Sixth Term Examination Papers **9465**

MATHEMATICS 1 Afternoon

MONDAY 21 JUNE 2010 Time: 3 hours

7609210635

Additional Materials: Answer Paper
Formulae Booklet

Candidates may **not** use a calculator

INSTRUCTIONS TO CANDIDATES

Please read this page carefully, but do not open this question paper until you are told that you may do so.

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

INFORMATION FOR CANDIDATES

Each question is marked out of 20. There is no restriction of choice.

You will be assessed on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

Calculators are not permitted.

Please wait to be told you may begin before turning this page.

This question paper consists of 7 printed pages and 1 blank page.

Section A: Pure Mathematics

1 Given that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv a\,(x - y + 2)^2 + b\,(cx + y)^2 + d\,,$$

find the values of the constants a, b, c and d.

Solve the simultaneous equations

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9\,,$$
$$6x^2 + 3y^2 - 8xy + 8x - 8y = 14\,.$$

2 The curve $y = \left(\dfrac{x-a}{x-b}\right)\mathrm{e}^x$, where a and b are constants, has two stationary points. Show that

$$a - b < 0 \quad \text{or} \quad a - b > 4\,.$$

(i) Show that, in the case $a = 0$ and $b = \frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote, and sketch the curve.

(ii) Sketch the curve in the case $a = \frac{9}{2}$ and $b = 0$.

3 Show that

$$\sin(x + y) - \sin(x - y) = 2\cos x\,\sin y$$

and deduce that

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A + B)\,\sin\tfrac{1}{2}(A - B)\,.$$

Show also that

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A + B)\,\sin\tfrac{1}{2}(A - B)\,.$$

The points P, Q, R and S have coordinates $(a\cos p, b\sin p)$, $(a\cos q, b\sin q)$, $(a\cos r, b\sin r)$ and $(a\cos s, b\sin s)$ respectively, where $0 \leqslant p < q < r < s < 2\pi$, and a and b are positive.

Given that neither of the lines PQ and SR is vertical, show that these lines are parallel if and only if

$$r + s - p - q = 2\pi\,.$$

4 Use the substitution $x = \dfrac{1}{t^2 - 1}$, where $t > 1$, to show that, for $x > 0$,

$$\int \frac{1}{\sqrt{x\,(x+1)}}\;\mathrm{d}x = 2\ln\left(\sqrt{x} + \sqrt{x+1}\,\right) + c\,.$$

[**Note:** You may use without proof the result $\displaystyle\int \frac{1}{t^2 - a^2}\,\mathrm{d}t = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + \text{constant.}$]

The section of the curve

$$y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$$

between $x = \frac{1}{8}$ and $x = \frac{9}{16}$ is rotated through 360^o about the x-axis. Show that the volume enclosed is $2\pi \ln \frac{5}{4}$.

5 By considering the expansion of $(1+x)^n$ where n is a positive integer, or otherwise, show that:

(i) $\displaystyle\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n\,;$

(ii) $\displaystyle\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}\,;$

(iii) $\displaystyle\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}\left(2^{n+1} - 1\right)\,;$

(iv) $\displaystyle\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n} = n\,(n+1)\,2^{n-2}\,.$

6 Show that, if $y = \mathrm{e}^x$, then

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0\,. \qquad (*)$$

In order to find other solutions of this differential equation, now let $y = u\mathrm{e}^x$, where u is a function of x. By substituting this into $(*)$, show that

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0\,. \qquad (**)$$

By setting $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$, where A and B are any constants.

7 Relative to a fixed origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\mathbf{c}$ given by

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}\,,$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\mathbf{p}$ and the lines OB and AC meet at the point Q with position vector $\mathbf{q}$. Show that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta}\,,$$

and write down $\mathbf{q}$ in terms of α, β and $\mathbf{b}$.

Show further that the point R with position vector $\mathbf{r}$ given by

$$\mathbf{r} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta}\,,$$

lies on the lines OC and AB.

The lines OB and PR intersect at the point S. Prove that $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.

8 **(i)** Suppose that a, b and c are integers that satisfy the equation

$$a^3 + 3b^3 = 9c^3.$$

Explain why a must be divisible by 3, and show further that both b and c must also be divisible by 3. Hence show that the only integer solution is $a = b = c = 0$.

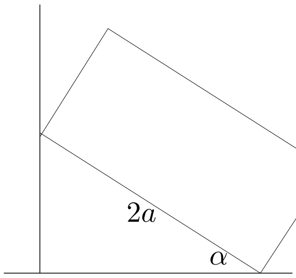
(ii) Suppose that p, q and r are integers that satisfy the equation

$$p^4 + 2q^4 = 5r^4\,.$$

By considering the possible final digit of each term, or otherwise, show that p and q are divisible by 5. Hence show that the only integer solution is $p = q = r = 0$.

Section B: Mechanics

9



The diagram shows a uniform rectangular lamina with sides of lengths $2a$ and $2b$ leaning against a rough vertical wall, with one corner resting on a rough horizontal plane. The plane of the lamina is vertical and perpendicular to the wall, and one edge makes an angle of α with the horizontal plane. Show that the centre of mass of the lamina is a distance $a\cos\alpha + b\sin\alpha$ from the wall.

The coefficients of friction at the two points of contact are each μ and the friction is limiting at both contacts. Show that

$$a\cos(2\lambda + \alpha) = b\sin\alpha\,,$$

where $\tan\lambda = \mu$.

Show also that if the lamina is square, then $\lambda = \frac{\pi}{4} - \alpha$.

10 A particle P moves so that, at time t, its displacement $\mathbf{r}$ from a fixed origin is given by

$$\mathbf{r} = \left(\mathrm{e}^t\cos t\right)\mathbf{i} + \left(\mathrm{e}^t\sin t\right)\mathbf{j}\,.$$

Show that the velocity of the particle always makes an angle of $\frac{\pi}{4}$ with the particle's displacement, and that the acceleration of the particle is always perpendicular to its displacement. Sketch the path of the particle for $0 \leqslant t \leqslant \pi$.

A second particle Q moves on the same path, passing through each point on the path a fixed time T after P does. Show that the distance between P and Q is proportional to e^t.

11 Two particles of masses m and M, with $M > m$, lie in a smooth circular groove on a horizontal plane. The coefficient of restitution between the particles is e. The particles are initially projected round the groove with the same speed u but in opposite directions. Find the speeds of the particles after they collide for the first time and show that they will both change direction if $2em > M - m$.

After a further $2n$ collisions, the speed of the particle of mass m is v and the speed of the particle of mass M is V. Given that at each collision both particles change their directions of motion, explain why

$$mv - MV = u(M - m),$$

and find v and V in terms of m, M, e, u and n.

Section C: Probability and Statistics

12 A discrete random variable X takes only positive integer values. Define $\mathrm{E}(X)$ for this case, and show that

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} \mathrm{P}\left(X \geqslant n\right).$$

I am collecting toy penguins from cereal boxes. Each box contains either one daddy penguin or one mummy penguin. The probability that a given box contains a daddy penguin is p and the probability that a given box contains a mummy penguin is q, where $p \neq 0$, $q \neq 0$ and $p + q = 1$.

Let X be the number of boxes that I need to open to get at least one of each kind of penguin. Show that $\mathrm{P}(X \geqslant 4) = p^3 + q^3$, and that

$$\mathrm{E}(X) = \frac{1}{pq} - 1.$$

Hence show that $\mathrm{E}(X) \geqslant 3$.

13 The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$p\mathrm{e}^{2\lambda} - \mathrm{e}^{\lambda} + 1 = 0.$$

Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p.

Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.

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