

**9470**

**STEP**

**SIXTH TERM EXAMINATION PAPERS**

**FURTHER MATHEMATICS**

**PAPER A**

Thursday

**28 JUNE 1990**

Morning

3 hours

Additional materials provided by the Syndicate/Board:

1. Mathematical Formulae and tables MF (STEP) 1

Additional materials provided by the candidate:

2. Electronic calculator

UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE  
OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD  
UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE

**UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE**

**OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD**

**Instructions to candidates:**

*All questions carry equal weight.*

*Candidates are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.*

*Candidates may attempt as many questions as they wish but marks will be assessed on the **six** questions best answered.*

---

**This Question Paper consists of 6 printed pages and 2 blank pages.**

1 Prove that both  $x^4 - 2x^3 + x^2$  and  $x^2 - 8x + 17$  are non-negative for all real  $x$ . By considering the intervals  $x \leq 0$ ,  $0 < x \leq 2$  and  $x > 2$  separately, or otherwise, prove that the equation

$$x^4 - 2x^3 + x^2 - 8x + 17 = 0$$

has no real roots.

Prove that the equation  $x^4 - x^3 + x^2 - 4x + 4 = 0$  has no real roots.

2 Prove that if  $A + B + C + D = \pi$ , then

$$\sin(A + B) \sin(A + D) - \sin B \sin D = \sin A \sin C.$$

The points  $P, Q, R$  and  $S$  lie, in that order, on a circle of centre  $O$ . Prove that

$$PQ \cdot RS + QR \cdot PS = PR \cdot QS.$$

3 Sketch the curves given by

$$y = x^3 - 2bx^2 + c^2x,$$

where  $b$  and  $c$  are non-negative, in the cases:

- (i)  $2b < c\sqrt{3}$ ;           (ii)  $2b = c\sqrt{3} \neq 0$ ;           (iii)  $c\sqrt{3} < 2b < 2c$ ;           (iv)  $b = c \neq 0$ ;  
(v)  $b > c > 0$ ;           (vi)  $c = 0, b \neq 0$ ;           (vii)  $c = b = 0$ .

Sketch also the curves given by

$$y^2 = x^3 - 2bx^2 + c^2x$$

in the cases (i), (v) and (vii).

4 A plane contains  $n$  distinct given lines, no two of which are parallel, and no three of which intersect at a point. By first considering the cases  $n = 1, 2, 3$  and  $4$ , provide and justify, by induction or otherwise, a formula for the number of line segments (including the infinite segments).

Prove also that the plane is divided into  $\frac{1}{2}(n^2 + n + 2)$  regions (including those extending to infinity).

5 The distinct points  $L, M, P$  and  $Q$  of the Argand diagram lie on a circle  $S$  centred on the origin and the corresponding complex numbers are  $l, m, p$  and  $q$ . By considering the perpendicular bisectors of the chords, or otherwise, prove that the chord  $LM$  is perpendicular to the chord  $PQ$  if and only if  $lm + pq = 0$ .

Let  $A_1, A_2$  and  $A_3$  be three given distinct points on  $S$ . For any given point  $A'_1$  on  $S$ , the points  $A'_2, A'_3$  and  $A''_1$  are chosen on  $S$  such that  $A'_1A'_2, A'_2A'_3$  and  $A'_3A''_1$  are perpendicular to  $A_1A_2, A_2A_3$  and  $A_3A_1$ , respectively. Show that for exactly two positions of  $A'_1$ , the points  $A'_1$  and  $A''_1$  coincide.

If, instead,  $A_1, A_2, A_3$  and  $A_4$  are four given distinct points on  $S$  and, for any given point  $A'_1$ , the points  $A'_2, A'_3, A'_4$  and  $A''_1$  are chosen on  $S$  such that  $A'_1A'_2, A'_2A'_3, A'_3A'_4$  and  $A'_4A''_1$  are respectively perpendicular to  $A_1A_2, A_2A_3, A_3A_4$  and  $A_4A_1$ , show that  $A'_1$  coincides with  $A''_1$ .

Give the corresponding result for  $n$  distinct points on  $S$ .

6 Let  $a, b, c, d, p$  and  $q$  be positive integers. Prove that:

(i) if  $b > 1$  and  $c > 1$ , then  $bc \geq 2c \geq 2 + c$ ;

(ii) if  $a < b$  and  $d < c$ , then  $bc - ad \geq a + c$ ;

(iii) if  $\frac{a}{b} < p < \frac{c}{d}$ , then  $(bc - ad)p \geq a + c$ ;

(iv) if  $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ , then  $p \geq \frac{a + c}{bc - ad}$  and  $q \geq \frac{b + d}{bc - ad}$ .

Hence find all fractions with denominators less than 20 which lie between  $8/9$  and  $9/10$ .

7 A damped system with feedback is modelled by the equation

$$f'(t) + f(t) - kf(t-1) = 0, \quad (\dagger)$$

where  $k$  is a given non-zero constant. Show that (non-zero) solutions for  $f$  of the form  $f(t) = Ae^{pt}$ , where  $A$  and  $p$  are constants, are possible provided  $p$  satisfies

$$p + 1 = ke^{-p}. \quad (*)$$

Show also, by means of a sketch, or otherwise, that equation (\*) can have 0, 1 or 2 real roots, depending on the value of  $k$ , and find the set of values of  $k$  for which such solutions of (†) exist. For what set of values of  $k$  do such solutions tend to zero as  $t \rightarrow +\infty$ ?

8 The functions  $x$  and  $y$  are related by

$$x(t) = \int_0^t y(u) du,$$

so that  $x'(t) = y(t)$ . Show that

$$\int_0^1 x(t)y(t)dt = \frac{1}{2}[x(1)]^2.$$

In addition, it is given that  $y(t)$  satisfies

$$y'' + (y^2 - 1)y' + y = 0, \quad (*)$$

with  $y(0) = y(1)$  and  $y'(0) = y'(1)$ . By integrating (\*), prove that  $x(1) = 0$ .

By multiplying (\*) by  $x(t)$  and integrating by parts, prove the relation

$$\int_0^1 [y(t)]^2 dt = \frac{1}{3} \int_0^1 [y(t)]^4 dt.$$

Prove also the relation

$$\int_0^1 [y'(t)]^2 dt = \int_0^1 [y(t)]^2 dt.$$

9 Show by means of a sketch that the parabola  $r(1 + \cos \theta) = 1$  cuts the interior of the cardioid  $r = 4(1 + \cos \theta)$  into two parts.

Show that the total length of the boundary of the part that includes the point  $r = 1, \theta = 0$  is  $18\sqrt{3} + \ln(2 + \sqrt{3})$ .

10 Two square matrices  $\mathbf{A}$  and  $\mathbf{B}$  satisfy  $\mathbf{AB} = \mathbf{0}$ . Show that either  $\det \mathbf{A} = 0$  or  $\det \mathbf{B} = 0$  or  $\det \mathbf{A} = \det \mathbf{B} = 0$ . If  $\det \mathbf{B} \neq 0$ , what must  $\mathbf{A}$  be? Give an example to show that the condition  $\det \mathbf{A} = \det \mathbf{B} = 0$  is not sufficient for the equation  $\mathbf{AB} = \mathbf{0}$  to hold.

Find real numbers  $p, q$  and  $r$  such that

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = (\mathbf{M} + p\mathbf{I})(\mathbf{M} + q\mathbf{I})(\mathbf{M} + r\mathbf{I}),$$

where  $\mathbf{M}$  is any square matrix and  $\mathbf{I}$  is the appropriate identity matrix.

Hence, or otherwise, find all matrices  $\mathbf{M}$  of the form  $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$  which satisfy the equation

$$\mathbf{M}^3 + 2\mathbf{M}^2 - 5\mathbf{M} - 6\mathbf{I} = \mathbf{0}.$$

11 A disc is free to rotate in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is  $mk^2$ . Along one diameter is a narrow groove in which a particle of mass  $m$  slides freely. At time  $t = 0$ , the disc is rotating with angular speed  $\Omega$ , and the particle is at a distance  $a$  from the axis and is moving towards the axis with speed  $V$ , where  $k^2V^2 = \Omega^2a^2(k^2 + a^2)$ . Show that, at a later time  $t$ , while the particle is still moving towards the axis, the angular speed  $\omega$  of the disc and the distance  $r$  of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \quad \text{and} \quad \frac{dr}{dt} = -\frac{\Omega r(k^2 + a^2)}{k(k^2 + r^2)^{\frac{1}{2}}}.$$

Deduce that

$$k \frac{dr}{d\theta} = -r(k^2 + r^2)^{\frac{1}{2}},$$

where  $\theta$  is the angle through which the disc has turned at time  $t$ . By making the substitution  $u = 1/r$ , or otherwise, show that  $r \sinh(\theta + \alpha) = k$ , where  $\sinh \alpha = k/a$ . Hence, or otherwise, show that the particle never reaches the axis.

12 A straight staircase consists of  $N$  smooth horizontal stairs each of height  $h$ . A particle slides over the top stair at speed  $U$ , with velocity perpendicular to the edge of the stair, and then falls down the staircase, bouncing once on every stair. The coefficient of restitution between the particle and each stair is  $e$ , where  $e < 1$ . Show that the horizontal distance  $d_n$  travelled between the  $n$ th and  $(n + 1)$ th bounces is given by

$$d_n = U \left( \frac{2h}{g} \right)^{\frac{1}{2}} (e\alpha_n + \alpha_{n+1}),$$

where  $\alpha_n = \left( \frac{1 - e^{2n}}{1 - e^2} \right)^{\frac{1}{2}}$ .

If  $N$  is very large, show that  $U$  must satisfy

$$U = \left( \frac{L^2 g}{2h} \right)^{\frac{1}{2}} \left( \frac{1 - e}{1 + e} \right)^{\frac{1}{2}},$$

where  $L$  is the horizontal distance between the edges of successive stairs.

13 A thin non-uniform rod  $PQ$  of length  $2a$  has its centre of gravity a distance  $a + d$  from  $P$ . It hangs (not vertically) in equilibrium suspended from a small smooth peg  $O$  by means of a light inextensible string of length  $2b$  which passes over the peg and is attached at its ends to  $P$  and  $Q$ . Express  $OP$  and  $OQ$  in terms of  $a$ ,  $b$  and  $d$ . By considering the angle  $POQ$ , or otherwise, show that  $d < a^2/b$ .

14 The identical uniform smooth spherical marbles  $A_1, A_2, \dots, A_n$ , where  $n \geq 3$ , each of mass  $m$ , lie in that order in a smooth straight trough, with each marble touching the next. The marble  $A_{n+1}$ , which is similar to  $A_n$  but has mass  $\lambda m$ , is placed in the trough so that it touches  $A_n$ . Another marble  $A_0$ , identical to  $A_n$ , slides along the trough with speed  $u$  and hits  $A_1$ . It is given that kinetic energy is conserved throughout.

(i) Show that if  $\lambda < 1$ , there is a possible subsequent motion in which only  $A_n$  and  $A_{n+1}$  move (and  $A_0$  is reduced to rest), but that if  $\lambda > 1$ , such a motion is not possible.

(ii) If  $\lambda > 1$ , show that a subsequent motion in which only  $A_{n-1}, A_n$  and  $A_{n+1}$  move is not possible.

(iii) If  $\lambda > 1$ , find a possible subsequent motion in which only two marbles move.

15 A target consists of a disc of unit radius and centre  $O$ . A certain marksman never misses the target, and the probability of any given shot hitting the target within a distance  $t$  from  $O$  is  $t^2$ , where  $0 \leq t \leq 1$ . The marksman fires  $n$  shots independently. The random variable  $Y$  is the radius of the smallest circle, with centre  $O$ , which encloses all the shots. Show that the probability density function of  $Y$  is  $2ny^{2n-1}$  and find the expected area of the circle.

The shot which is furthest from  $O$  is rejected. Show that the expected area of the smallest circle, with centre  $O$ , which encloses the remaining  $(n-1)$  shots is  $\left(\frac{n-1}{n+1}\right)\pi$ .

16 Each day, I choose at random between my brown trousers, my grey trousers and my expensive but fashionable designer jeans. Also in my wardrobe, I have a black silk tie, a rather smart brown and fawn polka-dot tie, my regimental tie and an elegant powder-blue cravat tie which I was given for Christmas. With the brown or grey trousers, I choose ties at random, except of course that I don't wear the cravat with the brown trousers nor the polka-dot tie with the grey trousers. With the jeans, the choice depends on whether it is Sunday or one of the six weekdays: on the weekdays, half the time I wear a cream-coloured sweat-shirt with  $E = mc^2$  on the front and no tie; otherwise, and on Sundays (when I always wear a tie), I just pick at random from my four ties.

This morning, I received through the post a compromising photograph of me. I often receive such photographs and they are equally likely to have been taken on any day of the week. However, in this particular photograph I am wearing my black tie. Show that, on the basis of this information, the probability that it was taken on a Sunday is  $11/68$ .

I should have mentioned that on Mondays I lecture on calculus, and therefore always wear my jeans (so that the lectures are easier to understand). Find, in the light of this new information, the probability that the photograph was taken on a Sunday.

[The phrase 'at random' means 'with equal probability'.]