SIXTH TERM EXAMINATION PAPERS

administered by the Oxford and Cambridge Schools Examination Board
on behalf of the Cambridge Colleges

9475

MATHEMATICS III

Friday 1 July 1994, afternoon

3 hours

Additional materials :
script paper; graph paper; MF(STEP)1.
To be brought by candidate: electronic calculator;
standard geometrical instruments.

All questions carry equal weight.

Candidates are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.

Candidates may attempt as many questions as they wish but marks will be assessed on the six questions best answered.

Mathematical Formulae and tables (MF STEP) are provided.

Electronic calculators may be used.

This Question Paper consists of 7 printed pages and 1 blank page.

Section A: Pure Mathematics

1 Calculate

$$\int_0^x \operatorname{sech} t \, dt.$$

Find the reduction formula involving I_n and I_{n-2}, where

$$I_n = \int_0^x \operatorname{sech}^n t \, dt$$

and, hence or otherwise, find I_5 and I_6.

2(i) By setting $y = x + x^{-1}$, find the solutions of

$$x^4 + 10x^3 + 26x^2 + 10x + 1 = 0.$$

(ii) Solve

$$x^4 + x^3 - 10x^2 - 4x + 16 = 0.$$

3 Describe geometrically the possible intersections of a plane with a sphere.

Let P_1 and P_2 be the planes with equations

$$3x - y - 1 = 0,$$
$$x - y + 1 = 0,$$

respectively, and let S_1 and S_2 be the spheres with equations

$$x^2 + y^2 + z^2 = 7,$$
$$x^2 + y^2 + z^2 - 6y - 4z + 10 = 0,$$

respectively. Let C_1 be the intersection of P_1 and S_1, let C_2 be the intersection of P_2 and S_2 and let L be the intersection of P_1 and P_2. Find the points where L meets each of S_1 and S_2. Determine, giving your reasons, whether the circles C_1 and C_2 are linked.

4 Find the two solutions of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4y$$

which pass through the point $(a,\, b^2)$, where $b \neq 0$.

Find two distinct points $(a_1,\, 1)$ and $(a_2,\, 1)$ such that one of the solutions through each of them also passes through the origin. Show that the graphs of these two solutions coincide and sketch their common graph, together with the other solutions through $(a_1,\, 1)$ and $(a_2,\, 1)$.

Now sketch sufficient members of the family of solutions (for varying a and b) to indicate the general behaviour. Use your sketch to identify a common tangent, and comment briefly on its relevance to the differential equation.

5 The function f is given by $\mathrm{f}(x) = \sin^{-1} x$ for $-1 < x < 1$. Prove that

$$(1 - x^2)\mathrm{f}''(x) - x\mathrm{f}'(x) = 0.$$

Prove also that

$$(1 - x^2)\mathrm{f}^{(n+2)}(x) - (2n+1)x\mathrm{f}^{(n+1)}(x) - n^2\mathrm{f}^{(n)}(x) = 0,$$

for all $n > 0$, where $\mathrm{f}^{(n)}$ denotes the nth derivative of f. Hence express $\mathrm{f}(x)$ as a Maclaurin series.

The function g is given by

$$\mathrm{g}(x) = \ln \sqrt{\frac{1+x}{1-x}},$$

for $-1 < x < 1$. Write down a power series expansion for $\mathrm{g}(x)$, and show that the coefficient of x^{2n+1} is greater than that in the expansion of f, for each $n > 0$.

6 The four points A, B, C, D in the Argand diagram (complex plane) correspond to the complex numbers a, b, c, d respectively. The point P_1 is mapped to P_2 by rotating about A through $\pi/2$ radians. Then P_2 is mapped to P_3 by rotating about B through $\pi/2$ radians, P_3 is mapped to P_4 by rotating about C through $\pi/2$ radians and P_4 is mapped to P_5 by rotating about D through $\pi/2$ radians, each rotation being in the positive sense. If z_i is the complex number corresponding to P_i, find z_5 in terms of a, b, c, d and z_1.

Show that P_5 will coincide with P_1, irrespective of the choice of the latter if, and only if, $a - c = i(b - d)$ and interpret this condition geometrically.

The points A, B and C are now chosen to be distinct points on the unit circle and the angle of rotation is changed to θ, where $\theta \neq 0$, on each occasion. Find the necessary and sufficient condition on θ and the points A, B and C for P_4 always to coincide with P_1.

7 Let S_3 be the group of permutations of three objects and Z_6 be the group of integers under addition modulo 6. List all the elements of each group, stating the order of each element. State, with reasons, whether S_3 is isomorphic with Z_6.

Let C_6 be the group of 6th roots of unity. That is, $C_6 = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$ where $\alpha = e^{\pi i/3}$ and the group operation is complex multiplication. Prove that C_6 is isomorphic with Z_6. Is there any (multiplicative or additive) subgroup of the complex numbers which is isomorphic with S_3? Give a reason for your answer.

8 Let a, b, c, d, p, q, r and s be real numbers. By considering the determinant of the matrix product

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} \begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix},$$

where z_1, z_2, z_3 and z_4 are suitably chosen complex numbers, find expressions L_1, L_2, L_3 and L_4, each of which is linear in a, b, c and d and also linear in p, q, r and s, such that

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = {L_1}^2 + {L_2}^2 + {L_3}^2 + {L_4}^2.$$

Section B: Mechanics

9 A smooth, axially symmetric bowl has its vertical cross-sections determined by $s = 2\sqrt{ky}$, where s is the arc-length measured from its lowest point V, and y is the height above V. A particle is released from rest at a point on the surface at a height h above V. Explain why

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + 2gy$$

is constant.

Show that the time for the particle to reach V is

$$\pi\sqrt{\frac{k}{2g}}.$$

Two elastic particles of mass m and αm, where $\alpha < 1$, are released simultaneously from opposite sides of the bowl at heights $\alpha^2 h$ and h respectively. If the coefficient of restitution between the particles is α, describe the subsequent motion.

10 The island of Gammaland is totally flat and subject to a constant wind of w kh^{-1}. blowing from the West. Its southernmost shore stretches almost indefinitely, due east and west, from the coastal city of Alphaport. A novice pilot is making her first solo flight from Alphaport to the town of Betaville which lies north-east of Alphaport. Her instructor has given her the correct heading to reach Betaville, flying at the plane's recommended airspeed of v kh^{-1}, where $v > w$.

On reaching Betaport the pilot returns with the opposite heading to that of the outward flight and, so featureless is Gammaland, that she only realises her error as she crosses the coast with Alphaport nowhere in sight. Assuming that she then turns West along the coast, and that her outward flight took t hours, show that her return flight takes

$$\left(\frac{v+w}{v-w}\right)t \text{ hours.}$$

If Betaville is d kilometres from Alphaport, show that, with the correct heading, the return flight should have taken

$$t + \frac{\sqrt{2}wd}{v^2 - w^2} \text{ hours.}$$

11 A step-ladder has two sections AB and AC, each of length $4a$, smoothly hinged at A and connected by a light elastic rope DE, of natural length $a/4$ and modulus W, where D is on AB, E is on AC and $AD = AE = a$. The section AB, which contains the steps, is uniform and of weight W and the weight of AC is negligible.

The step-ladder rests on a smooth horizontal floor and a man of weight $4W$ carefully ascends it to stand on a rung distant βa from the end of the ladder resting on the floor. Find the height above the floor of the rung on which the man is standing when β is the maximum value at which equilibrium is possible.

Section C: Probability & Statistics

12 In certain forms of Tennis two players A and B serve alternate games. Player A has probability p_A of winning a game in which she serves and probability p_B of winning a game in which player B serves. Player B has probability $q_B = 1 - p_B$ of winning a game in which she serves and probability $q_A = 1 - p_A$ of winning a game in which player A serves. In Shortened Tennis the first player to lead by 2 games wins the match. Find the probability P_{short} that A wins a Shortened Tennis match in which she serves first and show that it is the same as if B serves first.

In Standard Tennis the first player to lead by 2 or more games after 4 or more games have been played wins the match. Show that the probability that the match is decided in 4 games is

$$p_A^2 p_B^2 + q_A^2 q_B^2 + 2(p_A p_B + q_A q_B)(p_A q_B + q_A p_B).$$

If $p_A = p_B = p$ and $q_A = q_B = q$, find the probability P_{stan} that A wins a Standard Tennis match in which she serves first. Show that

$$P_{\text{stan}} - P_{\text{short}} = \frac{p^2 q^2 (p-q)}{(p^2+q^2)}.$$

13 During his performance a trapeze artist is supported by two identical ropes, either of which can bear his weight. Each rope is such that the time, in hours of performance, before it fails is exponentially distributed, independently of the other, with probability density function $\lambda \exp(-\lambda t)$ for $t \geqslant 0$ (and 0 for $t < 0$), for some $\lambda > 0$. A particular rope has already been in use for t_0 hours of performance. Find the distribution for the length of time the artist can continue to use it before it fails. Interpret and comment upon your result.

Before going on tour the artist insists that the management purchase two new ropes of the above type. Show that the probability density function of the time until both ropes fail is

$$\mathrm{f}(t) = \begin{cases} 2\lambda e^{-\lambda t}(1 - e^{-\lambda t}) & \text{if } t \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

If each performance lasts for h hours, find the probability that both ropes fail during the nth performance. Show that the probability that both ropes will fail during the same performance is $\tanh(\frac{\lambda h}{2})$.

14 Three points, P, Q and R, are independently randomly chosen on the perimeter of a circle. Prove that the probability that at least one of the angles of the triangle PQR will exceed $k\pi$ is $3(1-k)^2$ if $\frac{1}{2} \leqslant k \leqslant 1$. Find the probability if $\frac{1}{3} \leqslant k \leqslant \frac{1}{2}$.