

## Section A: Pure Mathematics

- 1 Sketch the curve with cartesian equation

$$y = \frac{2x(x^2 - 5)}{x^2 - 4}$$

and give the equations of the asymptotes and of the tangent to the curve at the origin.

Hence determine the number of real roots of the following equations:

- (i)  $3x(x^2 - 5) = (x^2 - 4)(x + 3)$ ;
- (ii)  $4x(x^2 - 5) = (x^2 - 4)(5x - 2)$ ;
- (iii)  $4x^2(x^2 - 5)^2 = (x^2 - 4)^2(x^2 + 1)$ .

- 2 Let

$$I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2 \theta}{1 - \sin \theta \sin 2\alpha} d\theta \quad \text{and} \quad J = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sec^2 \theta}{1 + \tan^2 \theta \cos^2 2\alpha} d\theta$$

where  $0 < \alpha < \frac{1}{4}\pi$ .

- (i) Show that  $I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2 \theta}{1 + \sin \theta \sin 2\alpha} d\theta$  and hence that  $2I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{2}{1 + \tan^2 \theta \cos^2 2\alpha} d\theta$ .
- (ii) Find  $J$ .
- (iii) By considering  $I \sin^2 2\alpha + J \cos^2 2\alpha$ , or otherwise, show that  $I = \frac{1}{2}\pi \sec^2 \alpha$ .
- (iv) Evaluate  $I$  in the case  $\frac{1}{4}\pi < \alpha < \frac{1}{2}\pi$ .

- 3 (i) Let

$$\tan x = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad \cot x = \frac{1}{x} + \sum_{n=0}^{\infty} b_n x^n$$

for  $0 < x < \frac{1}{2}\pi$ . Explain why  $a_n = 0$  for even  $n$ .

Prove the identity

$$\cot x - \tan x \equiv 2 \cot 2x$$

and show that

$$a_n = (1 - 2^{n+1})b_n.$$

- (ii) Let  $\operatorname{cosec} x = \frac{1}{x} + \sum_{n=0}^{\infty} c_n x^n$  for  $0 < x < \frac{1}{2}\pi$ . By considering  $\cot x + \tan x$ , or otherwise, show that

$$c_n = (2^{-n} - 1)b_n.$$

- (iii) Show that

$$\left(1 + x \sum_{n=0}^{\infty} b_n x^n\right)^2 + x^2 = \left(1 + x \sum_{n=0}^{\infty} c_n x^n\right)^2.$$

Deduce from this and the previous results that  $a_1 = 1$ , and find  $a_3$ .

- 4 The function  $f$  satisfies the identity

$$f(x) + f(y) \equiv f(x + y) \tag{*}$$

for all  $x$  and  $y$ . Show that  $2f(x) \equiv f(2x)$  and deduce that  $f''(0) = 0$ . By considering the Maclaurin series for  $f(x)$ , find the most general function that satisfies (\*).

[Do not consider issues of existence or convergence of Maclaurin series in this question.]

- (i) By considering the function  $G$ , defined by  $\ln(g(x)) = G(x)$ , find the most general function that, for all  $x$  and  $y$ , satisfies the identity

$$g(x)g(y) \equiv g(x + y).$$

- (ii) By considering the function  $H$ , defined by  $\ln(e^u) = H(u)$ , find the most general function that satisfies, for all positive  $x$  and  $y$ , the identity

$$h(x) + h(y) \equiv h(xy).$$

- (iii) Find the most general function  $t$  that, for all  $x$  and  $y$ , satisfies the identity

$$t(x) + t(y) \equiv t(z),$$

where  $z = \frac{x + y}{1 - xy}$ .

- 5 Show that the distinct complex numbers  $\alpha$ ,  $\beta$  and  $\gamma$  represent the vertices of an equilateral triangle (in clockwise or anti-clockwise order) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0.$$

Show that the roots of the equation

$$z^3 + az^2 + bz + c = 0 \tag{*}$$

represent the vertices of an equilateral triangle if and only if  $a^2 = 3b$ .

Under the transformation  $z = pw + q$ , where  $p$  and  $q$  are given complex numbers with  $p \neq 0$ , the equation (\*) becomes

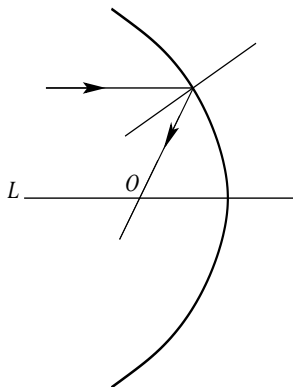
$$w^3 + Aw^2 + Bw + C = 0. \tag{**}$$

Show that if the roots of equation (\*) represent the vertices of an equilateral triangle, then the roots of equation (\*\*) also represent the vertices of an equilateral triangle.

- 6 Show that in polar coordinates the gradient of any curve at the point  $(r, \theta)$  is

$$\frac{\frac{dr}{d\theta} \tan \theta + r}{\frac{dr}{d\theta} - r \tan \theta}.$$

A mirror is designed so that if an incident ray of light is parallel to a fixed line  $L$  the reflected ray passes through a fixed point  $O$  on  $L$ . Prove that the mirror intersects any plane containing  $L$  in a parabola. You should assume that the angle between the incident ray and the normal is the same as the angle between the reflected ray and the normal.



- 7 (i) Solve the equation  $u^2 + 2u \sinh x - 1 = 0$  giving  $u$  in terms of  $x$ .

Find the solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} \sinh x - 1 = 0$$

that satisfies  $y = 0$  and  $\frac{dy}{dx} > 0$  at  $x = 0$ .

- (ii) Find the solution, not identically zero, of the differential equation

$$\sinh y \left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - \sinh y = 0$$

that satisfies  $y = 0$  at  $x = 0$ , expressing your solution in the form  $\cosh y = f(x)$ . Show that the asymptotes to the solution curve are  $y = \pm(-x + \ln 4)$ .

- 8  $\Delta$  is an operation that takes polynomials in  $x$  to polynomials in  $x$ ; that is, given any polynomial  $h(x)$ , there is a polynomial called  $\Delta h(x)$  which is obtained from  $h(x)$  using the rules that define  $\Delta$ . These rules are as follows:

(i)  $\Delta x = 1$ ;

(ii)  $\Delta(f(x) + g(x)) = \Delta f(x) + \Delta g(x)$  for any polynomials  $f(x)$  and  $g(x)$ ;

(iii)  $\Delta(\lambda f(x)) = \lambda \Delta f(x)$  for any constant  $\lambda$  and any polynomial  $f(x)$ ;

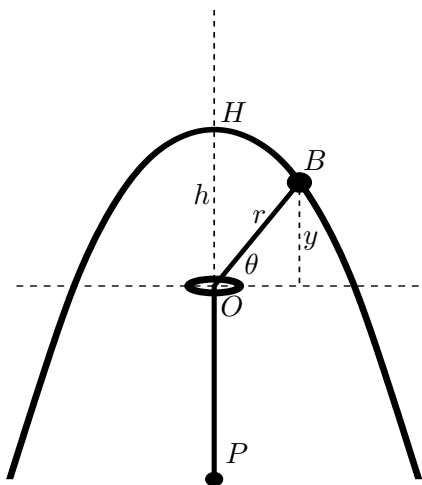
(iv)  $\Delta(f(x)g(x)) = f(x)\Delta g(x) + g(x)\Delta f(x)$  for any polynomials  $f(x)$  and  $g(x)$ .

Using these rules show that, if  $f(x)$  is a polynomial of degree zero (that is, a constant), then  $\Delta f(x) = 0$ . Calculate  $\Delta x^2$  and  $\Delta x^3$ .

Prove that  $\Delta h(x) \equiv \frac{dh(x)}{dx}$  for any polynomial  $h(x)$ . You should make it clear whenever you use one of the above rules in your proof.

## Section B: Mechanics

- 9 A long, light, inextensible string passes through a small, smooth ring fixed at the point  $O$ . One end of the string is attached to a particle  $P$  of mass  $m$  which hangs freely below  $O$ . The other end is attached to a bead,  $B$ , also of mass  $m$ , which is threaded on a smooth rigid wire fixed in the same vertical plane as  $O$ . The distance  $OB$  is  $r$ , the distance  $OH$  is  $h$  and the height of the bead above the horizontal plane through  $O$  is  $y$ , as shown in the diagram.



The shape of the wire is such that the system can be in static equilibrium for all positions of the bead. By considering potential energy, show that the equation of the wire is  $y + r = 2h$ .

The bead is initially at  $H$ . It is then projected along the wire with initial speed  $V$ . Show that, in the subsequent motion,

$$\dot{\theta} = -\frac{h\dot{r}}{r\sqrt{rh - h^2}}$$

where  $\theta$  is given by  $\theta = \arcsin(y/r)$ .

Hence show that the speed of the particle  $P$  is  $V\left(\frac{r-h}{2r-h}\right)^{\frac{1}{2}}$ .

[Note that  $\arcsin \theta$  is another notation for  $\sin^{-1} \theta$ .]

- 10** A disc rotates freely in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is  $mk^2$  (where  $k > 0$ ). Along one diameter is a smooth narrow groove in which a particle of mass  $m$  slides freely. At time  $t = 0$ , the disc is rotating with angular speed  $\Omega$ , and the particle is a distance  $a$  from the axis and is moving with speed  $V$  along the groove, towards the axis, where  $k^2V^2 = \Omega^2a^2(k^2 + a^2)$ .

Show that, at a later time  $t$ , while the particle is still moving towards the axis, the angular speed  $\omega$  of the disc and the distance  $r$  of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \quad \text{and} \quad \left(\frac{dr}{dt}\right)^2 = \frac{\Omega^2r^2(k^2 + a^2)^2}{k^2(k^2 + r^2)}.$$

Deduce that

$$k\frac{dr}{d\theta} = -r(k^2 + r^2)^{\frac{1}{2}},$$

where  $\theta$  is the angle through which the disc has turned by time  $t$ .

By making the substitution  $u = k/r$ , or otherwise, show that  $r \sinh(\theta + \alpha) = k$ , where  $\sinh \alpha = k/a$ . Deduce that the particle never reaches the axis.

- 11** A lift of mass  $M$  and its counterweight of mass  $M$  are connected by a light inextensible cable which passes over a fixed frictionless pulley. The lift is constrained to move vertically between smooth guides. The distance between the floor and the ceiling of the lift is  $h$ . Initially, the lift is at rest, and the distance between the top of the lift and the pulley is greater than  $h$ . A small tile of mass  $m$  becomes detached from the ceiling of the lift and falls to the floor of the lift. Show that the speed of the tile just before the impact is

$$\sqrt{\frac{(2M - m)gh}{M}}.$$

The coefficient of restitution between the tile and the floor of the lift is  $e$ . Given that the magnitude of the impulsive force on the lift due to tension in the cable is equal to the magnitude of the impulsive force on the counterweight due to tension in the cable, show that the loss of energy of the system due to the impact is  $mgh(1 - e^2)$ . Comment on this result.

## Section C: Probability and Statistics

- 12** Fifty times a year, 1024 tourists disembark from a cruise liner at a port. From there they must travel to the city centre either by bus or by taxi. Tourists are equally likely to be directed to the bus station or to the taxi rank. Each bus of the bus company holds 32 passengers, and the company currently runs 15 buses. The company makes a profit of £1 for each passenger carried. It carries as many passengers as it can, with any excess being (eventually) transported by taxi. Show that the largest annual licence fee, in pounds, that the company should consider paying to be allowed to run an extra bus is approximately

$$1600\Phi(2) - \frac{800}{\sqrt{2\pi}}(1 - e^{-2}),$$

where  $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}t^2} dt$ .

[You should not consider continuity corrections.]

- 13** Two points are chosen independently at random on the perimeter (including the diameter) of a semicircle of unit radius. The area of the triangle whose vertices are these two points and the midpoint of the diameter is denoted by the random variable  $A$ . Show that the expected value of  $A$  is  $(2 + \pi)^{-1}$ .
- 14** For any random variables  $X_1$  and  $X_2$ , state the relationship between  $E(aX_1 + bX_2)$  and  $E(X_1)$  and  $E(X_2)$ , where  $a$  and  $b$  are constants. If  $X_1$  and  $X_2$  are independent, state the relationship between  $E(X_1X_2)$  and  $E(X_1)$  and  $E(X_2)$ .

An industrial process produces rectangular plates. The length and the breadth of the plates are modelled by independent random variables  $X_1$  and  $X_2$  with non-zero means  $\mu_1$  and  $\mu_2$  and non-zero standard deviations  $\sigma_1$  and  $\sigma_2$ , respectively. Using the results in the paragraph above, and without quoting a formula for  $\text{Var}(aX_1 + bX_2)$ , find the means and standard deviations of the perimeter  $P$  and area  $A$  of the plates. Show that  $P$  and  $A$  are not independent.

The random variable  $Z$  is defined by  $Z = P - \alpha A$ , where  $\alpha$  is a constant. Show that  $Z$  and  $A$  are not independent if

$$\alpha \neq \frac{2(\mu_1\sigma_2^2 + \mu_2\sigma_1^2)}{\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2}.$$

Given that  $X_1$  and  $X_2$  can each take values 1 and 3 only, and that they each take these values with probability  $\frac{1}{2}$ , show that  $Z$  and  $A$  are not independent for any value of  $\alpha$ .