

OXFORD CAMBRIDGE AND RSA EXAMINATIONS
Sixth Term Examination Papers
administered on behalf of the Cambridge Colleges

MATHEMATICS I

9465

Wednesday

29 JUNE 2005

Afternoon

3 hours

Additional materials:

- Answer paper
- Graph paper
- Formulae booklet

Candidates may not use electronic calculators

TIME 3 hours

INSTRUCTIONS TO CANDIDATES

- Write your name, Centre number and candidate number in the spaces on the answer paper/ answer booklet.
- Begin each answer on a new page.

INFORMATION FOR CANDIDATES

- Each question is marked out of 20. There is no restriction of choice.
- You will be assessed on the **six** questions for which you gain the highest marks.
- You are advised to concentrate on no more than **six** questions. Little credit will be given to fragmentary answers.
- You are provided with Mathematical Formulae and Tables.
- **Electronic calculators are not permitted.**

This question paper consists of 8 printed pages.

Section A: Pure Mathematics

- 1 47231 is a five-digit number whose digits sum to $4 + 7 + 2 + 3 + 1 = 17$.
- (i) Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.
- (ii) How many five-digit numbers are there whose digits sum to 39?

- 2 The point P has coordinates $(p^2, 2p)$ and the point Q has coordinates $(q^2, 2q)$, where p and q are non-zero and $p \neq q$. The curve C is given by $y^2 = 4x$. The point R is the intersection of the tangent to C at P and the tangent to C at Q . Show that R has coordinates $(pq, p + q)$.

The point S is the intersection of the normal to C at P and the normal to C at Q . If p and q are such that $(1, 0)$ lies on the line PQ , show that S has coordinates $(p^2 + q^2 + 1, p + q)$, and that the quadrilateral $PSQR$ is a rectangle.

- 3 In this question a and b are distinct, non-zero real numbers, and c is a real number.

- (i) Show that, if a and b are either both positive or both negative, then the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1$$

has two distinct real solutions.

- (ii) Show that the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c$$

has exactly one real solution if $c^2 = -\frac{4ab}{(a-b)^2}$. Show that this condition can be

written $c^2 = 1 - \left(\frac{a+b}{a-b}\right)^2$ and deduce that it can only hold if $0 < c^2 \leq 1$.

- 4 (a) Given that $\cos \theta = \frac{3}{5}$ and that $\frac{3\pi}{2} \leq \theta \leq 2\pi$, show that $\sin 2\theta = -\frac{24}{25}$, and evaluate $\cos 3\theta$.

(b) Prove the identity $\tan 3\theta \equiv \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$.

Hence evaluate $\tan \theta$, given that $\tan 3\theta = \frac{11}{2}$ and that $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$.

- 5 (i) Evaluate the integral

$$\int_0^1 (x+1)^{k-1} dx$$

in the cases $k \neq 0$ and $k = 0$.

Deduce that $\frac{2^k - 1}{k} \approx \ln 2$ when $k \approx 0$.

- (ii) Evaluate the integral

$$\int_0^1 x(x+1)^m dx$$

in the different cases that arise according to the value of m .

- 6 (i) The point A has coordinates $(5, 16)$ and the point B has coordinates $(-4, 4)$. The variable point P has coordinates (x, y) and moves on a path such that $AP = 2BP$. Show that the Cartesian equation of the path of P is

$$(x+7)^2 + y^2 = 100.$$

- (ii) The point C has coordinates $(a, 0)$ and the point D has coordinates $(b, 0)$. The variable point Q moves on a path such that

$$QC = k \times QD,$$

where $k > 1$. Given that the path of Q is the same as the path of P , show that

$$\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}.$$

Show further that $(a+7)(b+7) = 100$, in the case $a \neq b$.

- 7 The notation $\prod_{r=1}^n f(r)$ denotes the product $f(1) \times f(2) \times f(3) \times \cdots \times f(n)$.

Simplify the following products as far as possible:

(i) $\prod_{r=1}^n \left(\frac{r+1}{r} \right);$

(ii) $\prod_{r=2}^n \left(\frac{r^2-1}{r^2} \right);$

(iii) $\prod_{r=1}^n \left(\cos \frac{2\pi}{n} + \sin \frac{2\pi}{n} \cot \frac{(2r-1)\pi}{n} \right),$ where n is even.

- 8 Show that, if $y^2 = x^k f(x)$, then $2xy \frac{dy}{dx} = ky^2 + x^{k+1} \frac{df}{dx}$.

- (i) By setting $k = 1$ in this result, find the solution of the differential equation

$$2xy \frac{dy}{dx} = y^2 + x^2 - 1$$

for which $y = 2$ when $x = 1$. Describe geometrically this solution.

- (ii) Find the solution of the differential equation

$$2x^2y \frac{dy}{dx} = 2 \ln(x) - xy^2$$

for which $y = 1$ when $x = 1$.

Section B: Mechanics

- 9 A non-uniform rod AB has weight W and length $3l$. When the rod is suspended horizontally in equilibrium by vertical strings attached to the ends A and B , the tension in the string attached to A is T .

When instead the rod is held in equilibrium in a horizontal position by means of a smooth pivot at a distance l from A and a vertical string attached to B , the tension in the string is T . Show that $5T = 2W$.

When instead the end B of the rod rests on rough horizontal ground and the rod is held in equilibrium at an angle θ to the horizontal by means of a string that is perpendicular to the rod and attached to A , the tension in the string is $\frac{1}{2}T$. Calculate θ and find the smallest value of coefficient of friction between the rod and the ground that will prevent slipping.

- 10 Three collinear, non-touching particles A , B and C have masses a , b and c , respectively, and are at rest on a smooth horizontal surface. The particle A is given an initial velocity u towards B . These particles collide, giving B a velocity v towards C . These two particles then collide, giving C a velocity w .

The coefficient of restitution is e in both collisions. Determine an expression for v , and show that

$$w = \frac{abu(1+e)^2}{(a+b)(b+c)}.$$

Determine the final velocities of each of the three particles in the cases:

- (i) $\frac{a}{b} = \frac{b}{c} = e$;
- (ii) $\frac{b}{a} = \frac{c}{b} = e$.

- 11 A particle moves so that \mathbf{r} , its displacement from a fixed origin at time t , is given by

$$\mathbf{r} = (\sin 2t)\mathbf{i} + (2 \cos t)\mathbf{j},$$

where $0 \leq t < 2\pi$.

- (i) Show that the particle passes through the origin exactly twice.
- (ii) Determine the times when the velocity of the particle is perpendicular to its displacement.
- (iii) Show that, when the particle is not at the origin, its velocity is never parallel to its displacement.
- (iv) Determine the maximum distance of the particle from the origin, and sketch the path of the particle.

Section C: Probability and Statistics

- 12 (a) The probability that a hobbit smokes a pipe is 0.7 and the probability that a hobbit wears a hat is 0.4. The probability that a hobbit smokes a pipe but does not wear a hat is p . Determine the range of values of p consistent with this information.
- (b) The probability that a wizard wears a hat is 0.7; the probability that a wizard wears a cloak is 0.8; and the probability that a wizard wears a ring is 0.4. The probability that a wizard does not wear a hat, does not wear a cloak and does not wear a ring is 0.05. The probability that a wizard wears a hat, a cloak and also a ring is 0.1. Determine the probability that a wizard wears exactly two of a hat, a cloak, and a ring.
- The probability that a wizard wears a hat but not a ring, **given** that he wears a cloak, is q . Determine the range of values of q consistent with this information.

- 13 The random variable X has mean μ and standard deviation σ . The distribution of X is symmetrical about μ and satisfies:

$$P(X \leq \mu + \sigma) = a \quad \text{and} \quad P(X \leq \mu + \frac{1}{2}\sigma) = b,$$

where a and b are fixed numbers. Do not assume that X is Normally distributed.

- (a) Determine expressions (in terms of a and b) for

$$P\left(\mu - \frac{1}{2}\sigma \leq X \leq \mu + \sigma\right) \quad \text{and} \quad P\left(X \leq \mu + \frac{1}{2}\sigma \mid X \geq \mu - \frac{1}{2}\sigma\right).$$

- (b) My local supermarket sells cartons of skimmed milk and cartons of full-fat milk: 60% of the cartons it sells contain skimmed milk, and the rest contain full-fat milk.
- The volume of skimmed milk in a carton is modelled by X ml, with $\mu = 500$ and $\sigma = 10$. The volume of full fat milk in a carton is modelled by X ml, with $\mu = 495$ and $\sigma = 10$.
- (i) Today, I bought one carton of milk, chosen at random, from this supermarket. When I get home, I find that it contains less than 505 ml. Determine an expression (in terms of a and b) for the probability that this carton of milk contains more than 500 ml.
- (ii) Over the years, I have bought a very large number of cartons of milk, all chosen at random, from this supermarket. 70% of the cartons I have bought have contained at most 505 ml of milk. Of all the cartons that have contained at least 495 ml of milk, one third of them have contained full-fat milk. Use this information to estimate the values of a and b .

- 14 The random variable X can take the value $X = -1$, and also any value in the range $0 \leq X < \infty$. The distribution of X is given by

$$P(X = -1) = m, \quad P(0 \leq X \leq x) = k(1 - e^{-x}),$$

for any non-negative number x , where k and m are constants, and $m < \frac{1}{2}$.

- (i) Find k in terms of m .
- (ii) Show that $E(X) = 1 - 2m$.
- (iii) Find, in terms of m , $\text{Var}(X)$ and the median value of X .
- (iv) Given that

$$\int_0^{\infty} y^2 e^{-y^2} dy = \frac{1}{4} \sqrt{\pi},$$

find $E(|X|^{\frac{1}{2}})$ in terms of m .