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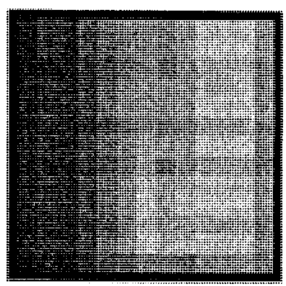
SIXTH TERM EXAMINATION PAPERS (STEP)

MATHEMATICS I, II AND III
2004 EXAMINATION
HINTS AND ANSWERS

STEP



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Section A: Pure Mathematics

1 (i) The quickest method is to use Pascal's triangle:

$$\left(3+2\sqrt{5}\right)^3 = 27 + 3\,(3)^2\left(2\sqrt{5}\right) + 3\,(3)\left(2\sqrt{5}\right)^2 + \left(2\sqrt{5}\right)^3$$

$$= 27 + 54\sqrt{5} + 180 + 40\sqrt{5} = 207 + 94\sqrt{5}$$

but for a small power such as 3 it is easy to compute $\left(3+2\sqrt{5}\right)^2 = 29 + 12\sqrt{5}$ and then multiply the answer by $3+2\sqrt{5}$. You might like to consider how to calculate e.g. $\left(3+2\sqrt{5}\right)^{11}$ efficiently, using the fact that $u^{11} = \left(\left(u^2\right)^2\right)^2 \times u^2 \times u$.

(ii) If $\sqrt[3]{99-70\sqrt{2}} = c - d\sqrt{2}$ then $99 - 70\sqrt{2} = \left(c - d\sqrt{2}\right)^3$.

By Pascal's triangle $\Rightarrow 99 - 70\sqrt{2} = c^3 - 3c^2d\sqrt{2} + 3c\left(d\sqrt{2}\right)^2 - \left(d\sqrt{2}\right)^3$.

Equating rational and irrational coefficients $\Rightarrow c^3 + 6cd^2 = 99$ and $3c^2d + 2d^3 = 70$. These equations are hard to solve algebraically, but the question tells you that c and d are **positive integers**, and since you can see that $c^3 < 99$ and $2d^3 < 70$ there aren't many values of c and d to try. You should quickly find that $c = 3$ and $d = 2$: remember to check that these values satisfy both equations.

(iii) Using the quadratic formula $x^3 = \dfrac{198 \pm \sqrt{198^2 - 4}}{2}$.

Notice how the discriminant is the difference of two squares:

$198^2 - 2^2 = (198-2)(198+2) = 196 \times 200 = 4 \times 49 \times 2 \times 100 = (2 \times 7 \times 10)^2 \times 2.$

$$\Rightarrow x^3 = \frac{198 \pm (2 \times 7 \times 10)\sqrt{2}}{2}$$

$$\Rightarrow x^3 = 99 \pm 70\sqrt{2}$$

$\Rightarrow x = 3 \pm 2\sqrt{2}$ using the answer to part (ii).

2 (i) Your graph should look like a staircase: if $0 \leqslant x < 1$ then $\sqrt{[x]} = 0$; if $1 \leqslant x < 2$ then $\sqrt{[x]} = 1$; if $2 \leqslant x < 3$ then $\sqrt{[x]} = \sqrt{2} \approx 1.4$; if $3 \leqslant x < 4$ then $\sqrt{[x]} = \sqrt{3} \approx 1.7$ etc.

Since a definite integral can be interpreted as the area between a graph and the x axis, it can be evaluated by adding up the areas of the rectangles under the graph, therefore

$$\int_0^a \sqrt{[x]}\,\mathrm{d}x = \left(1 \times \sqrt{0}\right) + \left(1 \times \sqrt{1}\right) + \left(1 \times \sqrt{2}\right) + \ldots + \left(1 \times \sqrt{a-1}\right) = \sum_{r=0}^{a-1} \sqrt{r}$$

(ii) A graph would be very helpful, as it will show the rectangles which need to be summed:

$$\int_0^a 2^{[x]}\,\mathrm{d}x = (1 \times 1) + (1 \times 2) + (1 \times 4) + \ldots + \left(1 \times 2^{a-1}\right).$$

This is the sum of a geometric progression, and so the definite integral $= 2^a - 1$.

(iii) If a is positive but not an integer, then the graph of $y = 2^{[x]}$ from 0 to a will be a staircase of rectangles each of width 1 from 0 to $[a]$, and then a thinner rectangle of width $a - [a]$. The rectangles of width 1 have total area $2^{[a]} - 1$ using your answer to part (ii), and the final rectangle has height $2^{[a]}$ and so has area $(a - [a]) \times 2^{[a]}$. Therefore, the definite integral $= \left(2^{[a]} - 1\right) + (a - [a]) \times 2^{[a]}$.

3 (i) If you substitute $x = 3$ into the given expression, it equals $27 - 45 + 18y + 3y^2 - 24y - 3y^2 + 18 + 6y = 0$, so by the Factor Theorem $x - 3$ is a factor of it.

If you expand $(x-3)(x+ay+b)(x+cy+d)$ you should get $x^3 + (a+c)x^2y + acxy^2 + (bc+ad-3a-3c)xy + (b+d-3)x^2 + (bd-3b-3d)x - 3(a+c)xy - 3acy^2 - 3bd$. Then equating coefficients:

$a + c = 2$

$b + d - 3 = -5$

$ac = 1$

$bd = 0$

$bc + ad - 3a - 3c = -8$

$bd - 3b - 3d = 6$

$bc + ad = -2$

The first and third equations tell you that $a = c = 1$ (since you are given that a, b, c and d are integers), and the second and fourth equations tell you that $b = 0,\ d = -2$ or $b = -2,\ d = 0$. Therefore, the given expression factorises as $(x-3)(x+y-2)(x+y)$.

A neat idea is to factorise the given expression (by division or inspection) as
$(x-3)\left(x^2 + y^2 + 2xy - 2x - 2y\right)$
and then solve the quadratic factor as an equation ($= 0$) in x with coefficients in y using the quadratic formula: $x = \dfrac{-(2y-2) \pm \sqrt{(2y-2)^2 - 4(y^2-2y)}}{2} = \dfrac{-2y+2 \pm \sqrt{4}}{2}$
$\Rightarrow x = -y + 2$ or $x = -y$, so the factors are $x - (-y+2)$ and $x - (-y)$.

(ii) Since $6y^3 - y^2 - 21y + 2x^2 + 12x - 4xy + x^2y - 5xy^2 + 10$ has no term in x^3, there must be a linear factor with no x term i.e. of the form $py + q$. Checking a few values of y shows that when $y = -2$, the given expression $= 0 \Rightarrow (y+2)(x+ay+b)(x+cy+d)$ is a factorisation. Notice how you shouldn't assume a factorisation of the form $(x \pm 6y + b)(x \pm y + d)$ as it might be of the form $(x \pm 3y + b)(x \pm 2y + d)$, but it is reasonable to assume that the coefficient of x in both factors is 1, since that will ensure a term $2x^2$.

The claimed factorisation is $(y+2)\left(x^2 + acy^2 + (a+c)xy + (b+d)x + (ad+bc)y + bd\right)$ so equating coefficients implies that

$ac = 6$

$ad + bc + 2ac = -1$

$bd + 2(ad+bc) = -21$

$b + d = 6$

$b + d + 2(a+c) = -4$

$a + c = -5$

$bd = 5$

The first and sixth equations tell you that $a = -2$ and $c = -3$ or vice versa, and the fourth and seventh equations tell you that $b = 5$ and $d = 1$ or vice versa. You need to use the second equation, which tells you that $ad + bc = -13$, to determine that the factorisation is $(y+2)(x-2y+1)(x-3y+5)$.

It may be instructive to find these factors using the quadratic formula, as described at the end of part (i).

4 The derivative of $\sec x$ is $\sec x \tan x$, which can be determined by differentiating $(\cos x)^{-1}$.

(i) Using the given substitution, and remembering not only to write the limits in terms of t but also to write $\mathrm{d}x$ in terms of $\mathrm{d}t$,

$$\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec t \tan t}{\sec^3 t\sqrt{\sec^2 t - 1}}\,\mathrm{d}t = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec t \tan t}{\sec^3 t\sqrt{\tan^2 t}}\,\mathrm{d}t = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2 t\,\mathrm{d}t$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 2t + 1}{2}\,\mathrm{d}t = \left[\frac{\sin 2t}{4} + \frac{t}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\sqrt{3}-2}{8} + \frac{\pi}{24}$$

using the identities $1 + \tan^2 t \equiv \sec^2 t$ and $\cos 2t \equiv 2\cos^2 t - 1$.

(ii) Since $(x+1)(x+3) = x^2 + 4x + 3 = (x+2)^2 - 1$, this integral in part (ii) looks similar to the integral in part (i).

So let $x + 2 = \sec t \Rightarrow \displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}}\,\mathrm{d}x = \int \frac{\sec t \tan t}{\sec t\sqrt{(\sec t - 1)(\sec t + 1)}}\,\mathrm{d}t$

$$= \int \frac{\sec t \tan t}{\sec t\sqrt{\sec^2 t - 1}}\,\mathrm{d}t = \int 1\,\mathrm{d}t = t + c = \operatorname{arcsec}(x+2) + c$$

(iii) Since $x^2 + 4x - 5 = (x+2)^2 - 9$, let $x + 2 = 3\sec t$.

Therefore $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}}\,\mathrm{d}x = \int \frac{3\sec t \tan t}{3\sec t\sqrt{9(\sec t)^2 - 9}}\,\mathrm{d}t$

$$= \int \frac{1}{3}\,\mathrm{d}t = \frac{t}{3} + c = \frac{1}{3}\operatorname{arcsec}\left(\frac{x+2}{3}\right) + c$$

5 Using the formula for the kth term of an arithmetic progression:

A: value of kth term $= 5k + 1$

B: value of kth term $= 5k + 2$

C: value of kth term $= 5k + 3$

D: value of kth term $= 5k + 4$

E: value of kth term $= 5k$

or equivalent.

Therefore, the sum of any term in B and any term in C can be written $(5k + 2) + (5n + 3) = 5(k + n + 1)$ which is a term in E. Notice that it is important to use k and n, to ensure that you are not only adding corresponding terms in B and C.

Similarly, the square of any term in B can be written $(5k + 2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4$ which is a term in D, and the square of any term in C can be written $(5k + 3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4$ which is also a term in D.

(i) Since $(5k)^2 = 5(5k^2)$, $(5k + 1)^2 = 5(5k^2 + 2k) + 1$ and $(5k + 4)^2 = 5(5k^2 + 8k + 3) + 1$, x^2 must be in E, A or D (including the results from above) i.e. $x^2 = 5n$, $5n + 1$ or $5n + 4$..

But $(5n + 1) + 5y = 5(n + y) + 1$, $(5n + 4) + 5y = 5(n + y) + 4$, and $5n + 5y = 5(n + y)$.

So x^2 + a term in E cannot be a term in C.

(ii) You know that x^2 must be in E, A or D, so $x^4 = (x^2)^2$ must be a term in A or E, as all terms in E square to give a term in E, and all terms in A and D square to give terms in A.

Therefore $2y^4$ must be a term in B or E.

No pair of terms from A, B or E can add to give a term in D (consider the "+4"), so the equation $x^4 + 2y^4 = 26\ 081\ 974$ has no solution for integer x and y.

6 The line joining A to the midpoint of BC has equation $y - q_1 = \left(\dfrac{\frac{q_2+q_3}{2} - q_1}{\frac{p_2+p_3}{2} - p_1} \right) (x - p_1)$

$$\Rightarrow y - q_1 = \left(\frac{q_2 + q_3 - 2q_1}{p_2 + p_3 - 2p_1} \right) (x - p_1)$$

The line joining B to the midpoint of AC has equation $y - q_2 = \left(\dfrac{q_1 + q_3 - 2q_2}{p_1 + p_3 - 2p_2} \right) (x - p_2)$

To find where the two lines meet you need to solve simultaneously these two equations. The many subscripts make it difficult to do this correctly, but the algebra required is not advanced. You should find that the lines meet at $\left(\dfrac{p_1 + p_2 + p_3}{3}, \dfrac{q_1 + q_2 + q_3}{3} \right)$.

You then need to verify that these coordinates satisfy the equation of the line joining C to the midpoint of AB, which is $y - q_3 = \left(\dfrac{q_1 + q_2 - 2q_3}{p_1 + p_2 - 2p_3} \right) (x - p_3)$

If AH is perpendicular to BC then $\dfrac{q_2 + q_3}{p_2 + p_3} \times \dfrac{q_3 - q_2}{p_3 - p_2} = -1$ since the product of the gradients of perpendicular lines is -1.

$$\Rightarrow q_3^2 - q_2^2 = -\left(p_3^2 - p_2^2\right) \Rightarrow p_2^2 + q_2^2 = p_3^2 + q_3^2$$

If the line BH intersects the line AC at right angles $\Rightarrow p_1^2 + q_1^2 = p_3^2 + q_3^2$. The instruction "write down" in the question suggests that very little work is needed to do so: see how this answer is structurally identical to the previous answer.

Therefore, if AH is perpendicular to BC and BH is perpendicular to AC

$$\Rightarrow p_2^2 + q_2^2 = p_3^2 + q_3^2 \text{ and } p_1^2 + q_1^2 = p_3^2 + q_3^2$$

$$\Rightarrow p_1^2 + q_1^2 = p_2^2 + q_2^2$$

$$\Rightarrow q_2^2 - q_1^2 = -\left(p_2^2 - p_1^2\right) \Rightarrow \frac{q_2 + q_1}{p_2 + p_1} \times \frac{q_2 - q_1}{p_2 - p_1} = -1$$

$\Rightarrow CH$ is perpendicular to AB

This question aims to show some interesting properties of triangles. The line joining a vertex of a triangle to the midpoint of the opposite side (e.g. A to the midpoint of BC) is called a median: you have proved that the three medians of any triangle are concurrent, i.e. they all meet at the same point with coordinates $\left(\dfrac{p_1 + p_2 + p_3}{3}, \dfrac{q_1 + q_2 + q_3}{3} \right)$. This point is called the centroid.

The line through a vertex which is perpendicular to the opposite side is called an altitude: the three altitudes of any triangle are concurrent, and they meet at a point called the orthocentre. In this question the triangle has been set up so that the orthocentre has simple coordinates $(p_1 + p_2 + p_3 , q_1 + q_2 + q_3)$. You should consider how this has been achieved: what geometric fact must be true if the coordinates of A, B and C satisfy $p_1^2 + q_1^2 = p_2^2 + q_2^2 = p_3^2 + q_3^2$?

7 (i) The relationship $a_0 = 2$, $a_1 = 7$ and $a_n - 7a_{n-1} + 10a_{n-2} = 0$ is called a recurrence relationship: given the values of a_0 and a_1 you can calculate that $a_2 = 29$, $a_3 = 133$ and so on. Therefore $\mathrm{f}(x) = 2 + 7x + 29x^2 + 133x^3 + ...$

$\Rightarrow \mathrm{f}(x) - 7x\mathrm{f}(x) + 10x^2\mathrm{f}(x) =$

$(2 + 7x + 29x^2 + 133x^3 + ...) -$

$(14x + 49x^2 + 203x^3 + 931x^4 + ...) +$

$(20x^2 + 70x^3 + 290x^4 + 1330x^5 + ...) = 2 - 7x$

because all the other terms cancel.

More formally, $\mathrm{f}(x) - 7x\mathrm{f}(x) + 10x^2\mathrm{f}(x) = \sum_{r=0}^{\infty} a_r x^r - \sum_{r=0}^{\infty} 7a_r x^{r+1} + \sum_{r=0}^{\infty} 10a_r x^{r+2}$

$= 2 + 7x + \sum_{r=0}^{\infty} a_{r+2}x^{r+2} - 14x - \sum_{r=0}^{\infty} 7a_{r+1}x^{r+2} + \sum_{r=0}^{\infty} 10a_r x^{r+2}$ (if this is unclear, expand the first few terms of each of the sigmas).

$= 2 - 7x + \sum_{r=0}^{\infty} (a_{r+2} - 7a_{r+1} + 10a_r)\, x^{r+2}$

$= 2 - 7x + \sum_{r=0}^{\infty} (0)\, x^{r+2}$

because of the given relationship $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geqslant 2$.

$\Rightarrow \mathrm{f}(x) = \dfrac{7 - 2x}{1 - 7x + 10x^2} = \dfrac{7 - 2x}{(1 - 2x)(1 - 5x)} = \dfrac{1}{1 - 2x} + \dfrac{1}{1 - 5x}$ by partial fractions.

Since $(1 - 2x)^{-1} = 1 + 2x + 4x^2 + 8x^3 + ... + 2^n x^n + ...$

and $(1 - 5x)^{-1} = 1 + 5x + 25x^2 + 125x^3 + ... + 5^n x^n + ...$

the coefficient of x^n is $2^n + 5^n$, therefore $a_n = 2^n + 5^n$.

(ii) In this part of the question you haven't been given the precise relationship between the terms of the sequence b_n, so you need to determine that first.

$n = 2 \Rightarrow b_2 = pb_1 + qb_0 \Rightarrow 40 = 10p + 5q$

$n = 3 \Rightarrow b_3 = pb_2 + qb_1 \Rightarrow 100 = 40p + 10q$

$\Rightarrow p = 1,\ q = 6 \Rightarrow b_n - b_{n-1} - 6b_{n-2} = 0$

Part (a) suggests consider $\mathrm{g}(x) - x\mathrm{g}(x) - 6x^2\mathrm{g}(x) =$

$(5 + 10x + 40x^2 + 100x^3 + ...) -$

$(5x + 10x^2 + 40x^3 + 100x^4 + ...) -$

$(30x^2 + 60x^3 + 240x^4 + 600x^5 + ...) = 5 + 5x$

This is sufficient, but you should try to write a convincing argument using sigma notation as above.

$\Rightarrow \mathrm{g}(x) = \dfrac{5 + 5x}{1 - x - 6x^2} = \dfrac{5 + 5x}{(1 + 2x)(1 - 3x)} = \dfrac{1}{1 + 2x} + \dfrac{4}{1 - 3x}$

$\Rightarrow \mathrm{g}(x) = \left(1 - 2x + 4x^2 - 8x^3 + ... + (-2)^n x^n + ...\right) + 4\left(1 + 3x + 9x^2 + 27x^3 + ... 3^n x^n + ...\right)$

$\Rightarrow b_n = (-2)^n + 4(3)^n$

8 (i) It is important to realise that it is not sufficient to choose a specific value of x_0 e.g. $x_0 = 4$, evaluate $x_1 = 4.4$ and then claim that the sequence is increasing for all n and for all $x_0 > 3$. When asked to prove that $a > b$, a good strategy may be to consider $a - b$, and try to see why $a - b$ is positive.

In this problem, therefore, consider

$$x_{n+1} - x_n = \frac{x_n^2 + 6}{5} - x_n = \frac{x_n^2 - 5x_n + 6}{5} = \frac{(x_n - 3)(x_n - 2)}{5}$$

which will be positive for any $x_n > 3$ (or any $x_n < 2$).

Let $n = 0$: you are told that $x_0 > 3$, so $x_1 - x_0 > 0$

$\Rightarrow x_1 > x_0 \Rightarrow x_1 > 3$ since $x_0 > 3$

$\Rightarrow x_2 - x_1 > 0 \Rightarrow x_2 > x_1$

$\Rightarrow x_2 > 3$ since $x_1 > 3$

$\Rightarrow x_3 - x_2 > 0 \Rightarrow x_3 > x_2$ and so on.

Therefore $x_{n+1} > x_n$ for all n.

(ii) As in part (i), consider

$$y_{n+1} - y_n = 5 - \frac{6}{y_n} - y_n = \frac{5y_n - 6 - y_n^2}{y_n} = \frac{-(y_n - 2)(y_n - 3)}{y_n}$$

which is positive for any y_n between 2 and 3.

Let $n = 0$: $2 < y_0 < 3 \Rightarrow y_1 - y_0 > 0 \Rightarrow y_1 > y_0$.

Also $3 - y_1 = 3 - \left(5 - \frac{6}{y_0}\right) = \frac{6}{y_0} - 2 > 0$ since $y_0 < 3$.

$\Rightarrow 3 - y_1 > 0 \Rightarrow y_1 < 3$

Therefore $2 < y_0 < y_1 < 3$, so $y_2 - y_1 > 0 \Rightarrow y_2 > y_1$.

Also $3 - y_2 = 3 - \left(5 - \frac{6}{y_1}\right) = \frac{6}{y_1} - 2 > 0$ since $y_1 < 3$.

So $2 < y_1 < y_2 < 3$ and so on.

Therefore, $y_{n+1} > y_n$ for all n, but $y_n < 3$ for all n.

Both of these arguments are most easily written using Mathematical Induction: if you are familiar with this method of proof, then you might like to prove these two results in this way.

Section B: Mechanics

9 At the maximum height H, the vertical velocity $= 0$.

Therefore $0 = (u\sin\theta)^2 - 2gH \Rightarrow H = \dfrac{u^2\sin^2\theta}{2g}$

At the point P, $ut\cos\theta = d$ and $ut\sin\theta - \frac{1}{2}gt^2 = \frac{1}{2}d$

$$\Rightarrow \frac{d}{2} = u\sin\theta \times \frac{d}{u\cos\theta} - \frac{g}{2}\left(\frac{d}{u\cos\theta}\right)^2 = d\tan\theta - \frac{gd^2\sec^2\theta}{2u^2}$$

$$\Rightarrow u^2 = \frac{gd\sec^2\theta}{2\tan\theta - 1} \quad \text{so} \quad \frac{gd\sec^2\theta}{2\tan\theta - 1} \times \frac{\sin^2\theta}{2g} < \frac{9d}{10}$$

because the projectile doesn't hit the roof of the tunnel and therefore its maximum height is less than $\frac{9}{10}d$.

$\Rightarrow 10\sec^2\theta\sin^2\theta < 18\,(2\tan\theta - 1)$

$\Rightarrow 5\tan^2\theta - 18\tan\theta + 9 < 0$

$\Rightarrow (5\tan\theta - 3)(\tan\theta - 3) < 0$

$\Rightarrow \dfrac{3}{5} < \tan\theta < 3$

$\Rightarrow \arctan\frac{3}{5} < \theta < \arctan 3$

10 (i) A good velocity time graph makes this question much easier to understand.

Since $v^2 = u^2 + 2(3a)(d_1)$ and $0 = v^2 - 2a(d_2)$,

$\Rightarrow 2ad_2 = u^2 + 6ad_1$

$\Rightarrow 6ad_1 < 2ad_2$ because $u^2 > 0$

$\Rightarrow 3d_1 < d_2$

(ii) A velocity time graph is **very** helpful here.

The particle takes time $\dfrac{-u}{3a}$ to accelerate from initial velocity $u < 0$ to rest, in which time it travels a **distance** of $\dfrac{u^2}{6a}$. You should not calculate this distance using $v^2 = u^2 + 2as$ because s is the displacement of the particle not the distance it travels, but instead should look at the (positive) area of the relevant triangle on the graph.

Then the particle travels a distance D where $v^2 = 0 + 2(3a)(D)$, so that $D = \dfrac{v^2}{6a}$.

So $d_1 = \dfrac{u^2 + v^2}{6a}$.

Still $d_2 = \dfrac{v^2}{2a}$, so $3d_1 = \dfrac{u^2 + v^2}{2a} > \dfrac{v^2}{2a} = d_2$.

Also, $v > |u| \Rightarrow v^2 > u^2$, so $3d_1 = \dfrac{u^2 + v^2}{2a} < \dfrac{v^2 + v^2}{2a} = 2d_2$

If $u > 0$ then the total time taken by the journey $= \dfrac{v-u}{3a} + \dfrac{v}{a} = \dfrac{4v-u}{3a} = \dfrac{4v-|u|}{3a}$ since $u > 0 \Rightarrow u = |u|$.

Therefore, average speed $= \dfrac{\text{total distance}}{\text{total time}} = \dfrac{\frac{v^2-u^2}{6a} + \frac{v^2}{2a}}{\frac{4v-|u|}{3a}} = \dfrac{4v^2 - u^2}{2(4v - |u|)}$.

If $u < 0$ then total time taken by the journey $= \dfrac{v+|u|}{3a} + \dfrac{v}{a} = \dfrac{4v+|u|}{3a}$.

Note the use of $|u|$ to ensure that you are adding two positive quantities: the expression $v+u$ is actually a subtraction if u is negative.

$$\Rightarrow \text{average speed} = \frac{\frac{v^2+u^2}{6a} + \frac{v^2}{2a}}{\frac{4v+|u|}{3a}} = \frac{4v^2 + u^2}{2(4v + |u|)}.$$

As in question 8, you can consider the difference between these two:

$$\frac{4v^2 - u^2}{2(4v - |u|)} - \frac{4v^2 + u^2}{2(4v + |u|)} = \frac{(4v^2 - u^2)(4v + |u|) - (4v^2 + u^2)(4v - |u|)}{2(4v - |u|)(4v + |u|)}$$

$$= \frac{-8u^2v + 8v^2|u|}{2(4v - |u|)(4v + |u|)} = \frac{8v|u|(v - |u|)}{2(4v - |u|)(4v + |u|)} > 0 \text{ since } v > |u|.$$

11 A good diagram is essential here. Let AB be the left ladder. Let the normal reaction at A be R, and the frictional resistance be $F_1 \leqslant \mu R$: recall $F_1 = \mu R$ only if the friction is limiting e.g. the ladder AB is about to slip - and you don't know that this is the case. Let the normal reaction at C be S, and the frictional resistance be $F_2 \leqslant \mu S$. Let $AB = BC = l$.

The easiest strategy is to take moments about B, and therefore not worry about the nature of the reaction force acting there between the ladders.

Taking moments about B of the forces acting on BC

$$\Rightarrow S \times \frac{l}{2} = 4W \times \frac{l}{4} + F_2 \times \frac{l\sqrt{3}}{2}$$

Taking moments about B of the forces acting on AB

$$\Rightarrow R \times \frac{l}{2} = W \times \frac{l}{4} + 7W \times \frac{2l}{3} \times \frac{1}{2} + F_2 \times \frac{l\sqrt{3}}{2}$$

Therefore, $S = 2W + F_2\sqrt{3}$ and $6R = 31W + 6F_1\sqrt{3}$.

Resolving vertically all forces acting on the system $\Rightarrow R + S = 12W$.

Resolving horizontally all forces acting on the system $\Rightarrow F_1 = F_2$.

$$\Rightarrow S - 2W = \frac{6R - 31W}{6}$$

$$\Rightarrow 6S = 6R - 19W$$

$$\Rightarrow 72W - 6R = 6R - 19W$$

$$\Rightarrow R = \frac{91W}{12} \text{ and } S = \frac{53W}{12}$$

$$\Rightarrow F_1 = \frac{29W\sqrt{3}}{36} = F_2$$

BC slips as $S < R$, and $\mu = \dfrac{F_2}{S} = \dfrac{29\sqrt{3}}{159}$.

If the reaction between the two ladders at the hinge B is included, it is best to consider a horizontal and vertical component of the reaction force acting on each ladder. There is no reason to assume that the reaction force will act purely horizontally: not all the forces act symmetrically on the ladders. The horizontal components will be equal in magnitude but in opposite directions, and the vertical components will be similar. Let the reaction at B acting on AB be X horizontally (to the left) and Y upwards; let the reaction acting on BC be X to the right and Y downwards. It would be instructive now, for example, to take moments about A and C, and compare the merits of the two solutions.

Section C: Probability and Statistics

12 A tree diagram is very helpful, if only to clarify the language of the question.

$$P\,(\text{made by } A \mid \text{perfect}) = \frac{P\,(\text{made by } A \text{ and perfect})}{P\,(\text{perfect})} = \frac{2}{5}$$

$$\Rightarrow \frac{\lambda p}{\lambda p + (1-\lambda)\,q} = \frac{2}{5} \Rightarrow 5\lambda p = 2\lambda p + 2\,(1-\lambda)\,q$$

$$\Rightarrow \lambda = \frac{2q}{3p+2q}$$

Now, $P\,(\text{made by } A \mid \text{declared to be perfect}) = \dfrac{2}{5}$

$$\Rightarrow \frac{\frac{3}{4}\lambda p + \frac{1}{4}\lambda\,(1-p)}{\frac{3}{4}\lambda p + \frac{1}{4}\lambda\,(1-p) + \frac{3}{4}\,(1-\lambda)\,q + \frac{1}{4}\,(1-\lambda)\,(1-q)} = \frac{2}{5}$$

since those that are declared to be perfect are either those that are perfect and have been correctly assigned (with probability $\frac{3}{4}$) or those that are imperfect but have been incorrectly assigned (with probability $\frac{1}{4}$).

$$\Rightarrow 15\lambda p + 5\lambda\,(1-p) = 6\lambda p + 2\lambda\,(1-p) + 6\,(1-\lambda)\,q + 2\,(1-\lambda)\,(1-q)$$

$$\Rightarrow \lambda = \frac{4q+2}{6p+4q+5}$$

13 (i) $P\,(\text{maximum of 3 numbers} \leqslant 0.8) = P(\textbf{all} \text{ three numbers} \leqslant 0.8) = 0.8^3 = 0.512$

Similarly, $\mathrm{F}(x)$ = the cumulative distribution function of $X = P\,(X \leqslant x) = x^3$

$\Rightarrow \mathrm{f}(x)$ = the probability density function of $X = \mathrm{F}'(x) = 3x^2$

$$\Rightarrow E\,(X) = \int_0^1 3x^3 \, \mathrm{d}x = \frac{3}{4}.$$

A similar argument would work to calculate $E\,(Y)$, where Y is the minimum of the numbers drawn.

(ii) If H_0 is true, then $P\,(X \leqslant 0.8) = 0.8^N$.

We require the minimum N such that $0.8^N < 0.05$ since then H_0 will be rejected.

$$\Rightarrow \left(\frac{2^3}{10}\right)^N < \frac{1}{20} \Rightarrow 2^{3N} < \frac{10^N}{20} \Rightarrow 2^{3N} < \frac{10^N}{2 \times 10}$$

$\Rightarrow 2^{3N+1} < 10^{N-1} \Rightarrow 2^{3N+1} < \left(2^{\frac{10}{3}}\right)^{N-1}$ using the given approximation $10^3 \approx 2^{10}$

$$\Rightarrow 3N + 1 < \frac{10\,(N-1)}{3} \Rightarrow 9N + 3 < 10N - 10$$

$\Rightarrow N > 13$

$\Rightarrow N = 14$

If $a = 0.8$ then $P\,(H_0 \text{ is rejected}) = 1$

If $a = 0.9$ then $P\,(H_0 \text{ is rejected}) = \left(\frac{8}{9}\right)^{14}$

14 (i) The first pirate will have some gold if he does not take a lead coin first.

$$\Rightarrow P\,(\text{first pirate has some gold }) = \frac{n}{n+2}$$

(ii) There are at least two different approaches that can be used to answer this part. The easiest argument, if you are familiar with it, is to see this as a problem about the number of ways of rearranging a selection of objects.

You may recall that there are $m!$ ways of rearranging m different objects in a line. Here there are $n+2$ coins which can be rearranged in $(n+2)!$ ways, but the second pirate wants the two lead coins **not** to be adjacent. It is easier to count the number of ways to rearrange the $n+2$ coins so that the two lead ones **are** adjacent: there are $(n+1)! \times 2!$ ways of doing this, because you treat the two adjacent lead coins as a single item and therefore rearrange the remaining $n+1$ coins, but the two lead coins can be swapped around $2!$ ways so that there are twice as many rearrangements.

Hence there are $(n+2)! - (n+1)! \times 2!$ ways of rearranging the $n+2$ coins so that the lead coins are not adjacent.

$$\Rightarrow P\,(\text{the second pirate has some gold}) = \frac{(n+2)! - 2!\,(n+1)!}{(n+2)!} = \frac{(n+1)!\,(n+2-2!)}{(n+2)!}$$

$$= \frac{(n+1)! \times n}{(n+2) \times (n+1)!} = \frac{n}{n+2}$$

An alternative argument is the following. If the second pirate has some gold coins, then the coins must have been taken from the chest in the order LG or GLG or $GGLG$ or $GGGLG$ etc, until $GGG...GGLGL$, where G represents taking a gold coin and L represents taking a lead coin. You should consider why the orders LGG, $LGGG$, $LGGGG$ etc do not need to be listed: a tree diagram will be helpful.

$$\text{Therefore, } P\,(\text{the second pirate has some gold}) = \left(\frac{2}{n+2} \times \frac{n}{n+1}\right) + \left(\frac{n}{n+2} \times \frac{2}{n+1} \times \frac{n-1}{n}\right) +$$

$$\left(\frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{2}{n} \times \frac{n-2}{n-1}\right) + \ldots + \left(\frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n} \times \ldots \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1}\right)$$

$$= \frac{2}{(n+2)\,(n+1)}\,[n + (n-1) + (n-2) + \ldots + 1]$$

$$= \frac{2}{(n+2)\,(n+1)}\left[\frac{n\,(n+1)}{2}\right]$$

$$= \frac{n}{n+2}$$

(iii) If all three pirates have some gold, then a gold coin was chosen first, a gold coin was chosen last, and when the remaining n coins were chosen the lead coins were not adjacent.

$$\Rightarrow P\,(\text{all three pirates have some gold}) = \frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n}$$

because the probability that the last coin is gold must equal the probability that the second coin is gold: once the second coin has been drawn it can be renamed as the last coin.

The third fraction has been deduced by adapting the answer from part (ii).

$$\text{This simplifies to } \frac{n-1}{n+1} \times \frac{n-2}{n+2} = \frac{n^2 - 3n + 2}{n^2 + 3n + 2}$$

OXFORD AND CAMBRIDGE AND RSA EXAMINATIONS

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Q1(i) Put the terms with radicals on one side and the terms without on the other and square. Repeat this strategy (S) and the equation $x^4 - 6x^3 + 9x^2 - 4x = 0(*)$ will be obtained. The roots of (*) are $x = 0, 1, 4$.

Squaring may introduce spurious roots, so these numbers must be checked to see that they are roots of the original equation. In fact, they are.

(ii) Application of S again leads to (*). Checking shows that $x = 0$, $x = 1$ are roots of the second equation but that $x = 4$ is not.

(iii)Again application of S leads to (*). Checking shows that $x = 1$, $x = 4$ are roots of the third equation but that $x = 0$ is not.

Q2 Write $Q \equiv x^2 - \alpha|x| + 2 = [|x| - \alpha/2]^2 + 2 - \alpha^2/4$.

Thus $\alpha < 2\sqrt{2} \Rightarrow 2 - \alpha^2/4 > 0 \Rightarrow Q > 0$ for all x.

It is therefore unnecessary to consider $x > 0$ and $x < 0$ separately and even more unnecessary to use calculus methods.

• if $\alpha = 3$ then $Q \equiv (|x| - 1)(|x| - 2)$, in which case the solution set of $Q < 0$ is $\{x : -2 < x < -1\} \cup \{x : 1 < x < 2\}$.

• The solutions in x of the equation $Q = 0$ are of the form $-x_2, -x_1, x_1, x_2,$ where $0 < x_1 < x_2$, so that $S = 2(x_2 - x_1)$. Use of the identity $x_2 - x_1 = \sqrt{(x_2 + x_1)^2 - 4x_1x_2}$ will lead immediately to $S = 2\sqrt{\alpha^2 - 8}$. Thus $S < 2\sqrt{\alpha^2} = 2\alpha$.

• The graph of S as a function of α is that part of the hyperbola $4\alpha^2 - S^2 = 32$ which is in the first quadrant. A sketch of this graph should, therefore, leave the other quadrants empty. It should also show the curve starting at the point $(2\sqrt{2}, 0)$ and asymptotically approaching the line $S = 2\alpha$.

Q3 The obtaining of dy/dx in the form required is a routine exercise in differentiation followed by some algebra.

Setting $dy/dx = 0$ shows that there are stationary points where $x = -2/3, 1/2, 2$. Moreover $d^2y/dx^2 = (x-2)^3(12x+1)$ + a term which is necessarily zero when $x = -2/3, 1/2, 2$. Thus d^2y/dx^2 is positive when $x = -2/3$ and negative when $x = 1/2$, so that C has a minimum at $(-2/3, -8192/729)$ and a maximum at $(1/2, 243/64)$. (Note that it is unnecessary to determine a simplified version of d^2y/dx^2 before inserting values of x.)

The argument $d^2y/dx^2 = 0$ at $x = 2 \Rightarrow C$ has a point of inflexion at $(2, 0)$ is false. In fact, in the neighbourhood of this point, $y \approx 6(x-2)^4$, so that it is obvious that C has a minimum there.

The sketch of C must have correct overall shape, location and orientation, and also show correct forms at (0,0), (2,0) and at ∞.

(i) This sketch may be deduced from that of C. It has symmetry about the x - axis and no part of it appears in the region $-1 < x < 0$.

(ii) This sketch may also be deduced from that of C. It has symmetry about the y - axis and no part of it appears in the region $y < 0$.

Q4 It is important to realise at the outset that α is a constant defined by a and b and that β is a constant defined by a, b and w. Variable angles θ/ϕ are needed to define the orientation of the rod/table in the general situation.

(i) Clearly, for all $\theta \in (0,\ \pi/2)$, it is necessary that $f(\theta) \geq L$, where $f(\theta) = a\csc\theta + b\sec\theta$. Setting $f'(\theta) = 0$ will then lead to the required result.

(ii) Here, for all $\phi \in (0,\ \pi/2)$, it is necessary that $y \geq l$, where y is such that $b = (y-x)\cos\phi + w\sin\phi$ and x is such that $a = x\sin\phi + w\cos\phi$. (Other formulations are possible.) Elimination of x leads to $y = a\csc\phi + b\sec\phi - 2w\csc 2\phi$

Setting $y'(\phi) = 0$ plus some further working will then produce the required result.

Q5 Using the integration by parts rule it is easy to establish the results $\int_0^\pi x\sin x\,dx = \pi$ and $\int_0^\pi x\cos x\,dx = -2$.

- Write $\sin(x+t) = \sin x\cos t + \sin t\cos x$ and the result $f(t) = t + A\sin t + B\cos t$, where A and B are as defined in the question, follows immediately.
- Hence write $t + A\sin t + B\cos t = t + \int_0^\pi (x + A\sin x + B\cos x)\sin(x+t)\,dx$ $(***)$ so that as

$\int_0^\pi x\sin(x+t)\,dx = \ldots = \pi\cos t - 2\sin t,$

$\int_0^\pi \sin x\sin(x+t)\,dx = \ldots = (\pi/2)\cos t,$

$\int_0^\pi \cos x\sin(x+t)\,dx = \ldots = (\pi/2)\sin t,$

then, by considering the coefficients of $\cos t$ and $\sin t$ on both sides of (***), it follows that

$A = -2 + (\pi/2)B,\ B = \pi + (\pi/2)A \Rightarrow A = -2,\ B = 0.$

Alternatively, equations for A and B can be obtained by putting $t = 0$ and $t = \pi/2$ in (***).

Q6 From the data it follows that the component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is $3\mathbf{a}$.

Hence $\mathbf{p} = 4\mathbf{a}$ and $\mathbf{q} = \mathbf{b} - 3\mathbf{a}$.

- Again from the data, it follows that $(\mathbf{c}.\mathbf{a})\mathbf{a} = -2\mathbf{a}$ and

$|\mathbf{q}|^2 = \mathbf{b}.\mathbf{b} - 6\mathbf{a}.\mathbf{b} + 9\mathbf{a}.\mathbf{a} = 25 - 18 + 9 = 16 \Rightarrow |\mathbf{q}| = 4,$ so that

$\left[(\mathbf{c}.\mathbf{q})/|\mathbf{q}|^2\right]\mathbf{q} = (1/2)\mathbf{b} - (3/2)\mathbf{a}.$

Thus $\mathrm{P} = 2\mathrm{a}$, $\mathrm{Q} = -(9/2)\mathrm{a} + (3/2)\mathrm{b}$, $\mathrm{R} = (7/2)\mathrm{a} - (1/2)\mathrm{b} + \mathrm{c}$.

Q7 Good sketch graphs of $y = x$ and $y = 2\sin x$, in the same diagram and over the interval $[0, \pi]$, will readily show that the equation $f(x) = 0$ has exactly one root in the interval $[\pi/2, \pi]$.

- $f(3\pi/4) = \sqrt{2} - 3\pi/4$ has the same sign as $2 - 9\pi^2/16 \approx 2 - 45/8 = -29/8 < 0$. Hence as $f(\pi/2) = 2 - \pi/2 > 0$ and $f(\pi) = -\pi < 0$, then $I_1 = [\pi/2,\ 3\pi/4]$.
- $x = \sin 5\pi/8 \Rightarrow 2x\sqrt{1-x^2} = \sin 3\pi/4 = 1/\sqrt{2} \Rightarrow 8x^4 - 8x^2 + 1 = 0\,(*) \Rightarrow x^2 = 1/2 + 1/(2\sqrt{2}) \approx 0.85$. (**). The sign of $f(5\pi/8)$ is the same as that of $4x^2 - 25\pi^2/64 \approx 17/5 - 125/32 = -81/625 < 0$. Hence $I_2 = [\pi/2,\ 5\pi/8]$.
- A good approximation to $x = \sin 9\pi/16$ may also be obtained in a similar way. In fact, it will be found that $f(9\pi/16) > 0$ so that $I_3 = [9\pi/16,\ 5\pi/8]$.

Q8(i) Integration leads to the general solution $t = A - \ln(1-x)$ and $x(0) = 0 \Rightarrow A = 0$. Thus $x = 1 - e^{-t}$.

(ii) Obviously, $(1-x)^{1/2} < (1+x)^{1/2}$ for all $x \in (0, 1]$. Hence multiplying this inequality through by $(1-x)^{1/2}$ leads immediately to the required result.

Arguments which go in the wrong direction, e.g., $1 - x < (1-x^2)^{1/2} \Rightarrow \ldots \Rightarrow x - x^2 > 0$, etc., are invalid. It may be possible to salvage them by replacing $'\Rightarrow'$ by $'\Leftarrow'$.

In the case $n = 2$, the substitution $x = \sin y$ will lead to $t = y + B$ and hence to $t = \sin^{-1}(x) + B$ as the general solution. In particular, $x(0) = 0 \Rightarrow B = 0 \Rightarrow x = \sin t$.

Note that the question does not allow the use of the standard form $\int (1-x^2)^{-1/2}\,dx = \sin^{-1}(x) +$ an arbitrary constant, without proof.

(iii) If G_n is the graph of x for $0 \le x \le 1$, then the given inequality shows that the gradient of G_3 is greater than the gradient of G_2 for each x in this interval. (The inequality of (ii) shows that the same is true of G_2 in relation to G_1.) These considerations will help to clarify ideas when drawing sketches of G_n for $n = 1, 2, 3$ in the same diagram. In particular, the sketch of G_3 should make it clear that once x reaches the value 1 it remains there.

Q9 For each of the two given situations, it is essential that a properly annotated diagram consistent with a possible state of equilibrium is supplied.

In the first situation, taking moments about the point of contact of the hemisphere with the floor leads to

$$mgr\cos\alpha = Mg\,(p\sin\alpha - q\cos\alpha) \Rightarrow \tan\alpha = (Mq + mr)/Mp.$$

A similar argument applied to the second situation leads to

$$mgr\cos\beta = Mg\,(p\sin\beta + q\cos\beta) \Rightarrow \tan\beta = (mr - Mq)/Mp.$$

It is then easy to see that

$$\tan(\alpha+\beta) = (\tan\alpha + \tan\beta)/(1-\tan\alpha\tan\beta) = 2mMrp/\left[M^2(p^2+q^2) - m^2r^2\right].$$

If the sense of the rotation is taken into account then β should be changed to $-\beta$.

Q10 If the retardation of the particles when moving up the plane is $a_1\, ms^{-2}$, then $4g(\sqrt{3}/2)(1/5\sqrt{3}) + 2g = 4a_1 \Rightarrow a_1 = 6$, so that P comes to rest after 1 second at D where $AD = 3\,m$.

If the acceleration of P down the slope is $a_2\, ms^{-2}$, then $-4g(\sqrt{3}/2)(1/5\sqrt{3}) + 2g = 4a_2 \Rightarrow a_2 = 4$.

Hence if P and Q meet at time τ, then $3 - 2(\tau-1)^2 = 6(\tau - T) - 3(\tau - T)^2$

$\Rightarrow \ldots \Rightarrow \tau^2 - (2+6T)\tau + 3T^2 + 6T + 1 = 0 \Rightarrow \ldots \Rightarrow \tau = 1 + (3-\sqrt{6})T$.

Note that the condition $T < 1 + \sqrt{3/2}$ ensures that the collision takes place before P returns to A.

(ii) A possible solution is first to show that $T = 1 + \sqrt{2/3} \Rightarrow \tau = 2$.

Hence as $v_P(2) = 4\ ms^{-2}$, $v_Q(2) = 2\sqrt{6}\ ms^{-2}$ then the total KE at $t = 2$ of P and Q = 80 j.

Further, gain in PE at $t = 2$ since start of motion = 40 j so that energy lost due to friction = 144 - 80 - 40 = 24 j.

Alternatively, and more directly, the work done against friction up to the moment of collision = frictional force opposing motion of P (or Q) $\times 6$ j $= 4 \times 6 = 24$ j.

Q11 (i) At full engine power, the equation of motion of A is $Pv^{1/2} - kv = m(dv/dt)$.

The result $\int 1/(Pv^{1/2} - kv)\, dv = -(2/k)\ln(P - kv^{1/2})$ + constant, together with use of the condition $v(0) = 0$, followed by some algebra will lead to $v_A = (P^2/k^2)\left(1 - e^{-kt/2m}\right)^2$ (*), where v_A is the velocity of A at time t.

To obtain v_B, the velocity of B at time t, substitute $2m$ for m and $2P$ for P in (*). Thus $v_B = (4P^2/k^2)\left(1 - e^{-kt/4m}\right)^2$

(ii) $9v_A = 4v_B \Rightarrow 9\left(1 - e^{-kt/2m}\right)^2 = 16\left(1 - e^{-kt/4m}\right)^2 \Rightarrow 9\left(1 + e^{-kt/4m}\right)^2 = 16$

$\Rightarrow \ldots \Rightarrow e^{-kt/4m} = 1/3 \Rightarrow v_A = 64P^2/81k^2$ and $v_B = 16P^2/9k^2$.

(iii) The equation of motion of A is now $m(dv_A/dt) = -kv_A$, where t is now measured from the instant at which the engine of A is switched off. Since the velocity of A at the start of this phase of the motion is $64P^2/81k^2$, then subsequently $v_A = (64P^2/81k^2)e^{-kt/m}$. By a similar argument the result $v_B = (16P^2/9k^2)e^{-kt/2m}$ will be obtained. Elimination of t will then lead to $k^2v_B^2 = 4P^2v_A$.

Q12 This question generates seven separate tasks and so it is especially important to set out responses in an orderly way.

- The sketch is unimodal and falls entirely in the first quadrant of the $x - y$ plane. In particular, $y'(0+) > 0$ and y is asymptotic to $y = 0$ as $x \to \infty$.

• For $f(x)$, the constant k is determined by $\int_0^a kxe{-}x^2\,dx = 1 \Rightarrow \ldots \Rightarrow k = 2a/\left(1 - e^{-a}\right)$.

• For the mode, note first that $f'(x) = k\left[1 - 2ax^2\right]e^{-2ax^2}$ which is zero when $x = \sqrt{1/2a}$.

As $a < 1/2 \Rightarrow x = \sqrt{1/2a} > 1$ and $f'(x) > 0$ for any $x \in [0,1]$, then in this case m= 1.

On the other hand, $a \geq 1/2 \Rightarrow \sqrt{1/2a} \in [0,1]$ in which case $m = \sqrt{1/2a}$.

• To determine h, set $F(h) = 1/2$, where $F(x) = \int_0^x f(y)\,dy$. This leads to $k/2a - (k/2a)e^{-ah^2} = 1/2 \Rightarrow \ldots \Rightarrow h = \sqrt{(1/a)\ln[2/(1+e^{-a})]}$.

• $a > -\ln\left(2e^{-1/2} - 1\right) \Rightarrow \ldots \Rightarrow e^{1/2} < 2/(1+e^{-a}) \Rightarrow \ldots \Rightarrow h > m$.

• $e > 1 \Rightarrow e^{-1/2} < 1 \Rightarrow 2e^{-1/2} - 1 < e^{-1/2} \Rightarrow \ln(2e^{-1/2} - 1) < -1/2 \Rightarrow -\ln(2e^{-1/2} - 1) > 1/2$.

• $P(X > m|X < h)P(X < h) = P(X > m \cap X < h) \Rightarrow P(X > m|X < h) = [1/2 - F(1/\sqrt{2a})]/(1/2) = 1 - (k/a)\left[1 - e^{-1/2}\right] = \ldots = (2e^{-1/2} - e^{-a} - 1)/(1 - e^{-a})$.

Q13 If W_n pounds is the gain from draw n, then $E(W_{n+1}) = (b-r-n)/(b-n)\times 1 + r/(b-n)\times(-n)$ which is zero if $n = (b-r)/(r+1) = \xi$, say.

• W_{n+1} increases as n increases for $n < \xi$, and W_{n+1} decreases as n increases for $n > \xi$. Hence W_n maximum when $n = [\xi] + 1 = n_c$, say, so that optimal stopping n is n_c.

• For $r = 1$ and b even, $n_c = b/2$, in which case $P(\text{first } n_c - 1 \text{ draws are all white}) = (b - n_c + 1)/b = 1/2 + 1/b$.

Thus expected total reward $= (1/2 - 1/b) \times 0 + (1/2 + 1/b)\left[(b/2)/((b/2)+1)\right] \times n_c = \ldots = b/4$ pounds.

• For $r = 1$ and b odd, $n_c = b/2 + 1/2$ so that now $P(\text{first } n_c - 1 \text{ draws are all white}) = 1/2 + 1/b$.

Hence expected total reward $= (1/2 + 1/2b) \times [(b/2 - 1/2)/(b + 1/2)] \times (b+1)/2 = \ldots = (b^2 - 1)/4b$ pounds.

Q14 The introductory result may be explained by means of a diagram. Alternatively, replacing B by $B \cup C$ in $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ will lead to the displayed result almost immediately.

• $P_r = P(\text{at least one pudding contains no sixpence}) = 3\left[(2/3)^r - (1/3)^r\right]$.

• $P_5 = 31/81 > 1/3$, $P_6 = 7/27 < 1/3 \Rightarrow \min(r) = 6$.

• With $r = 6$, let A be the event that each pudding contains ≥ 1 sixpences and let B be the event that each pudding contains 2 sixpences. Then,

$P(A) = 1 - 7/27 = 20/27$,

$P(A \cap B) = P(B) = \ldots = 10/81$,

$P(B|A) = (10/81)/(20/27) = 1/6$.

Section A: Pure Mathematics

1 The substitution $u = \cosh x$ should suggest itself (because of the factor of $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \sinh x$ in t numerator), and the resulting integral can be tackled by splitting the integrand into part fractions:

$$\int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,\mathrm{d}x = \int_1^{\cosh a} \frac{\mathrm{d}u}{2u^2-1} = \frac{1}{2}\int_1^{\cosh a} \frac{1}{\sqrt{2}u-1} - \frac{1}{\sqrt{2}u+1}\,\mathrm{d}u$$

$$= \frac{1}{2\sqrt{2}}\left[\ln\left(\sqrt{2}u-1\right) - \ln\left(\sqrt{2}u+1\right)\right]_1^{\cosh a} = \frac{1}{2\sqrt{2}}\left(\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)$$

Similarly, substituting $u = \sinh x$, and then recognising an arctan integral:

$$\int_0^a \frac{\cosh x}{1+2\sinh^2 x}\,\mathrm{d}x = \int_0^{\sinh a} \frac{\mathrm{d}u}{1+2u^2} = \frac{1}{\sqrt{2}}\left[\arctan\left(\sqrt{2}u\right)\right]_0^{\sinh a} = \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\sinh a\right)$$

To show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1+2\sinh^2 x}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

note that

a $\cosh^2 x = \sinh^2 x + 1$, so $2\cosh^2 x - 1 = 1 + 2\sinh^2 x$, and the integral required is t second minus the first of those calculated earlier, as $a \to \infty$.

b as $a \to \infty$, $\sinh a \to \infty$, so $\arctan\left(\sqrt{2}\sinh a\right) \to \dfrac{\pi}{2}$

c as $a \to \infty$, $\cosh a \to \infty$, so $\dfrac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1} \to 1$ and $\ln\left(\dfrac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) \to 0.$

Substituting $u = \mathrm{e}^x$, so that $\cosh x = \frac{1}{2}\left(u + \dfrac{1}{u}\right)$ and $\sinh x = \frac{1}{2}\left(u - \dfrac{1}{u}\right)$:

$$\int_0^\infty \frac{\cosh x - \sinh x}{1+2\sinh^2 x}\,\mathrm{d}x = \int_1^\infty \frac{\left(u+\frac{1}{u}\right) - \left(u - \frac{1}{u}\right)}{1+\frac{1}{2}\left(u^2 - 2 + \frac{1}{u^2}\right)}\,\frac{1}{u}\,\mathrm{d}u = \int_1^\infty \frac{2}{1+u^4}\,\mathrm{d}u$$

so $\displaystyle\int_1^\infty \frac{1}{1+u^4}\,\mathrm{d}u = \frac{\pi}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$

2 (i) Inspection of the denominator shows that the vertical asymptotes are at $x = 0$, $x = 4$, and the third term in $\mathrm{f}(x)$ tends to zero as $|x| \to \infty$, so the oblique asymptote is just $y = x - 4$.

The oblique asymptote meets the curve when $\dfrac{16\,(2x+1)^2}{x^2\,(x-4)} = 0$ or $(2x+1)^2 = 0$, hence there is a double root at $x = -\frac{1}{2}$ and hence the asymptote touches rather than crosses the curve at $(-\frac{1}{2}, -\frac{9}{2})$, so is a tangent there.

(ii) $\mathrm{f}(x) = 0$ when $x^2(x-4)^2 - 16(2x+1)^2 = 0$.

The left hand side of this equation is a difference of two squares, so factorises to give $(x(x-4) - 4(2x+1))\,(x(x-4) + 4(2x+1)) = 0$; that is, $\left(x^2 - 12x - 4\right)(x+2)^2 = 0$, which has a double root at $x = -2$.

(iii) On your sketch you should show:

the double root at $(-2, 0)$ — the curve has a local maximum here and touches the x-axis;

the remaining roots (solutions of $x^2 - 12x - 4 = 0$) at $x = 6 \pm 2\sqrt{10}$;

the curve approaching the oblique asymptote $y = x - 4$ from below as $x \to \infty$, approaching it from above as $x \to -\infty$ and touching it at $(-\frac{1}{2}, -\frac{9}{2})$;

$\mathrm{f}(x) \to \infty$ as $x \to 0$ from above or below, $\mathrm{f}(x) \to +\infty$ as $x \to 4$ from below and $\mathrm{f}(x) \to -\infty$ as $x \to 4$ from above;

local minima at some x value with $0 < x < 4$ and with $y > 0$ and at some x value with $-2 < x < -\frac{1}{2}$ and with $-\frac{9}{2} > y > x - 4$ — note that this second minimum is not at the point of tangency with the oblique asymptote.

3 The sketch should show a curve with increasing gradient: because the gradient is increasing, the curve lies below the chord joining $(a, \mathrm{f}(a))$ and $(b, \mathrm{f}(b))$ and above the tangent to the curve at $\left(\dfrac{a+b}{2}, \mathrm{f}\left(\dfrac{a+b}{2}\right)\right)$. The illustration is clearer if $\mathrm{f}(x) > 0$ for $a \leqslant x \leqslant b$: then the area of the trapezium cut off by the chord and the lines $x = a$, $x = b$ and $y = 0$, which is $(b-a)\dfrac{\mathrm{f}(a)+\mathrm{f}(b)}{2}$, is larger than the area represented by the integral and the area of the trapezium cut off by the tangent and the lines $x = a$, $x = b$ and $y = 0$, which is $(b-a)\mathrm{f}\left(\dfrac{a+b}{2}\right)$, is smaller than the area represented by the integral.

Choose $\mathrm{f}(x) = \dfrac{1}{x^2}$, checking that this has $\mathrm{f}''(x) > 0$, $a = n-1$ and $b = n$ to get the quoted result.

Take the sum from $n = 2$ to ∞ of each term in the inequality: the left hand sum is directly as quoted; in the middle sum, you need to notice that it telescopes, so that all the terms except the first cancel in pairs; in the right hand sum, each reciprocal square occurs twice, cancelling the factor of $\frac{1}{2}$, except the first.

For the next part, observe that $\dfrac{1}{(n+1)^2} < \dfrac{1}{n^2}$, so $\dfrac{1}{2}\left(\dfrac{1}{n^2} + \dfrac{1}{(n+1)^2}\right) < \dfrac{1}{2}\left(\dfrac{1}{n^2} + \dfrac{1}{n^2}\right) = \dfrac{1}{n^2}$.

Finally, combine the two previous results to get

$$2\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) < 1 < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots,$$

so that if $S = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, then $2\left(S - 1 - \dfrac{1}{2^2}\right) < 1 < S - \dfrac{1}{2}$; rearranging these inequalities gives the required bounds on S.

4 If circle n has centre O_n then $OO_n = \dfrac{r_n}{\sin\alpha}$, $OO_{n+1} = \dfrac{r_{n+1}}{\sin\alpha}$ and $OO_n - OO_{n+1} = r_n + r_{n+1}$.

Substituting and multiplying by $\sin\alpha$ gives $r_n - r_{n+1} = \sin\alpha(r_n + r_{n+1})$ which simplifies to the required result.

This result then implies that $r_n = \left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right)^n r_0$, so the total area is

$$S = \frac{1}{2}\pi r_0^2 + \pi\left(\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right) r_0\right)^2 + \pi\left(\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2 r_0\right)^2 + \pi\left(\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^3 r_0\right)^2 + \dots$$

which is almost a geometric series with common ratio $\left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right)^2$, so

$$S = \frac{\pi r_0^2}{1-\left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right)^2} - \frac{1}{2}\pi r_0^2 = \left(\frac{(1+\sin\alpha)^2}{4\sin\alpha} - \frac{1}{2}\right)\pi r_0^2 = \frac{1+\sin^2\alpha}{4\sin\alpha}\pi r_0^2.$$

Area T of triangle $OAB = \frac{1}{2}AB \times OO_0 = \dfrac{r_0}{\cos\alpha}\dfrac{r_0}{\sin\alpha}$,

so $\dfrac{S}{T} = \dfrac{\pi}{4}\cos\alpha\left(1+\sin^2\alpha\right) = \dfrac{\pi}{4}\cos\alpha\left(2-\cos^2\alpha\right)$.

By differentiation, the maximum $\dfrac{S}{T}$ occurs where $2 - 3\cos^2\alpha = 0$ (not $\sin\alpha = 0$) and equals

$$\frac{\pi}{4}\sqrt{\frac{2}{3}}\left(2-\frac{2}{3}\right) = \frac{\pi}{3}\sqrt{\frac{2}{3}} > \sqrt{\frac{2}{3}} = \sqrt{\frac{16}{24}} > \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

5 If $\cos(x-\alpha) = \cos\beta$ then $x-\alpha = 2n\pi \pm \beta$ so $x = \alpha \pm \beta + 2n\pi$ so $\tan x = \tan(\alpha \pm \beta)$ however, for example, $x = \pi$, $\alpha = \beta = 0$ has $\tan x = \tan\pi = \tan 0 = \tan(\alpha+\beta)$ but $\cos(x-\alpha) = \cos\pi = -1 \neq 1 = \cos\beta$.

a Writing $\cos x - 7\sin x = R\cos(x-\alpha)$ requires $R = \sqrt{50} = 5\sqrt{2}$ and $\tan\alpha = -7$, so $\cos(x-\alpha) = \cos\beta$, where $\cos\beta = \dfrac{1}{\sqrt{2}}$, so we can take $\tan\beta = 1$.

Hence $\tan x = \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{-7\pm 1}{1 \pm 7} = -\frac{3}{4}$ or $\frac{4}{3}$.

The first of these gives $x = \frac{1}{2}\pi + \omega$ or $x = \frac{3}{2}\pi + \omega$ (since $\arctan\frac{3}{4} = \frac{\pi}{2} - \arctan\frac{4}{3}$)and the second $x = \omega$ or $x = \pi + \omega$. However, the first solution in each case does not satisfy the original equation (both have $\sin x > 0$, so $\cos x - 7\sin x < 1$), so $x = \frac{3}{2}\pi + \omega$ or $\pi + \omega$.

b proceeding as in (i), $\cos(x-\alpha) = \cos\beta$, where $\tan\alpha = \frac{11}{2}$ and $R = 5\sqrt{5}$, so $\cos\beta = \dfrac{2}{\sqrt{5}}$ and so $\tan\beta = \frac{1}{2}$. Hence $\tan x = \dfrac{22 \pm 2}{4 \mp 11} = -\frac{24}{7}$ or $\frac{4}{3}$.

Notice that $\tan 2\omega = \dfrac{2\tan\omega}{1-\tan^2\omega} = -\frac{24}{7}$ so the solutions are $x = \omega$ and $x = 2\omega$, again eliminating the other two possibilities, $\omega + \pi$ and $2\omega + \pi$, by checking in the original equation.

6 $F_n - F_{n-1} = w_n^2 + w_{n-1}^2 - 4w_n w_{n-1} - w_{n-1}^2 - w_{n-2}^2 + 4w_{n-1}w_{n-2} = w_n^2 - w_{n-2}^2 - 4w_{n-1}(w_n - w_{n-2})$
$= (w_n - w_{n-2})\,(w_n + w_{n-2} - 4w_{n-1}.)$ $\quad (+)$

(i) Let w_n be u_n; then $u_n + u_{n-2} - 4u_{n-1} = 0$, so $F_n - F_{n-1} = 0$ for $n \geqslant 2$, by (+), but $F_1 = u_1^2 + u_0^2 - 4u_1u_0 = -3$ so $F_n = -3$ for $n \geqslant 1$

(ii) In this part, let w_n be v_n.

(a) $v_1^2 + 1 = 4v_1 - 3 \Rightarrow (v_1 - 2)^2 = 0 \Rightarrow v_1 = 2$
$F_n = v_n^2 + v_{n-1}^2 - 4v_nv_{n-1} = -3$ for $n \geqslant 1$
$\Rightarrow v_n - v_{n-2} = 0$ or $v_n + v_{n-2} - 4v_{n-1} = 0$, for $n \geqslant 2$, by (+).

(b) Since $1, 2, 1, 2, \ldots$ satisfies $v_n - v_{n-2} = 0$ for $n \geqslant 2$, F_n is constant, by (+) and since $v_0 = 1, v_1 = 2$ that constant is -3, so the sequence satisfies (*).

(c) The sequence $1, 2, 7, 2, \ldots$, with period 4, satisfies $v_n - v_{n-2} = 0$ for odd $n \geqslant 2$ and $v_n + v_{n-2} - 4v_{n-1} = 0$ for even $n \geqslant 2$, so F_n is constant, by (+), and since $v_0 = 1, v_1 = 2$ that constant is -3, so the sequence satisfies (*).

7 On $0 \leqslant t \leqslant 1$, the integrand is non-negative and $0 \leqslant \dfrac{t}{t+1} = 1 - \dfrac{1}{t+1} \leqslant \frac{1}{2}$, so

$$I_{n+1} = \int_0^1 \frac{t^n}{(t+1)^{n+1}}\,\mathrm{d}t = \int_0^1 \frac{t}{t+1}\frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t < \frac{1}{2}\int_0^1 \tfrac{1}{2}\frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t = \tfrac{1}{2}I_n.$$

Integration by parts gives

$$I_{n+1} = \left[-\frac{t^n}{n(t+1)^n}\right]_0^1 + \int_0^1 \frac{nt^{n-1}}{n(t+1)^n}\,\mathrm{d}t = -\frac{1}{n2^n} + I_n,$$

so $I_n > 2I_{n+1} = -\dfrac{1}{n2^{n-1}} + 2I_n \quad \Rightarrow \quad I_n < \dfrac{1}{n2^{n-1}} \quad (*)$.

Since $\dfrac{1}{2^r} = I_r - I_{r+1}$, $\displaystyle\sum_{r=1}^{n} \frac{1}{r2^r} = (I_1 - I_2) + (I_2 - I_3) + \ldots + (I_n - I_{n+1}) = I_1 - I_{n+1}$, and $I_1 = \displaystyle\int_0^1 \frac{1}{t+1}\,\mathrm{d}t = \ln 2$, so $\ln 2 = \displaystyle\sum_{r=1}^{n} \frac{1}{r2^r} + I_{n+1}$.

Hence $\ln 2 > \displaystyle\sum_{r=1}^{3} \frac{1}{r2^r} = \frac{2}{3}$ and, by inequality $(*)$, $\ln 2 = \displaystyle\sum_{r=1}^{2} \frac{1}{r2^r} + I_3 < \sum_{r=1}^{2} \frac{1}{r2^r} + \frac{1}{3 \cdot 2^2} = \frac{17}{24}$.

8 If $u = y^2$ then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{f}(x)y^2 + 2\mathrm{g}(x) = 2\mathrm{f}(x)u + 2\mathrm{g}(x)$,

which is a linear differential equation for $u(x)$.

In this case, $\mathrm{f}(x) = \dfrac{1}{x}$, $\mathrm{g}(x) = -1$ so the differential equation is $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{2u}{x} - 2$.

The integrating factor is $\mathrm{e}^{\int \frac{2}{x}\mathrm{d}x} = \mathrm{e}^{2\ln x} = \dfrac{1}{x^2}$, giving $\dfrac{1}{x^2}\dfrac{\mathrm{d}u}{\mathrm{d}x} - \dfrac{2u}{x^3} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{u}{x^2}\right) = \dfrac{-2}{x^2}$

so that $\dfrac{u}{x^2} = \displaystyle\int \frac{-2}{x^2}\mathrm{d}x = \frac{2}{x} + c$ or $u = y^2 = cx^2 + 2x$.

The solution curves which pass through $(1,\ 1)$, $(2,\ 2)$ and $(4,\ 4)$ are $y^2 + (x-1)^2 = 1$, $y^2 = 2x$ and $(x+2)^2 - 2y^2 = 4$ respectively. In drawing these curves it should be made clear that all of them pass through the origin, and that this is their only point of intersection; that the first is a circle with centre $(1, 0)$, the second a parabola and the third an hyperbola with centre $(-2, 0)$ and asymptotes $y = \pm\dfrac{x+2}{\sqrt{2}}$.

Section B: Mechanics

9 Let angle AOM be 2θ, so APM $= \theta$, and let R, F be the normal reaction and frictional forces of the hoop on the mouse.

The forces on the hoop are its weight, the force on the hoop from its suspension, and the reaction on the hoop to the forces R and F of the hoop on the mouse. For the hoop to be in equilibrium, the net moment of these forces about the point of suspension must be zero, but the lines of action of the weight of the hoop, and the force on it from its suspension, pass through the point of suspension, so have zero moment about it. Thus equilibrium of hoop requires the net moment of the reactions to R and F about the point of suspension of the hoop to be zero; that is, $F \times \text{PM}\cos\theta - R \times \text{PM}\sin\theta = 0$, or $F = R\tan\theta$.

For the mouse (of mass m, say) to have constant speed u, its equations of motion are:

resolving radially inward, $R - mg\cos 2\theta = \dfrac{mu^2}{a}$ and resolving tangentially, $F - mg\sin 2\theta = 0$. Combining these three equations gives $mg\sin 2\theta\cos\theta = \left(mg\cos 2\theta + \dfrac{mu^2}{a}\right)\sin\theta$ which reduces to $u^2 = ag$ using the double angle identities.

To maintain a speed u with $u^2 = ag$ requires $R = mg(\cos 2\theta + 1) = 2mg\cos^2\theta$ and $F = mg\sin 2\theta = 2mg\cos^2\theta\tan\theta$ which is greater than μR if θ exceeds $\arctan\mu$ and hence angle AOM exceeds $2\arctan\mu$, so, initially, the hoop will begin to rotate in the opposite sense to the mouse's motion round the circle.

10 For $6a \leqslant x \leqslant 7a$, $\ddot{x} = g + 6g\dfrac{(7a-x)}{2a} - 6g\dfrac{(x-6a)}{6a} = \dfrac{4g}{a}(7a-x)$

and for $7a \leqslant x \leqslant 9a$, $\ddot{x} = g - 6g\dfrac{(x-6a)}{6a} = \dfrac{g}{a}(7a-x)$.

Notice that these both describe simple harmonic motion with $x = 7a$ as the equilibrium position so that, for $6a \leqslant x \leqslant 7a$,

$$x = 7a + A\cos\sqrt{\frac{4g}{a}}t + B\sin\sqrt{\frac{4g}{a}}t \text{ and } \dot{x} = -\sqrt{\frac{4g}{a}}A\sin\sqrt{\frac{4g}{a}}t + \sqrt{\frac{4g}{a}}B\cos\sqrt{\frac{4g}{a}}t$$

and initial conditions $x = 6a, \dot{x} = 0$ at $t = 0$ then give $A = -a, B = 0$.

Let the particle pass through $x = 7a$ at $t = t_0$; then $\sqrt{\dfrac{4g}{a}}t_0 = \dfrac{\pi}{2}$ and, at this point, $\dot{x} = \sqrt{4ga}$.

For $7a \leqslant x \leqslant 9a$, similarly $x = 7a + A\cos\sqrt{\dfrac{g}{a}}(t-t_0) + B\sin\sqrt{\dfrac{g}{a}}(t-t_0)$. The initial conditions are $x = 7a, \dot{x} = \sqrt{4ga}$ at $t = t_0$, which give $A = 0, B = 2a$.

Finally, $x = 9a$ when $\sqrt{\dfrac{g}{a}}(t-t_0) = \dfrac{\pi}{2}$; that is, when $t = \dfrac{\pi}{2}\sqrt{\dfrac{a}{4g}} + \dfrac{\pi}{2}\sqrt{\dfrac{a}{g}} = \dfrac{3\pi}{4}\sqrt{\dfrac{a}{g}}$.

11 Since z is initially, and hence always, positive, Newton's Law gives $2\ddot{x}_1 = -\frac{2}{z^3}$ and $\ddot{x}_2 = \frac{2}{z^3}$, so that $\ddot{z} = \ddot{x}_2 - \ddot{x}_1 = \frac{3}{z^3}$.

Writing this equation as $v\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{3}{z^3}$ and integrating with respect to z, we have $\frac{v^2}{2} = -\frac{3}{2z^2} + c$ so $v = \pm\sqrt{2c - \frac{3}{z^2}}$ where the initial condition $v = -1$ when $z = 1$ requires the negative sign to be chosen and c=2.

Writing $v = \frac{\mathrm{d}z}{\mathrm{d}t}$ and separating the variables gives

$$\int \frac{\mathrm{d}z}{\sqrt{4 - \frac{3}{z^2}}} = -\int \mathrm{d}t \quad \text{or} \quad c - t = \int \frac{z\mathrm{d}z}{\sqrt{4z^2 - 3}} = \tfrac{1}{4}\sqrt{4z^2 - 3}$$

so that $\sqrt{4z^2 - 3} = 1 - 4t$, using the initial condition $z = 1$ at $t = 0$ to determine c.

Then $z = \sqrt{4t^2 - 2t + 1}$ as required.

Defining $w = x_2 + 2x_1$, $\ddot{w} = \ddot{x}_2 + 2\ddot{x}_1 = 0$ so that $\dot{w} = a$, $w = at + b$. Initially, $\dot{x}_1 = 1$ and $\dot{x}_2 = 0$ so $a = 2$; $x_1 = 0$ and $x_2 = 1$ so $b = 1$. This gives

$$x_1 = \tfrac{1}{3}(w - z) = \tfrac{1}{3}\left(2t + 1 - \sqrt{4t^2 - 2t + 1}\right) \text{ and } x_2 = \tfrac{1}{3}(w + 2z) = \tfrac{1}{3}\left(2t + 1 + 2\sqrt{4t^2 - 2t + 1}\right).$$

It is worth noting, though not required by the question, that $\dot{x}_1 \to \frac{1}{2}$, $\dot{x}_2 \to 2$ as $t \to \infty$.

Section C: Statistics

12 For C_1, we have $\mathrm{P}(0) = \dfrac{m-1}{m}$ and $\mathrm{P}(1) = \dfrac{1}{m}$, so that $\mathrm{E}\,[C_1] = 0 \times \dfrac{m-1}{m} + 1 \times \dfrac{1}{m} = \dfrac{1}{m}$ and $\mathrm{Var}\,[C_1] = \left(0^2 \times \dfrac{m-1}{m} + 1^2 \times \dfrac{1}{m}\right) - \left(\dfrac{1}{m}\right)^2 = \dfrac{m-1}{m^2}$

$\mathrm{Cov}\,[C_1, C_2] = 1^2 \times \mathrm{P}(C_1 = C_2 = 1) - \mathrm{E}[C_1]\mathrm{E}[C_1]$ (since the other terms in the expectation of C_1C_2 are all zero). $\mathrm{P}(C_1 = C_2 = 1) = \mathrm{P}(\text{players 1 and 2 get their own shirts}) = \dfrac{1}{m}\dfrac{1}{m-1}$,

so $\mathrm{Cov}\,[C_1, C_2] = \dfrac{1}{m(m-1)} - \left(\dfrac{1}{m}\right)^2 = \dfrac{1}{m^2(m-1)}$

$\mathrm{E}\,[N] = \mathrm{E}\,[C_1] + \mathrm{E}\,[C_2] + \ldots = m \cdot \dfrac{1}{m} = 1$ and $\mathrm{Var}\,[N] = \mathrm{Var}\,[C_1] + \mathrm{Var}\,[C_2] + \ldots + \mathrm{Cov}\,[C_1, C_2] + \mathrm{Cov}\,[C_1, C_3] + \mathrm{Cov}\,[C_2, C_1] + \ldots = m \cdot \mathrm{Var}\,[C_1] + m(m-1) \cdot \mathrm{Cov}\,[C_1, C_2] = 1$.

A normal approximation with mean and standard deviation both equal to 1 is not likely to be appropriate as the approximation would give high probability to negative values of N, which are impossible. A Poisson approximation might be reasonable as mean = variance.

There are 9 arrangements where no player wears his own shirt out of 24 permutations,while the Poisson approximation to $\mathrm{P}(0)$, with mean 1, is e^{-1}.

The relative error is $\dfrac{\frac{9}{24} - \mathrm{e}^{-1}}{\frac{9}{24}} \approx 1 - \dfrac{800}{3 \times 272} = \dfrac{2}{102} \approx 2\%$.

13 (i) P(a competitor drops out in round r) $= p^{r-1}(1-p)$

so P(all three drop out in round r)$= \left(p^{r-1}(1-p)\right)^3$,

so P(all three drop out in the same round) $\equiv P_3 = \sum_{r=1}^{\infty}\left(p^{r-1}(1-p)\right)^3$

This is a geometric series with common ratio p^3 and first term $(1-p)^3$

so $P_3 = \dfrac{(1-p)^3}{1-p^3}$.

(ii) The probability that a competitor survives round $r-1$ is p^{r-1}, so the probability that a competitor drops out in round $r-1$ or earlier (that is, before round r) is $1-p^{r-1}$. Therefore the probability that two competitors drop out in round r and the third earlier is $3\times\left(p^{r-1}(1-p)\right)^2\times\left(1-p^{r-1}\right)$, where the factor of three is required, because any of the three could be the one to drop out earliest.

(iii) From (ii), Pr(two drop out in same round and the third earlier) $\equiv P_2$

$= \sum_{r=2}^{\infty} 3\left(p^{r-1}(1-p)\right)^2\left(1-p^{r-1}\right) = 3(1-p)^2\sum_{r=2}^{\infty}\left(p^{2(r-1)}-p^{3(r-1)}\right)$

$= 3(1-p)^2\left(\dfrac{p^2}{1-p^2}-\dfrac{p^3}{1-p^3}\right)$, summing to infinity two geometric series with first terms p^2 and p^3 and common ratios p^2 and p^3 respectively.

Pr(the grand prize is awarded) $= 1-P_2-P_3$, which simplifies to $\dfrac{3p\left(1+p^2\right)}{(1+p)\left(1+p+p^2\right)}$, using the factorisation $1-p^3=(1-p)(1+p+p^2)$.

14 The test is appropriate because, if H_0 were true, $\bar{x}$ would have a higher probability of being in the region stated than if H_1 were true.

Under H_0, $\bar{X}$ has a Normal distribution with mean μ and standard deviation $\dfrac{\sigma_0}{\sqrt{n}}$ so

$$\alpha = \mathrm{P}(|\bar{X} - \mu| > c) = 2\left(1 - \Phi\left(\frac{c}{\frac{\sigma_0}{\sqrt{n}}}\right)\right)$$

so $\Phi\left(\dfrac{c}{\frac{\sigma_0}{\sqrt{n}}}\right) = 1 - \dfrac{\alpha}{2}$, so $\dfrac{c}{\frac{\sigma_0}{\sqrt{n}}} = z_\alpha$ or $c = \dfrac{\sigma_0 z_\alpha}{\sqrt{n}}$.

Under H_1, $\bar{X}$ has a Normal distribution with mean μ and standard deviation $\dfrac{\sigma_1}{\sqrt{n}}$ so

$$\beta = \mathrm{P}(|\bar{X} - \mu| < c) = 1 - 2\left(1 - \Phi\left(\frac{c}{\frac{\sigma_1}{\sqrt{n}}}\right)\right) = 2\Phi\left(\frac{c}{\frac{\sigma_1}{\sqrt{n}}}\right) - 1 = 2\Phi\left(\frac{\sigma_0 z_\alpha}{\sigma_1}\right) - 1,$$

so β is independent of n.

$$\beta < 0.05 \quad \Rightarrow \quad \Phi\left(\frac{\sigma_0 z_\alpha}{\sigma_1}\right) < \frac{1 + 0.05}{2} = 0.525 \quad \Rightarrow \quad \frac{\sigma_0 z_\alpha}{\sigma_1} < 0.063.$$

$$\alpha < 0.05 \quad \Rightarrow \quad z_\alpha > 1.960.$$

For these both to hold, we must have $0.063 > \dfrac{\sigma_0 z_\alpha}{\sigma_1} > \dfrac{1.960\sigma_0}{\sigma_1}$ or $\dfrac{\sigma_1}{\sigma_0} > \dfrac{1.960}{.063} = \dfrac{280}{9} \approx 30.$

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