Section A: Pure Mathematics

1 To nine decimal places, $\log_{10} 2 = 0.301029996$ and $\log_{10} 3 = 0.477121255$.

(i) Calculate $\log_{10} 5$ and $\log_{10} 6$ to three decimal places. By taking logs, or otherwise, show that

$$5 \times 10^{47} < 3^{100} < 6 \times 10^{47}.$$

Hence write down the first digit of 3^{100}.

(ii) Find the first digit of each of the following numbers: 2^{1000}; $2^{10\,000}$; and $2^{100\,000}$.

2 Show that the coefficient of x^{-12} in the expansion of

$$\left(x^4 - \frac{1}{x^2}\right)^5 \left(x - \frac{1}{x}\right)^6$$

is -15, and calculate the coefficient of x^2.

Hence, or otherwise, calculate the coefficients of x^4 and x^{38} in the expansion of

$$(x^2 - 1)^{11}(x^4 + x^2 + 1)^5.$$

3 For any number x, the largest integer less than or equal to x is denoted by $[x]$. For example, $[3.7] = 3$ and $[4] = 4$.

Sketch the graph of $y = [x]$ for $0 \leqslant x < 5$ and evaluate

$$\int_0^5 [x] \; \mathrm{d}x.$$

Sketch the graph of $y = [\,\mathrm{e}^x]$ for $0 \leqslant x < \ln n$, where n is an integer, and show that

$$\int_0^{\ln n} [\,\mathrm{e}^x] \; \mathrm{d}x = n \ln n - \ln(n!).$$

4 (i) Show that, for $0 \leqslant x \leqslant 1$, the largest value of $\dfrac{x^6}{(x^2+1)^4}$ is $\dfrac{1}{16}$.

(ii) Find constants A, B, C and D such that, for all x,

$$\frac{1}{(x^2+1)^4} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) + \frac{Dx^6}{(x^2+1)^4}.$$

(iii) Hence, or otherwise, prove that

$$\frac{11}{24} \leqslant \int_0^1 \frac{1}{(x^2+1)^4}\,\mathrm{d}x \leqslant \frac{11}{24} + \frac{1}{16}\,.$$

5 Arthur and Bertha stand at a point O on an inclined plane. The steepest line in the plane through O makes an angle θ with the horizontal. Arthur walks uphill at a steady pace in a straight line which makes an angle α with the steepest line. Bertha walks uphill at the same speed in a straight line which makes an angle β with the steepest line (and is on the same side of the steepest line as Arthur). Show that, when Arthur has walked a distance d, the distance between Arthur and Bertha is $2d|\sin\frac{1}{2}(\alpha-\beta)|$. Show also that, if $\alpha \neq \beta$, the line joining Arthur and Bertha makes an angle ϕ with the vertical, where

$$\cos\phi = \sin\theta\sin\tfrac{1}{2}(\alpha+\beta).$$

6 Show that

$$x^2 - y^2 + x + 3y - 2 = (x-y+2)(x+y-1)$$

and hence, or otherwise, indicate by means of a sketch the region of the x-y plane for which

$$x^2 - y^2 + x + 3y > 2.$$

Sketch also the region of the x-y plane for which

$$x^2 - 4y^2 + 3x - 2y < -2.$$

Give the coordinates of a point for which both inequalities are satisfied or explain why no such point exists.

7 Let

$$\mathrm{f}(x) = ax - \frac{x^3}{1+x^2},$$

where a is a constant. Show that, if $a \geqslant 9/8$, then $\mathrm{f}'(x) \geqslant 0$ for all x.

8 Show that

$$\int_{-1}^{1} |\, x\,\mathrm{e}^x \,|\,\mathrm{d}x = -\int_{-1}^{0} x\,\mathrm{e}^x\,\mathrm{d}x + \int_{0}^{1} x\,\mathrm{e}^x\,\mathrm{d}x$$

and hence evaluate the integral.

Evaluate the following integrals:

(i) $\displaystyle\int_0^4 |\, x^3 - 2x^2 - x + 2 \,|\;\mathrm{d}x\,;$

(ii) $\displaystyle\int_{-\pi}^{\pi} |\, \sin x + \cos x \,|\;\mathrm{d}x\,.$

Section B: Mechanics

9 A child is playing with a toy cannon on the floor of a long railway carriage. The carriage is moving horizontally in a northerly direction with acceleration a. The child points the cannon southward at an angle θ to the horizontal and fires a toy shell which leaves the cannon at speed V. Find, in terms of a and g, the value of $\tan 2\theta$ for which the cannon has maximum range (in the carriage).

If a is small compared with g, show that the value of θ which gives the maximum range is approximately

$$\frac{\pi}{4} + \frac{a}{2g},$$

and show that the maximum range is approximately $\dfrac{V^2}{g} + \dfrac{V^2 a}{g^2}$.

10 Three particles P_1, P_2 and P_3 of masses m_1, m_2 and m_3 respectively lie at rest in a straight line on a smooth horizontal table. P_1 is projected with speed v towards P_2 and brought to rest by the collision. After P_2 collides with P_3, the latter moves forward with speed v. The coefficients of restitution in the first and second collisions are e and e', respectively. Show that

$$e' = \frac{m_2 + m_3 - m_1}{m_1}.$$

Show that $2m_1 \geqslant m_2 + m_3 \geqslant m_1$ for such collisions to be possible.

If m_1, m_3 and v are fixed, find, in terms of m_1, m_3 and v, the largest and smallest possible values for the final energy of the system.

11 A rod AB of length 0.81 m and mass 5 kg is in equilibrium with the end A on a rough floor and the end B against a very rough vertical wall. The rod is in a vertical plane perpendicular to the wall and is inclined at 45° to the horizontal. The centre of gravity of the rod is at G, where $AG = 0.21$ m. The coefficient of friction between the rod and the floor is 0.2, and the coefficient of friction between the rod and the wall is 1.0. Show that the friction cannot be limiting at both A and B.

A mass of 5 kg is attached to the rod at the point P such that now the friction is limiting at both A and B. Determine the length of AP.

Section C: Probability and Statistics

12 I have k different keys on my key ring. When I come home at night I try one key after another until I find the key that fits my front door. What is the probability that I find the correct key in exactly n attempts in each of the following three cases?

(i) At each attempt, I choose a key that I have not tried before but otherwise each choice is equally likely.

(ii) At each attempt, I choose a key from all my keys and each of the k choices is equally likely.

(iii) At the first attempt, I choose from all my keys and each of the k choices is equally likely. Thereafter, I choose from the keys that I did not try the previous time but otherwise each choice is equally likely.

13 Every person carries two genes which can each be either of type A or of type B. It is known that 81% of the population are AA (i.e. both genes are of type A), 18% are AB (i.e. there is one gene of type A and one of type B) and 1% are BB. A child inherits one gene from each of its parents. If one parent is AA, the child inherits a gene of type A from that parent; if the parent is BB, the child inherits a gene of type B from that parent; if the parent is AB, the inherited gene is equally likely to be A or B.

(i) Given that two AB parents have four children, show that the probability that two of them are AA and two of them are BB is $3/128$.

(ii) My mother is AB and I am AA. Find the probability that my father is AB.

14 The random variable X is uniformly distributed on the interval $[-1, 1]$. Find $\mathrm{E}(X^2)$ and $\mathrm{Var}\,(X^2)$.

A second random variable Y, independent of X, is also uniformly distributed on $[-1, 1]$, and $Z = Y - X$. Find $\mathrm{E}(Z^2)$ and show that $\mathrm{Var}\,(Z^2) = 7\,\mathrm{Var}\,(X^2)$.