is reasonably routine to those candidates who are conversant with standard rigid body results and are confident with integration..

11 The first part requires two separate equations of motion to establish the acceleration of the connected parts; this is standard work. The second extends the usual situation to take in impulsive tensions. The “given” assumption can be justified but is provided as a hint.

12 This question requires the build up of a suitable model with appropriate approximation. This has to be followed through by converting a discrete sum into an approximate integral.

13 This question is another with no structuring. It requires an analysis that breaks the problem into separate parts, which must be recombined at the end.

14 This was a long but straightforward question for those who could handle their definitions with confidence.

STEP Mathematics (9465/9470/9475)
June 2006 Assessment Series

Unit Threshold Marks

Unit	Maximum Mark	S	1	2	3	U
9465	120	82	67	50	38	0
9470	120	85	60	49	31	0
9475	120	80	60	49	31	0

The cumulative percentage of candidates achieving each grade was as follows:

Unit	S	1	2	3	U
9465	7.7	21.3	44.2	68.1	100.0
9470	12.9	41.4	57.2	82.4	100.0
9475	12.2	38.6	59.2	78.7	100.0

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