

Section A: Pure Mathematics

- 1** Two curves have equations $x^4 + y^4 = u$ and $xy = v$, where u and v are positive constants. State the equations of the lines of symmetry of each curve.

The curves intersect at the distinct points A , B , C and D (taken anticlockwise from A). The coordinates of A are (α, β) , where $\alpha > \beta > 0$. Write down, in terms of α and β , the coordinates of B , C and D .

Show that the quadrilateral $ABCD$ is a rectangle and find its area in terms of u and v only. Verify that, for the case $u = 81$ and $v = 4$, the area is 14.

- 2** The curve C has equation

$$y = a^{\sin(\pi e^x)},$$

where $a > 1$.

- (i) Find the coordinates of the stationary points on C .
- (ii) Use the approximations $e^t \approx 1 + t$ and $\sin t \approx t$ (both valid for small values of t) to show that

$$y \approx 1 - \pi x \ln a$$

for small values of x .

- (iii) Sketch C .
- (iv) By approximating C by means of straight lines joining consecutive stationary points, show that the area between C and the x -axis between the k th and $(k + 1)$ th maxima is approximately

$$\left(\frac{a^2 + 1}{2a}\right) \ln \left(1 + \left(k - \frac{3}{4}\right)^{-1}\right).$$

3 Prove that

$$\tan\left(\frac{1}{4}\pi - \frac{1}{2}x\right) \equiv \sec x - \tan x. \quad (*)$$

(i) Use (*) to find the value of $\tan \frac{1}{8}\pi$. Hence show that

$$\tan \frac{11}{24}\pi = \frac{\sqrt{3} + \sqrt{2} - 1}{\sqrt{3} - \sqrt{6} + 1}.$$

(ii) Show that

$$\frac{\sqrt{3} + \sqrt{2} - 1}{\sqrt{3} - \sqrt{6} + 1} = 2 + \sqrt{2} + \sqrt{3} + \sqrt{6}.$$

(iii) Use (*) to show that

$$\tan \frac{1}{48}\pi = \sqrt{16 + 10\sqrt{2} + 8\sqrt{3} + 6\sqrt{6}} - 2 - \sqrt{2} - \sqrt{3} - \sqrt{6}.$$

4 The polynomial $p(x)$ is of degree 9 and $p(x) - 1$ is exactly divisible by $(x - 1)^5$.

(i) Find the value of $p(1)$.

(ii) Show that $p'(x)$ is exactly divisible by $(x - 1)^4$.

(iii) Given also that $p(x) + 1$ is exactly divisible by $(x + 1)^5$, find $p(x)$.

5 Expand and simplify $(\sqrt{x-1} + 1)^2$.

(i) Evaluate

$$\int_5^{10} \frac{\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1}} dx.$$

(ii) Find the total area between the curve

$$y = \frac{\sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1}}$$

and the x -axis between the points $x = \frac{5}{4}$ and $x = 10$.

(iii) Evaluate

$$\int_{\frac{5}{4}}^{10} \frac{\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}} + 2}{\sqrt{x^2 - 1}} dx.$$

- 6** The Fibonacci sequence F_1, F_2, F_3, \dots is defined by $F_1 = 1, F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1} \quad (n \geq 2).$$

Write down the values of F_3, F_4, \dots, F_{10} .

Let $S = \sum_{i=1}^{\infty} \frac{1}{F_i}$.

- (i) Show that $\frac{1}{F_i} > \frac{1}{2F_{i-1}}$ for $i \geq 4$ and deduce that $S > 3$.

Show also that $S < 3\frac{2}{3}$.

- (ii) Show further that $3.2 < S < 3.5$.

- 7** Let $y = (x - a)^n e^{bx} \sqrt{1 + x^2}$, where n and a are constants and b is a non-zero constant. Show that

$$\frac{dy}{dx} = \frac{(x - a)^{n-1} e^{bx} q(x)}{\sqrt{1 + x^2}},$$

where $q(x)$ is a cubic polynomial.

Using this result, determine:

- (i) $\int \frac{(x - 4)^{14} e^{4x} (4x^3 - 1)}{\sqrt{1 + x^2}} dx$;
- (ii) $\int \frac{(x - 1)^{21} e^{12x} (12x^4 - x^2 - 11)}{\sqrt{1 + x^2}} dx$;
- (iii) $\int \frac{(x - 2)^6 e^{4x} (4x^4 + x^3 - 2)}{\sqrt{1 + x^2}} dx$.

- 8** The non-collinear points A, B and C have position vectors \mathbf{a}, \mathbf{b} and \mathbf{c} , respectively. The points P and Q have position vectors \mathbf{p} and \mathbf{q} , respectively, given by

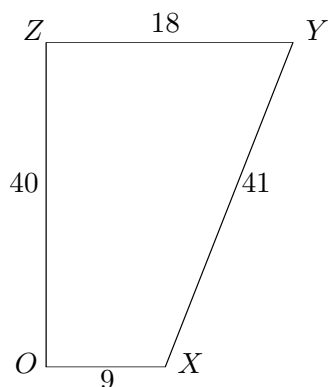
$$\mathbf{p} = \lambda \mathbf{a} + (1 - \lambda) \mathbf{b} \quad \text{and} \quad \mathbf{q} = \mu \mathbf{a} + (1 - \mu) \mathbf{c}$$

where $0 < \lambda < 1$ and $\mu > 1$. Draw a diagram showing A, B, C, P and Q .

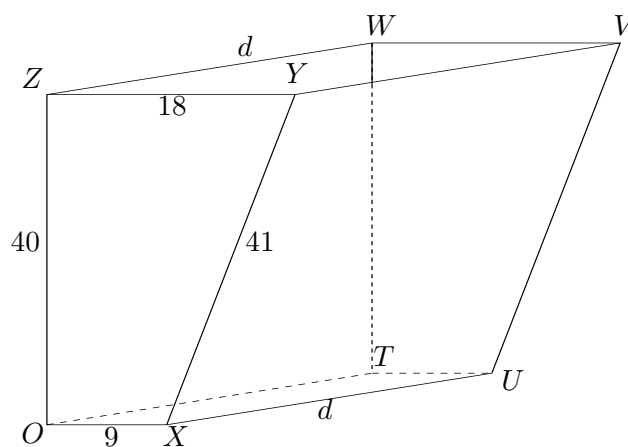
Given that $CQ \times BP = AB \times AC$, find μ in terms of λ , and show that, for all values of λ , the line PQ passes through the fixed point D , with position vector \mathbf{d} given by $\mathbf{d} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$. What can be said about the quadrilateral $ABDC$?

Section B: Mechanics

- 9 (i) A uniform lamina $OXYZ$ is in the shape of the trapezium shown in the diagram. It is right-angled at O and Z , and OX is parallel to YZ . The lengths of the sides are given by $OX = 9$ cm, $XY = 41$ cm, $YZ = 18$ cm and $ZO = 40$ cm. Show that its centre of mass is a distance 7 cm from the edge OZ .



- (ii) The diagram shows a tank with no lid made of thin sheet metal. The base $OXUT$, the back $OTWZ$ and the front $XUVY$ are rectangular, and each end is a trapezium as in part (i). The width of the tank is d cm.

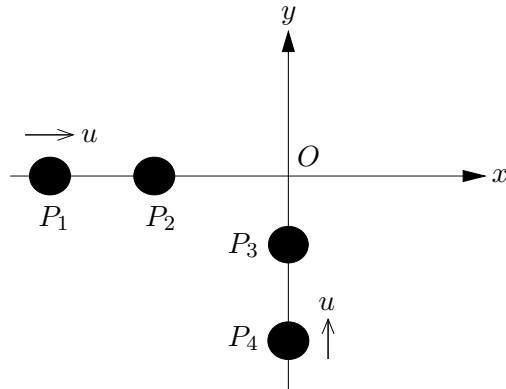


Show that the centre of mass of the tank, when empty, is a distance

$$\frac{3(140 + 11d)}{5(12 + d)} \text{ cm}$$

from the back of the tank.

The tank is then filled with a liquid. The mass per unit volume of this liquid is k times the mass per unit area of the sheet metal. In the case $d = 20$, find an expression for the distance of the centre of mass of the filled tank from the back of the tank.



Four particles P_1 , P_2 , P_3 and P_4 , of masses m_1 , m_2 , m_3 and m_4 , respectively, are arranged on smooth horizontal axes as shown in the diagram.

Initially, P_2 and P_3 are stationary, and both P_1 and P_4 are moving towards O with speed u . Then P_1 and P_2 collide, at the same moment as P_4 and P_3 collide. Subsequently, P_2 and P_3 collide at O , as do P_1 and P_4 some time later. The coefficient of restitution between each pair of particles is e , and $e > 0$.

Show that initially P_2 and P_3 are equidistant from O .

- 11** A train consists of an engine and n trucks. It is travelling along a straight horizontal section of track. The mass of the engine and of each truck is M . The resistance to motion of the engine and of each truck is R , which is constant. The maximum power at which the engine can work is P .

Obtain an expression for the acceleration of the train when its speed is v and the engine is working at maximum power.

The train starts from rest with the engine working at maximum power. Obtain an expression for the time T taken to reach a given speed V , and show that this speed is only achievable if

$$P > (n + 1)RV.$$

- (i) In the case when $(n + 1)RV/P$ is small, use the approximation $\ln(1 - x) \approx -x - \frac{1}{2}x^2$ (valid for small x) to obtain the approximation

$$PT \approx \frac{1}{2}(n + 1)MV^2$$

and interpret this result.

- (ii) In the general case, the distance moved from rest in time T is X . Write down, with explanation, an equation relating P , T , X , M , V , R and n and hence show that

$$X = \frac{2PT - (n + 1)MV^2}{2(n + 1)R}.$$

Section C: Probability and Statistics

- 12** A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ ke^{-2x^2} & \text{for } 0 \leq x < \infty, \end{cases}$$

where k is a constant.

- (i) Sketch the graph of $f(x)$.
 - (ii) Find the value of k .
 - (iii) Determine $E(X)$ and $\text{Var}(X)$.
 - (iv) Use statistical tables to find, to three significant figures, the median value of X .
- 13** Satellites are launched using two different types of rocket: the Andover and the Basingstoke. The Andover has four engines and the Basingstoke has six. Each engine has a probability p of failing during any given launch. After the launch, the rockets are retrieved and repaired by replacing some or all of the engines. The cost of replacing each engine is K .

For the Andover, if more than one engine fails, all four engines are replaced. Otherwise, only the failed engine (if there is one) is replaced. Show that the expected repair cost for a single launch using the Andover is

$$4Kp(1 + q + q^2 - 2q^3) \quad (q = 1 - p) \quad (*)$$

For the Basingstoke, if more than two engines fail, all six engines are replaced. Otherwise only the failed engines (if there are any) are replaced. Find, in a form similar to (*), the expected repair cost for a single launch using the Basingstoke.

Find the values of p for which the expected repair cost for the Andover is $\frac{2}{3}$ of the expected repair cost for the Basingstoke.