9470 **STEP JULY**

SIXTH TERM EXAMINATION PAPERS

FURTHER MATHEMATICS
PAPER A

First examination in 1987

(time) *(date)* 3 hours

Additional materials provided by the Syndicate/Board:
Mathematical Formulae and tables MF (STEP) 1.

UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS SYNDICATE

OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION BOARD

SPECIMEN PAPER

Instructions to candidates:

All questions carry equal weight and you may attempt as many as you wish.

Candidates are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.

This Question Paper consists of 6 printed pages and 2 blank pages.

1 Given that

$$f(x) = \frac{3x^2 + 2(a+b)x + ab}{x^3 + (a+b)x^2 + abx}, \quad \text{where } a \text{ and } b \text{ are non-zero}$$

express $f(x)$ in partial fractions, considering any special case which may arise.

If x, a and b are positive integers, show that $f(x)$ takes the value 1 for only a finite number of values of x, a and b.

2 Given that x, y and z satisfy the equations

$$x^2 - yz = a,$$
$$y^2 - zx = b,$$
$$z^2 - xy = c,$$

where a, b and c are positive distinct real numbers, show that

$$\frac{y-z}{b-c} = \frac{z-x}{c-a} = \frac{x-y}{a-b} = \frac{1}{x+y+z}.$$

By considering

$$(y-z)^2 + (z-x)^2 + (x-y)^2,$$

or otherwise, show that

$$x + y + z = \Delta,$$

where

$$\Delta^2 = \frac{a^2 + b^2 + c^2 - bc - ca - ab}{a+b+c}.$$

Hence solve the given equations for x, y and z.

3 Prove de Moivre's theorem, that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

where n is a positive integer.

Find all real numbers x and y which satisfy

$$x^3\cos 3y + 2x^2\cos 2y + 2x\cos y = -1,$$
$$x^3\sin 3y + 2x^2\sin 2y + 2x\sin y = 0.$$

4 (*a*) Show that

$$\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x),$$

when principal values only are considered.

(*b*) Show that

$$\sinh^{-1}(\tan y) = \tanh^{-1}(\sin y),$$

when $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$.

Sketch the graphs of $\sinh^{-1}(\tan y)$ and $\tanh^{-1}(\sin y)$ in the interval $-\pi < y < \pi$ and find the relationship between the two expressions when $\frac{1}{2}\pi < y < \pi$.

5 Explain, by means of a sketch, or otherwise, why

$$\sum_{r=n}^{\infty} r^{-2} > \int_n^{\infty} x^{-2}\,dx > \sum_{r=n+1}^{\infty} r^{-2}.$$

Deduce that

$$n^{-1} > A - \sum_{r=1}^{n} r^{-2} > (n+1)^{-1},$$

where $A = \sum_{1}^{\infty} r^{-2}$.

Find the smallest value of n for which $\sum_{1}^{n} r^{-2}$ approximates A with an error of less than 10^{-4}.

Show that, for this n,

$$(n+1)^{-1} + \sum_{r=1}^{n} r^{-2}$$

approximates A with an error of less than 10^{-8}.

6 [In this question, standard properties of exponential, logarithmic and trigonometric functions should not be used.]

A function f satisfies

$$\frac{d}{dx}[f(x)] = f(x)$$

with $f(0) = 1$. For any fixed number a, show that

$$\frac{d}{dx}[f(a-x)f(x)] = 0,$$

and deduce that $f(x)f(y) = f(x+y)$ for all x and y.

Functions c and s satisfy

$$\frac{d}{dx}[c(x)] = -\,s(x) \quad \text{and} \quad \frac{d}{dx}[s(x)] = c(x),$$

with $s(0) = 0$ and $c(0) = 1$. Show that

$$c(x+y) = c(x)c(y) - s(x)s(y).$$

7 Let

$$I = \int_0^{\ln K} [e^x]\,dx,$$

where the notation $[y]$ means the largest integer less than or equal to y. Show that

$$I = N \ln K - \ln(N!),$$

where $N = [K]$.

8 Let S be the set of consecutive integers $1, 2, \ldots, (N-1)$, where $N \geqslant 3$, and let G be a subset of S which forms a group under multiplication modulo N. Show that if $(N-1) \in G$, then the order of $(N-1)$ is 2.

Let m and n be elements of G, with orders p and q respectively, such that $m + n = N$. Explaining your reasoning carefully, show that

(i) if p and q are both even, then $p = q$,

(ii) if p is even and q is odd, then $p = 2q$,

(iii) it is impossible for both p and q to be odd.

Now suppose that $G = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$, which may be assumed to form a group under multiplication modulo 21. Calculate the order of the elements 2 and 5 of this group. By making deductions about the orders of all other elements of G, or otherwise, prove that G is not isomorphic to the cyclic group of order 12.

9 (*a*) Let $\mathbf{a}$ and $\mathbf{b}$ be given vectors with $\mathbf{b} \neq \mathbf{0}$, and let $\mathbf{x}$ be a position vector. Find the condition for the sphere $|\mathbf{x}| = R$, where $R > 0$, and the plane $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0$ to intersect.

When this condition is satisfied, find the radius and the position vector of the centre of the circle in which the plane and sphere intersect.

(*b*) Let $\mathbf{c}$ be a given vector, with $\mathbf{c} \neq \mathbf{0}$. The vector $\mathbf{x}'$ is related to the vector $\mathbf{x}$ by

$$\mathbf{x}' = \mathbf{x} - \frac{2(\mathbf{x} \cdot \mathbf{c})\,\mathbf{c}}{|\mathbf{c}|^2}.$$

Interpret this relation geometrically.

10 The distant island of Amphibia is populated by speaking frogs and toads. They spend much of their time in small groups, making statements about themselves. Toads always tell the truth and frogs always lie. In each of the following four scenes from Amphibian life, decide which characters mentioned are frogs and which are toads, explaining your reasoning carefully:

(i) A: "Both B and myself are frogs."

(ii) C: "At least one of D and myself is a frog."

(iii) E: "Both G and H are toads."
G: "That is true."
H: "No, that is not true."

(iv) I and J talking about I, J and K:
I: "All of us are frogs."
J: "Exactly one of us is a toad."

11 Two points A and B are at a distance a apart on a horizontal plane. A particle of mass m is projected from A with speed V, at an angle of elevation of 45° to the line AB. Another particle, also of mass m, is projected from B with speed U at an angle of elevation of 30° to the line BA so that the two particles collide at the instant when each particle is at the highest point of its trajectory. Show that $U^2 = 2V^2$ and that

$$a = \frac{V^2}{2g}(1 + \sqrt{3}).$$

At impact the two particles coalesce. When the combined particle strikes the horizontal plane the velocity of the particle is inclined at an angle ϕ to the horizontal. Show that $\tan \phi = 1 + \sqrt{3}$.

12 A thin smooth wire in the form of a circle, of radius a and centre O, is fixed in a horizontal plane. Two small beads A and B, each of mass m, are threaded on the wire and are connected by a light straight spring of natural length $2a \sin\alpha$ and modulus λ, where $0 < \alpha < \pi/4$. The spring is compressed so that the angle AOB is 2β and the beads are then released from rest. Show that in the ensuing motion

$$ma\dot{\theta}^2 \sin\alpha + \lambda(\sin\theta - \sin\alpha)^2 = \lambda(\sin\beta - \sin\alpha)^2$$

where 2θ denotes the angle AOB at time t after release.

(i) If $\beta - \alpha$ is small, show that T, the period of oscillations, is given approximately by

$$T = 2\pi\sqrt{\left(\frac{ma \sin\alpha}{\lambda \cos^2\alpha}\right)},$$

(ii) If $\beta - \alpha$ is not small, write down an expression, in the form of a definite integral, for the exact period of oscillations, in each of the two cases

$$(a)\ \sin\beta > 2\sin\alpha - 1,$$
$$(b)\ \sin\beta < 2\sin\alpha - 1.$$

13 A chocolate orange consists of a solid sphere of uniform chocolate of mass M and radius a, sliced into segments by planes through its axis. It stands on a horizontal table with its axis vertical, and it is held together only by a narrow ribbon round its equator.

Show that the tension in the ribbon is at least $\frac{3}{32}Mg$.

(You may assume that the centre of mass of a segment of angle 2θ is at a distance $\dfrac{3\pi a \sin\theta}{16\theta}$ from the axis.)

14 A uniform disc of mass M and radius a is free to rotate in a horizontal plane about a fixed vertical axis through the centre, O, of the disc. A particle of mass $\frac{1}{2}M$ is attached by a light straight wire of length $a/2$ to the vertical axis at O, so that the particle can rotate freely about the vertical axis. The particle, initially at rest, is placed gently on the disc at time $t = 0$, when the disc is spinning with angular speed Ω. Relative motion between the particle and disc is opposed by a frictional force of magnitude $Mak(\omega_1 - \omega_2)$, where, at time t, ω_1 is the angular speed of the disc, ω_2 is the angular speed of the wire, and k is a constant. Derive equations for the rate of change of ω_1 and ω_2, and show that

$$4\omega_1 + \omega_2 = 4\Omega.$$

Show further that

$$\omega_1 = \frac{\Omega}{5}(4 + e^{-5kt}).$$

15 The King of Smorgasbrod wishes to raise as much money as possible by fining people who sell underweight cartons of kippers. The weight of a kipper is normally distributed with mean 200 grams and standard deviation 10 grams. Kippers are packed in cartons of 625, and vast quantities of them are sold.

Every day a carton is to be selected at random from each vendor of kippers. Three schemes for determining the fines are proposed:

1. Weigh the entire carton, and fine the vendor 1500 crowns if the average weight of a kipper is less than 199 grams.
2. Weigh 25 kippers selected at random from the carton and fine the vendor 100 crowns if the average weight of a kipper is less than 198 grams.
3. Remove kippers one at a time and at random from the carton until a kipper has been found which weighs *more* than 200 grams and fine the vendor $3n(n-1)$ crowns, where n is the number of kippers removed.

Determine which scheme the king should select, justifying your answer.

16 A tennis tournament is arranged for 2^n players. It is organised as a knockout tournament so that only the winners in any given round proceed to the next round. Opponents in each round except the final are drawn at random, and in any match either player has a probability of $\frac{1}{2}$ of winning. Two players are chosen at random before the draw for the first round. Find the probabilities that they play each other:

(i) in the first round,

(ii) in the final,

(iii) in the tournament.