SIXTH TERM EXAMINATION PAPERS

administered by the Oxford and Cambridge Schools Examination Board
on behalf of the Cambridge Colleges

9475

FURTHER MATHEMATICS

PAPER B

Friday 2 July 1993, afternoon

3 hours

Additional materials:
script paper; graph paper; MF(STEP)1.
To be brought by candidate: electronic calculator;
standard geometrical instruments.

All questions carry equal weight.

Candidates are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.

Candidates may attempt as many questions as they wish but marks will be assessed on the six questions best answered.

Mathematical Formulae and tables (MF STEP)1 are provided.

Electronic calculators may be used.

1 The curve P has the parametric equations

$$x = \sin\theta, \quad y = \cos 2\theta \qquad \text{for } -\pi/2 \leqslant \theta \leqslant \pi/2.$$

Show that P is part of the parabola $y = 1 - 2x^2$ and sketch P.

Show that the length of P is $\surd(17) + \frac{1}{4}\sinh^{-1} 4$.

Obtain the volume of the solid enclosed when P is rotated through 2π radians about the line $y = -1$.

2 The curve C has the equation $x^3 + y^3 = 3xy$.

(i) Show that there is no point of inflection on C. You may assume that the origin is not a point of inflection.

(ii) The part of C which lies in the first quadrant is a closed loop touching the axes at the origin. By converting to polar coordinates, or otherwise, evaluate the area of this loop.

3 The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{M}$ are given by

$$\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 13 & 5 \\ 3 & 8 & 7 \end{pmatrix},$$

where a, b, ..., r are real numbers. Given that $\mathbf{M} = \mathbf{AB}$, show that $a = 1$, $b = 4$, $c = 1$, $d = 3$, $e = -1$, $f = -2$, $p = 3$, $q = 2$, $r = -3$ gives the *unique* solution for $\mathbf{A}$ and $\mathbf{B}$. Evaluate $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

Hence, or otherwise, solve the simultaneous equations

$$\begin{aligned} x + 3y + 2z &= 7 \\ 4x + 13y + 5z &= 18 \\ 3x + 8y + 7z &= 25. \end{aligned}$$

4 Sum the following infinite series.

(i) $1+\frac{1}{3}\left(\frac{1}{2}\right)^2+\frac{1}{5}\left(\frac{1}{2}\right)^4+\cdots+\frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n}+\cdots$.

(ii) $2-x-x^3+2x^4-\cdots+2x^{4k}-x^{4k+1}-x^{4k+3}+\cdots$ where $|x|<1$.

(iii) $\displaystyle\sum_{r=2}^{\infty}\frac{r\,2^{r-2}}{3^{r-1}}$.

(iv) $\displaystyle\sum_{r=2}^{\infty}\frac{2}{r(r^2-1)}$.

5 The set S consists of ordered pairs of complex numbers (z_1, z_2) and a binary operation $\circ$ on S is defined by

$$(z_1, z_2)\circ(w_1, w_2) = (z_1w_1 - z_2w_2^*,\ z_1w_2 + z_2w_1^*).$$

Show that the operation $\circ$ is associative and determine whether it is commutative. Evaluate $(z,0)\circ(w,0)$, $(z,0)\circ(0,w)$, $(0,z)\circ(w,0)$ and $(0,z)\circ(0,w)$.

The set S_1 is the subset of S consisting of $A, B, \ldots, H$, where $A=(1,0)$, $B=(0,1)$, $C=(i,0)$, $D=(0,i)$, $E=(-1,0)$, $F=(0,-1)$, $G=(-i,0)$ and $H=(0,-i)$. Show that S_1 is closed under $\circ$ and that it has an identity element. Determine the inverse and order of each element of S_1. Show that S_1 is a group under $\circ$.
[You are not required to compute the multiplication table in full.]

Show that $\{A, B, E, F\}$ is a subgroup of S_1 and determine whether it is isomorphic to the group generated by the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ under matrix multiplication.

6 The point in the Argand diagram representing the complex number z lies on the circle with centre K and radius r, where K represents the complex number k. Show that

$$zz^* - kz^* - k^*z + kk^* - r^2 = 0.$$

The points P, Q_1 and Q_2 represent the complex numbers z, w_1 and w_2 respectively. The point P lies on the circle with OA as diameter, where O and A represent 0 and $2i$ respectively. Given that $w_1 = z/(z-1)$, find the equation of the locus L of Q_1 in terms of w_1 and describe the geometrical form of L.

Given that $w_2 = z^*$, show that the locus of Q_2 is also L. Determine the positions of P for which Q_1 coincides with Q_2.

7 The real numbers x and y satisfy the simultaneous equations

$$\sinh(2x) = \cosh y \qquad \text{and} \qquad \sinh(2y) = 2\cosh x.$$

Show that $\sinh^2 y$ is a root of the equation

$$4t^3 + 4t^2 - 4t - 1 = 0$$

and demonstrate that this gives at most one valid solution for y. Show that the relevant value of t lies between 0.7 and 0.8, and use an iterative process to find t to 6 decimal places.

Find y and hence find x, checking your answers and stating the final answers to four decimal places.

8 A square pyramid has its base vertices at the points A $(a,0,0)$, B $(0,a,0)$, C $(-a,0,0)$ and D $(0,-a,0)$, and its vertex at E $(0,0,a)$. The point P lies on AE with x-coordinate λa, where $0 < \lambda < 1$, and the point Q lies on CE with x-coordinate $-\mu a$, where $0 < \mu < 1$. The plane BPQ cuts DE at R and the y-coordinate of R is $-\gamma a$. Prove that

$$\gamma = \frac{\lambda\mu}{\lambda + \mu - \lambda\mu}.$$

Show that the quadrilateral $BPRQ$ cannot be a parallelogram.

9 For the real numbers a_1, a_2, a_3, ...,

(i) prove that $a_1^2 + a_2^2 \geqslant 2a_1a_2$,

(ii) prove that $a_1^2 + a_2^2 + a_3^2 \geqslant a_2a_3 + a_3a_1 + a_1a_2$,

(iii) prove that $3(a_1^2 + a_2^2 + a_3^2 + a_4^2) \geqslant 2(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)$,

(iv) state and prove a generalisation of (iii) to the case of n real numbers,

(v) prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 \geqslant \frac{2n}{n-1} \sum a_i a_j,$$

where the latter sum is taken over all pairs (i, j) with $1 \leqslant i < j \leqslant n$.

10 The transformation T of the point P in the x,y plane to the point P' is constructed as follows: Lines are drawn through P parallel to the lines $y = mx$ and $y = -mx$ to cut the line $y = kx$ at Q and R respectively, m and k being given constants. P' is the fourth vertex of the parallelogram $PQP'R$.

Show that if P is (x_1, y_1) then Q is

$$\left(\frac{mx_1 - y_1}{m-k}, \frac{k(mx_1 - y_1)}{m-k}\right).$$

Obtain the coordinates of P' in terms of x_1, y_1, m and k, and express T as a matrix transformation. Show that areas are transformed under T into areas of the same magnitude.

11 [In this question, all gravitational forces are to be neglected.] A rigid frame is constructed from 12 equal uniform rods, each of length a and mass m, forming the edges of a cube. Three of the edges are OA, OB and OC, and the vertices opposite O, A, B and C are O', A', B' and C' respectively. Forces act along the lines as follows, in the directions indicated by the order of the letters:

$2mg$ along OA,	mg along AC',	$\sqrt{2}mg$ along $O'A$,
$\sqrt{2}mg$ along OA',	$2mg$ along $C'B$,	mg along $A'C$.

Initially the frame is at rest and there are no other forces acting on it.

(i) The frame is freely pivoted at O. Show that the direction of the line about which it will start to rotate is $\begin{pmatrix}1\\1\\2\end{pmatrix}$ with respect to axes along OA, OB and OC respectively.

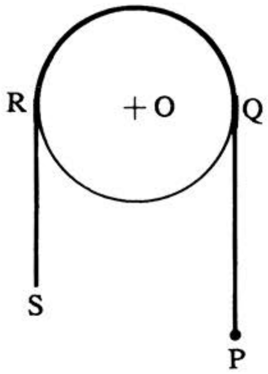
(ii) Show that the moment of inertia of the rod OA about the axis OO' is $2ma^2/9$ and about a parallel axis through its mid-point is $ma^2/18$. Hence find the moment of inertia of $B'C$ about OO' and show that the moment of inertia of the frame about OO' is $14ma^2/3$. If the frame is freely pivoted about the line OO' and the forces continue to act along the specified lines, find the initial angular acceleration of the frame.

12 $ABCD$ is a horizontal line with $AB = CD = a$ and $BC = 6a$. There are fixed smooth pegs at B and C. A uniform string of natural length $2a$ and modulus of elasticity kmg is stretched from A to D, passing over the pegs at B and C. A particle of mass m is attached to the midpoint P of the string. When the system is in equilibrium, P is a distance $a/4$ below BC. Evaluate k.

The particle is pulled down to a point Q, which is at a distance pa below the mid-point of BC, and is released from rest. P rises to a point R, which is at a distance $3a$ above BC. Show that $2p^2 - p - 17 = 0$.

Show also that the tension in the strings is less when the particle is at R than when the particle is at Q.

13



A uniform circular disc with radius a, mass $4m$ and centre O is freely mounted on a fixed horizontal axis which is perpendicular to its plane and passes through O. A uniform heavy chain PS of length $(4+\pi)a$, mass $(4+\pi)m$ and negligible thickness is hung over the rim of the disc as shown in the diagram: Q and R are the points of the chain at the same level as O. The contact between the chain and the rim of the disc is sufficiently rough to prevent slipping. Initially, the system is at rest with $PQ = RS = 2a$. A particle of mass m is attached to the chain at P and the system is released. By considering the energy of the system, show that when P has descended a distance x, its speed v is given by

$$(\pi + 7)av^2 = 2g(x^2 + ax).$$

By considering the part PQ of the chain as a body of variable mass, show that when S reaches R the tension in the chain at Q is

$$\frac{5\pi - 2}{\pi + 7}mg.$$

14 A particle rests at a point A on a horizontal table and is joined to a point O on the table by a taut inextensible string of length c. The particle is projected vertically upwards at a speed $64\sqrt{(6gc)}$. It next strikes the table at a point B and rebounds. The coefficient of restitution for any impact between the particle and the table is $\frac{1}{2}$. After rebounding at B, the particle will rebound alternately at A and B until the string becomes slack. Show that when the string becomes slack the particle is at height $c/2$ above the table.

Determine whether the first rebound *between* A and B is nearer to A or to B.

15 The probability of throwing a head with a certain coin is p and the probability of throwing a tail is $q = 1 - p$. The coin is thrown until at least two heads and at least two tails have been thrown; this happens when the coin has been thrown N times. Write down an expression for the probability that $N = n$.

Show that the expectation of N is

$$2\left(\frac{1}{pq} - 1 - pq\right).$$

16 The time taken for me to set an acceptable examination question it T hours. The distribution of T is a truncated normal distribution with probability density f where

$$\mathrm{f}(t) = \begin{cases} \dfrac{1}{k\sigma\sqrt{(2\pi)}} \exp\left(-\dfrac{1}{2}\left(\dfrac{t-\sigma}{\sigma}\right)^2\right) & \text{for } t \geqslant 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Sketch the graph of $\mathrm{f}(t)$. Show that k is approximately 0.841 and obtain the mean of T as a multiple of σ.

Over a period of years, I find that the mean setting time is 3 hours.

(i) Find the approximate probability that none of the 16 questions on next year's paper will take more than 4 hours to set.

(ii) Given that a particular question is unsatisfactory after 2 hours work, find the probability that it will still be unacceptable after a further 2 hours work.