Stats 2 Continuous Random Variable Questions

4 (a) A random variable X has probability density function defined by

$$f(x) = \begin{cases} k & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that $k = \dfrac{1}{b-a}$. *(1 mark)*

(ii) Prove, using integration, that $E(X) = \frac{1}{2}(a+b)$. *(4 marks)*

(b) The error, X grams, made when a shopkeeper weighs out loose sweets can be modelled by a rectangular distribution with the following probability density function:

$$f(x) = \begin{cases} k & -2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

(i) Write down the value of the mean, μ, of X. *(1 mark)*

(ii) Evaluate the standard deviation, σ, of X. *(2 marks)*

(iii) Hence find $P\left(X < \dfrac{2-\mu}{\sigma}\right)$. *(3 marks)*

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7 Engineering work on the railway network causes an increase in the journey time of commuters travelling into work each morning.

The increase in journey time, T hours, is modelled by a continuous random variable with probability density function

$$f(t) = \begin{cases} 4t(1-t^2) & 0 \leqslant t \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $E(T) = \dfrac{8}{15}$. *(3 marks)*

(b) (i) Find the cumulative distribution function, $F(t)$, for $0 \leqslant t \leqslant 1$. *(2 marks)*

(ii) Hence, or otherwise, for a commuter selected at random, find

$$P(\text{mean} < T < \text{median})$$ *(5 marks)*

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5 (a) The continuous random variable X follows a rectangular distribution with probability density function defined by

$$\mathrm{f}(x) = \begin{cases} \dfrac{1}{b} & 0 \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}$$

(i) Write down $\mathrm{E}(X)$. *(1 mark)*

(ii) Prove, using integration, that

$$\mathrm{Var}(X) = \tfrac{1}{12}b^2$$ *(5 marks)*

(b) At an athletics meeting, the error, in seconds, made in recording the time taken to complete the 10 000 metres race may be modelled by the random variable T, having the probability density function

$$\mathrm{f}(t) = \begin{cases} 5 & -0.1 \leqslant t \leqslant 0.1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $\mathrm{P}(|T| > 0.02)$. *(3 marks)*

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7 The continuous random variable X has probability density function defined by

$$\mathrm{f}(x) = \begin{cases} \tfrac{1}{5}(2x+1) & 0 \leqslant x \leqslant 1 \\ \tfrac{1}{15}(4-x)^2 & 1 < x \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the graph of f. *(2 marks)*

(b) (i) Show that the cumulative distribution function, $\mathrm{F}(x)$, for $0 \leqslant x \leqslant 1$ is

$$\mathrm{F}(x) = \tfrac{1}{5}x(x+1)$$ *(3 marks)*

(ii) Hence write down the value of $\mathrm{P}(X \leqslant 1)$. *(1 mark)*

(iii) Find the value of x for which $\mathrm{P}(X \geqslant x) = \frac{17}{20}$. *(5 marks)*

(iv) Find the lower quartile of the distribution. *(4 marks)*

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6 The waiting time, T minutes, before being served at a local newsagents can be modelled by a continuous random variable with probability density function

$$f(t) = \begin{cases} \frac{3}{8}t^2 & 0 \leqslant t < 1 \\ \frac{1}{16}(t+5) & 1 \leqslant t \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the graph of f. *(3 marks)*

(b) For a customer selected at random, calculate $P(T \geqslant 1)$. *(2 marks)*

(c) (i) Show that the cumulative distribution function for $1 \leqslant t \leqslant 3$ is given by

$$F(t) = \frac{1}{32}(t^2 + 10t - 7)$$ *(5 marks)*

(ii) Hence find the median waiting time. *(4 marks)*

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8 The continuous random variable X has the cumulative distribution function

$$F(x) = \begin{cases} 0 & x \leqslant -4 \\ \dfrac{x+4}{9} & -4 \leqslant x \leqslant 5 \\ 1 & x \geqslant 5 \end{cases}$$

(a) Determine the probability density function, $f(x)$, of X. *(2 marks)*

(b) Sketch the graph of f. *(2 marks)*

(c) Determine $P(X > 2)$. *(2 marks)*

(d) Evaluate the mean and variance of X. *(2 marks)*

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4 Students are each asked to measure the distance between two points to the nearest tenth of a metre.

(a) Given that the rounding error, X metres, in these measurements has a rectangular distribution, explain why its probability density function is

$$f(x) = \begin{cases} 10 & -0.05 < x \leqslant 0.05 \\ 0 & \text{otherwise} \end{cases}$$ *(3 marks)*

(b) Calculate $P(-0.01 < X < 0.02)$. *(2 marks)*

(c) Find the mean and the standard deviation of X. *(2 marks)*

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6 The continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} 3x^2 & 0 < x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine:

(i) $E\left(\frac{1}{X}\right)$; *(3 marks)*

(ii) $\operatorname{Var}\left(\frac{1}{X}\right)$. *(4 marks)*

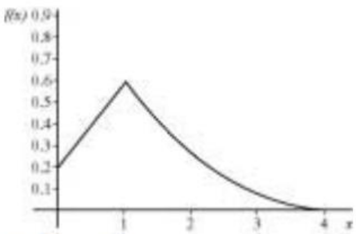
(b) Hence, or otherwise, find the mean and the variance of $\left(\frac{5 + 2X}{X}\right)$. *(5 marks)*

Stats 2 Continuous Random Variable Answers

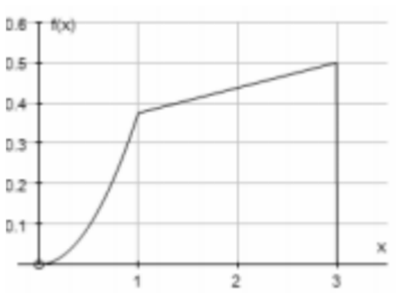
4(a)(i)	Area $= k(b-a) = 1$			
	$\Rightarrow \quad k = \dfrac{1}{b-a}$	E1	1	AG
(ii)	$\mathrm{E}(X) = \int_a^b kx \, \mathrm{d}x$	M1		
	$= \left(\dfrac{kx^2}{2}\right)\Big\vert_a^b$	A1		
	$= \dfrac{1}{2}k\left(b^2 - a^2\right)$			
	$= \dfrac{1}{2} \times \dfrac{1}{(b-a)} \times (b-a)(a+b)$	M1A1		(factors shown)
	$= \dfrac{1}{2}(a+b)$		4	AG
(b)(i)	$\mu = 1$	B1	1	
(ii)	$\sigma^2 = \mathrm{Var}(X) = \dfrac{1}{12}(b-a)^2$			
	$= \dfrac{1}{12} \times 6^2$	M1		
	$= 3$			
	$\therefore \sigma = \sqrt{3}$	A1	2	1.7321
(iii)	$\mathrm{P}\left(X < \dfrac{2-\mu}{\sigma}\right) = \mathrm{P}\left(X < \dfrac{1}{\sqrt{3}}\right)$	M1√		(on their μ and σ)
	$= \dfrac{1}{6} \times 2.577$	M1√		
	$= 0.430$	A1	3	cao
	Total		**11**	

7(a)	$E(T)=\int_0^1 t\,f(t)\,dt$			
	$=\int_0^1 4t^2(1-t^2)\,dt$	M1		
	$=\left(\frac{4t^3}{3}-\frac{4t^5}{5}\right)\Big\vert_0^1$	A1		
	$=\frac{4}{3}-\frac{4}{5}$	A1		
	$=\frac{8}{15}$		3	AG
(b)(i)	$F(t)=P(T\le t)=\int_0^t f(t)\,dt$			
	$=\int_0^t 4t(1-t^2)\,dt$	M1		
	$=(2t^2-t^4)\Big\vert_0^t$			
	$=2t^2-t^4$	A1	2	
(ii)	$P(\mu<T<m)=F(m)-F(\mu)$	M1		
	$\Downarrow$			
	$F(m)=0.5$	B1		
	$F(\mu)=F\left(\frac{8}{15}\right)=0.4880$	B1		
	$\therefore\ P(\mu<T<m)=0.5-0.4880$	M1√		0.5 – their $F(\mu)$
	$=0.012$	A1	5	
	Total		**10**	

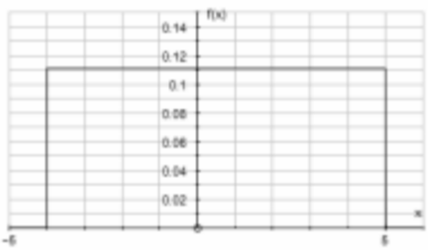
5(a)(i)	$\mathrm{E}(X)=\frac{1}{2}b$	B1	1	
(ii)	$\mathrm{E}(X^2)=\int_0^b \frac{1}{b}x^2\,\mathrm{d}x$	M1		
	$=\frac{1}{b}\left[\frac{x^3}{3}\right]_0^b$	A1		For correct integration
	$=\frac{1}{b}\left(\frac{b^3}{3}\right)$			
	$=\frac{1}{3}b^2$	A1		OE
	$\mathrm{Var}(X)=\frac{1}{3}b^2-\left(\frac{b}{2}\right)^2$	m1		Depending on using integration to get $\mathrm{E}(X^2)$
	$=\frac{1}{3}b^2-\frac{1}{4}b^2$			
	$=\frac{1}{12}b^2$	A1	5	AG
(b)	$\mathrm{P}(\lvert T\rvert>0.02)=1-\mathrm{P}(-0.02<T<0.02)$	M1		
	$=1-0.04\times 5$	M1		
	$=0.8$	A1	3	
	Total		**9**	

7(a)		B2	2	B1 for line segment (0,0.2) to (1,0.6) B1 for correctly shaped curve (1,0.6) to (4,0)
(b)(i)	for $0\le x\le 1$			
	$\mathrm{F}(x)=\int_0^x \frac{1}{5}(2x+1)\,\mathrm{d}x$	M1		Ignore limits
	$=\left[\frac{1}{5}(x^2+x)\right]_0^x$	A1		Ignore limits
	$=\frac{1}{5}x(x+1)$	A1	3	
(ii)	$\mathrm{P}(X\le 1)=\mathrm{F}(1)$			
	$=\frac{2}{5}$	B1	1	

(iii)	$\mathrm{P}(X \ge x) = \frac{17}{20} \quad \Rightarrow \quad \mathrm{F}(x) = \frac{3}{20}$	M1		
	$\frac{1}{5}x(x+1) = \frac{3}{20}$	m1		
	$x(x+1) = \frac{3}{4}$			
	$x^2 + x - \frac{3}{4} = 0$	A1		
	$\left(x - \frac{1}{2}\right)\left(x + \frac{3}{2}\right) = 0$	m1		Any valid method attempted
	$x = \frac{1}{2}$	A1	5	CAO
(iv)	Since $\mathrm{F}(1) = 0.4$, q lies in $0 \le r \le 1$			
	$\mathrm{F}(q) = \frac{1}{5}\left(q^2 + q\right) = 0.25$	M1		
	$\Rightarrow \quad q^2 + q = 1.25$			
	$q^2 + q - 1.25 = 0$	A1		
	$\Rightarrow \quad q = \frac{-1 \pm \sqrt{1 - 4\times(-1.25)}}{2}$	m1		
	$q = \frac{1}{2}\left(\sqrt{6} - 1\right) \quad (q > 0)$	A1	4	AWFW (0.724 to 0.725)
	Total		**15**	

6(a)		B1 B1 B1	3	for curve for line for axes
(b)	$\mathrm{P}(T \ge 1) = \frac{1}{2} \times \frac{7}{8} \times 2 = \frac{7}{8}$	M1A1	2	OE

6(c)(i)	For $1 \le t \le 3$			
	$\int_1^t \frac{1}{16}(t+5)\,\mathrm{d}t = \left[\frac{1}{32}t^2 + \frac{5}{16}t\right]_1^t$	M1A1		
	$\mathrm{F}(1) = \frac{1}{8}$	B1		
	$\mathrm{F}(t) = \frac{1}{8} + \frac{1}{32}t^2 + \frac{5}{16}t - \frac{11}{32}$	M1		Use of: $\mathrm{F}(t) = \mathrm{F}(1) + \int_1^t \frac{1}{16}(t+5)\,\mathrm{d}t$
	$\mathrm{F}(t) = \frac{1}{32}(t^2 + 10t - 7)$	A1	5	**AG**
	Alternative: $\int \frac{1}{16}(t+5)\,\mathrm{d}t$ $= \frac{1}{16}\left(\frac{1}{2}t^2 + 5t + c\right)$	(M1) (A1)		
	$\mathrm{F}(1) = \frac{1}{8}$	(B1)		
	$\Rightarrow c = -3.5$	(M1)		
	$\mathrm{F}(t) = \frac{1}{32}(t^2 + 10t - 7)$	(A1)		
(ii)	$\frac{1}{32}(m^2 + 10m - 7) = 0.5$	M1		
	$m^2 + 10m - 23 = 0$	A1		
	$m = \frac{-10 \pm \sqrt{192}}{2} = -5 \pm \sqrt{48}$ $= -5 \pm 4\sqrt{3}$	m1		(or any valid method)
	$(m > 0)$ $m = 4\sqrt{3} - 5 = 1.93$	A1	4	(1.9282)
	Total		**14**	

8(a)	$\mathrm{f}(x) = \begin{cases} \frac{1}{9} & -4 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$	M1 A1	2	
(b)		B1		horizontal line from –4 to 5
		B1	2	for drawn at $\frac{1}{9}$
(c)	$\mathrm{P}(X > 2) = \frac{1}{9} \times 3$	M1		$\mathrm{F}(5) - \mathrm{F}(2)$ $= 1 - \frac{2}{3}$
	$= \frac{1}{3}$	A1	2	$= \frac{1}{3}$

(d)	Mean $=\frac{1}{2}$	B1		
	Variance $=\frac{1}{12}\times 81$ $=6.75$	B1	2	
	Total		**8**	

4(a)	For a Rectangular Distribution $\mathrm{f}(x)=\begin{cases}\frac{1}{b-a} & a\le x\le b\\ 0 & \text{otherwise}\end{cases}$			
	$(-0.05, 0.05)\Rightarrow$	B1		$(\text{explain error} \pm 0.05)$
	$\frac{1}{b-a}=\frac{1}{0.05-(-0.05)}=\frac{1}{0.1}=10$	M1 A1	3	
	(Area $=10\times 0.1=1$)			
(b)	$\mathrm{P}(-0.01<X<0.02)=0.03\times 10=0.3$	M1 A1	2	
(c)	Mean $=0$	B1		CAO
	Standard deviation $=0.0289$	B1	2	$\frac{1}{20\sqrt{3}}$ OE
	Total		**7**	

6(a)(i)	$\mathrm{E}\left(\frac{1}{X}\right)=\int_0^1 \frac{1}{x}3x^2\,\mathrm{d}x=\int_0^1 3x\,\mathrm{d}x$	M1		
	$=\left[\frac{3x^2}{2}\right]_0^1=1.5$	A1 A1	3	CAO
(ii)	$\mathrm{E}\left(\frac{1}{X^2}\right)=\int_0^1 \frac{1}{x^2}3x^2\,\mathrm{d}x=\int_0^1 3\,\mathrm{d}x$	M1		
	$=[3x]_0^1=3.0$	A1		
	$\mathrm{Var}\left(\frac{1}{X}\right)=3.0-(1.5)^2$	m1		dependent on previous M1
	$=0.75$	A1√	4	[on their (i)] and Var >0
(b)	$\mathrm{E}\left(\frac{5+2X}{X}\right)=\mathrm{E}\left(\frac{5}{X}+2\right)$	M1		
	$=5\mathrm{E}\left(\frac{1}{X}\right)+2$	M1		
	$=5\times 1.5+2$ $=9.5$	A1		CAO
	$\mathrm{Var}\left(\frac{5+2X}{X}\right)=\mathrm{Var}\left(\frac{5}{X}+2\right)$			
	$=25\times\mathrm{Var}\left(\frac{1}{X}\right)$	M1		
	$=25\times 0.75$ $=18.75$	A1	5	CAO
	Total		**12**	