

**Sixth Term Examination Papers**

**9470**

**MATHEMATICS 2**

Morning

**WEDNESDAY 17 JUNE 2015**

Time: 3 hours

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Additional Materials: Answer Booklet  
Formulae Booklet

### **INSTRUCTIONS TO CANDIDATES**

**Please read this page carefully, but do not open this question paper until you are told that you may do so.**

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

### **INFORMATION FOR CANDIDATES**

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

**Calculators are not permitted.**

**Please wait to be told you may begin before turning this page.**

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This question paper consists of 8 printed pages and 4 blank pages.

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## Section A: Pure Mathematics

- 1 (i) By use of calculus, show that  $x - \ln(1 + x)$  is positive for all positive  $x$ . Use this result to show that

$$\sum_{k=1}^n \frac{1}{k} > \ln(n + 1).$$

- (ii) By considering  $x + \ln(1 - x)$ , show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2.$$

- 2 In the triangle  $ABC$ , angle  $BAC = \alpha$  and angle  $CBA = 2\alpha$ , where  $2\alpha$  is acute, and  $BC = x$ . Show that  $AB = (3 - 4 \sin^2 \alpha)x$ .

The point  $D$  is the midpoint of  $AB$  and the point  $E$  is the foot of the perpendicular from  $C$  to  $AB$ . Find an expression for  $DE$  in terms of  $x$ .

The point  $F$  lies on the perpendicular bisector of  $AB$  and is a distance  $x$  from  $C$ . The points  $F$  and  $B$  lie on the same side of the line through  $A$  and  $C$ . Show that the line  $FC$  trisects the angle  $ACB$ .

- 3 Three rods have lengths  $a$ ,  $b$  and  $c$ , where  $a < b < c$ . The three rods can be made into a triangle (possibly of zero area) if  $a + b \geq c$ .

Let  $T_n$  be the number of triangles that can be made with three rods chosen from  $n$  rods of lengths  $1, 2, 3, \dots, n$  (where  $n \geq 3$ ). Show that  $T_8 - T_7 = 2 + 4 + 6$  and evaluate  $T_8 - T_6$ . Write down expressions for  $T_{2m} - T_{2m-1}$  and  $T_{2m} - T_{2m-2}$ .

Prove by induction that  $T_{2m} = \frac{1}{6}m(m-1)(4m+1)$ , and find the corresponding result for an odd number of rods.

- 4 (i) The continuous function  $f$  is defined by

$$\tan f(x) = x \quad (-\infty < x < \infty)$$

and  $f(0) = \pi$ . Sketch the curve  $y = f(x)$ .

- (ii) The continuous function  $g$  is defined by

$$\tan g(x) = \frac{x}{1+x^2} \quad (-\infty < x < \infty)$$

and  $g(0) = \pi$ . Sketch the curves  $y = \frac{x}{1+x^2}$  and  $y = g(x)$ .

- (iii) The continuous function  $h$  is defined by  $h(0) = \pi$  and

$$\tan h(x) = \frac{x}{1-x^2} \quad (x \neq \pm 1).$$

(The values of  $h(x)$  at  $x = \pm 1$  are such that  $h(x)$  is continuous at these points.) Sketch the curves  $y = \frac{x}{1-x^2}$  and  $y = h(x)$ .

- 5 In this question, the arctan function satisfies  $0 \leq \arctan x < \frac{1}{2}\pi$  for  $x \geq 0$ .

- (i) Let

$$S_n = \sum_{m=1}^n \arctan \left( \frac{1}{2m^2} \right),$$

for  $n = 1, 2, 3, \dots$ . Prove by induction that

$$\tan S_n = \frac{n}{n+1}.$$

Prove also that

$$S_n = \arctan \frac{n}{n+1}.$$

- (ii) In a triangle  $ABC$ , the lengths of the sides  $AB$  and  $BC$  are  $4n^2$  and  $4n^4 - 1$ , respectively, and the angle at  $B$  is a right angle. Let angle  $BCA = 2\alpha_n$ . Show that

$$\sum_{n=1}^{\infty} \alpha_n = \frac{1}{4}\pi.$$

- 6 (i) Show that

$$\sec^2\left(\frac{1}{4}\pi - \frac{1}{2}x\right) = \frac{2}{1 + \sin x}.$$

Hence integrate  $\frac{1}{1 + \sin x}$  with respect to  $x$ .

- (ii) By means of the substitution  $y = \pi - x$ , show that

$$\int_0^\pi xf(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx,$$

where  $f$  is any function for which these integrals exist.

Hence evaluate

$$\int_0^\pi \frac{x}{1 + \sin x} dx.$$

- (iii) Evaluate

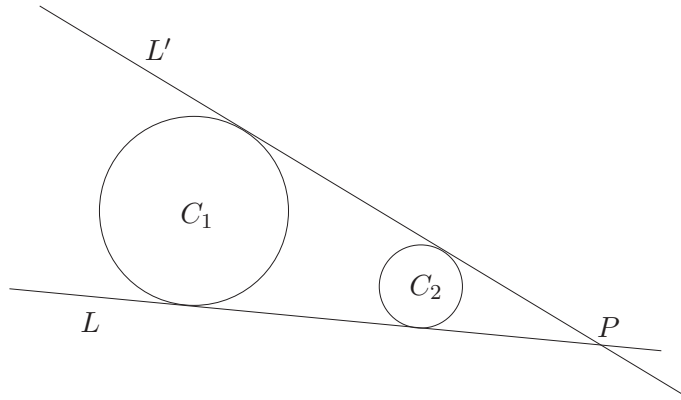
$$\int_0^\pi \frac{2x^3 - 3\pi x^2}{(1 + \sin x)^2} dx.$$

- 7 A circle  $C$  is said to be *bisected* by a curve  $X$  if  $X$  meets  $C$  in exactly two points and these points are diametrically opposite each other on  $C$ .

- (i) Let  $C$  be the circle of radius  $a$  in the  $x$ - $y$  plane with centre at the origin. Show, by giving its equation, that it is possible to find a circle of given radius  $r$  that bisects  $C$  provided  $r > a$ . Show that no circle of radius  $r$  bisects  $C$  if  $r \leq a$ .
- (ii) Let  $C_1$  and  $C_2$  be circles with centres at  $(-d, 0)$  and  $(d, 0)$  and radii  $a_1$  and  $a_2$ , respectively, where  $d > a_1$  and  $d > a_2$ . Let  $D$  be a circle of radius  $r$  that bisects both  $C_1$  and  $C_2$ . Show that the  $x$ -coordinate of the centre of  $D$  is  $\frac{a_2^2 - a_1^2}{4d}$ .

Obtain an expression in terms of  $d$ ,  $r$ ,  $a_1$  and  $a_2$  for the  $y$ -coordinate of the centre of  $D$ , and deduce that  $r$  must satisfy

$$16r^2d^2 \geq (4d^2 + (a_2 - a_1)^2)(4d^2 + (a_2 + a_1)^2).$$



The diagram above shows two non-overlapping circles  $C_1$  and  $C_2$  of different sizes. The lines  $L$  and  $L'$  are the two common tangents to  $C_1$  and  $C_2$  such that the two circles lie on the same side of each of the tangents. The lines  $L$  and  $L'$  intersect at the point  $P$  which is called the *focus* of  $C_1$  and  $C_2$ .

- (i) Let  $\mathbf{x}_1$  and  $\mathbf{x}_2$  be the position vectors of the centres of  $C_1$  and  $C_2$ , respectively. Show that the position vector of  $P$  is

$$\frac{r_1\mathbf{x}_2 - r_2\mathbf{x}_1}{r_1 - r_2},$$

where  $r_1$  and  $r_2$  are the radii of  $C_1$  and  $C_2$ , respectively.

- (ii) The circle  $C_3$  does not overlap either  $C_1$  or  $C_2$  and its radius,  $r_3$ , satisfies  $r_1 \neq r_3 \neq r_2$ . The focus of  $C_1$  and  $C_3$  is  $Q$ , and the focus of  $C_2$  and  $C_3$  is  $R$ . Show that  $P$ ,  $Q$  and  $R$  lie on the same straight line.
- (iii) Find a condition on  $r_1$ ,  $r_2$  and  $r_3$  for  $Q$  to lie half-way between  $P$  and  $R$ .

## Section B: Mechanics

**9** An equilateral triangle  $ABC$  is made of three light rods each of length  $a$ . It is free to rotate in a vertical plane about a horizontal axis through  $A$ . Particles of mass  $3m$  and  $5m$  are attached to  $B$  and  $C$  respectively. Initially, the system hangs in equilibrium with  $BC$  below  $A$ .

(i) Show that, initially, the angle  $\theta$  that  $BC$  makes with the horizontal is given by  $\sin \theta = \frac{1}{7}$ .

(ii) The triangle receives an impulse that imparts a speed  $v$  to the particle  $B$ . Find the minimum speed  $v_0$  such that the system will perform complete rotations if  $v > v_0$ .

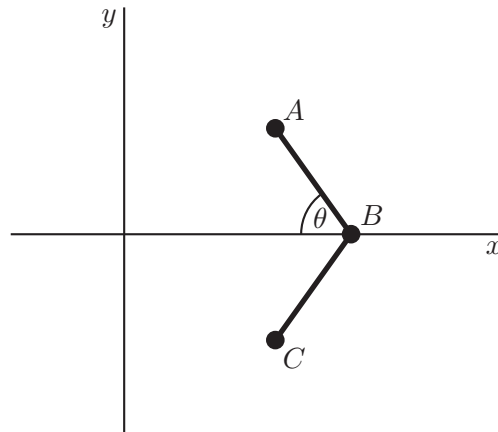
**10** A particle of mass  $m$  is pulled along the floor of a room in a straight line by a light string which is pulled at constant speed  $V$  through a hole in the ceiling. The floor is smooth and horizontal, and the height of the room is  $h$ . Find, in terms of  $V$  and  $\theta$ , the speed of the particle when the string makes an angle of  $\theta$  with the vertical (and the particle is still in contact with the floor). Find also the acceleration, in terms of  $V$ ,  $h$  and  $\theta$ .

Find the tension in the string and hence show that the particle will leave the floor when

$$\tan^4 \theta = \frac{V^2}{gh}.$$

- 11 Three particles,  $A$ ,  $B$  and  $C$ , each of mass  $m$ , lie on a smooth horizontal table. Particles  $A$  and  $C$  are attached to the two ends of a light inextensible string of length  $2a$  and particle  $B$  is attached to the midpoint of the string. Initially,  $A$ ,  $B$  and  $C$  are at rest at points  $(0, a)$ ,  $(0, 0)$  and  $(0, -a)$ , respectively.

An impulse is delivered to  $B$ , imparting to it a speed  $u$  in the positive  $x$  direction. The string remains taut throughout the subsequent motion.



- (i) At time  $t$ , the angle between the  $x$ -axis and the string joining  $A$  and  $B$  is  $\theta$ , as shown in the diagram, and  $B$  is at  $(x, 0)$ . Write down the coordinates of  $A$  in terms of  $x$ ,  $a$  and  $\theta$  at this time. Given that the velocity of  $B$  is  $(v, 0)$ , show that the velocity of  $A$  is  $(v + a\dot{\theta} \sin \theta, a\dot{\theta} \cos \theta)$ , where the dot denotes differentiation with respect to time.

- (ii) Show that, before  $A$  and  $C$  first collide,

$$3v + 2a\dot{\theta} \sin \theta = u \quad \text{and} \quad \dot{\theta}^2 = \frac{u^2}{a^2(3 - 2 \sin^2 \theta)}.$$

- (iii) When  $A$  and  $C$  collide, the collision is elastic (no energy is lost). At what value of  $\theta$  does the second collision between particles  $A$  and  $C$  occur? (You should justify your answer.)

- (iv) When  $v = 0$ , what are the possible values of  $\theta$ ? Is  $v = 0$  whenever  $\theta$  takes these values?



## Section C: Probability and Statistics

- 12** Four players  $A$ ,  $B$ ,  $C$  and  $D$  play a coin-tossing game with a fair coin. Each player chooses a sequence of heads and tails, as follows:

Player A: HHT; Player B: THH; Player C: TTH; Player D: HTT.

The coin is then tossed until one of these sequences occurs, in which case the corresponding player is the winner.

- (i) Show that, if only  $A$  and  $B$  play, then  $A$  has a probability of  $\frac{1}{4}$  of winning.
- (ii) If all four players play together, find the probabilities of each one winning.
- (iii) Only  $B$  and  $C$  play. What is the probability of  $C$  winning if the first two tosses are TT? Let the probabilities of  $C$  winning if the first two tosses are HT, TH and HH be  $p$ ,  $q$  and  $r$ , respectively. Show that  $p = \frac{1}{2} + \frac{1}{2}q$ . Find the probability that  $C$  wins.

- 13** The maximum height  $X$  of flood water each year on a certain river is a random variable with probability density function  $f$  given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\lambda$  is a positive constant.

It costs  $ky$  pounds each year to prepare for flood water of height  $y$  or less, where  $k$  is a positive constant and  $y \geq 0$ . If  $X \leq y$  no further costs are incurred but if  $X > y$  the additional cost of flood damage is  $a(X - y)$  pounds where  $a$  is a positive constant.

- (i) Let  $C$  be the total cost of dealing with the floods in the year. Show that the expectation of  $C$  is given by

$$E(C) = ky + \frac{a}{\lambda} e^{-\lambda y}.$$

How should  $y$  be chosen in order to minimise  $E(C)$ , in the different cases that arise according to the value of  $a/k$ ?

- (ii) Find the variance of  $C$ , and show that the more that is spent on preparing for flood water in advance the smaller this variance.

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