MEI

Mathematics in Education and Industry

MEI Structured Mathematics

Module Summary Sheets

Mechanics 3
(Version B: Reference to new book)

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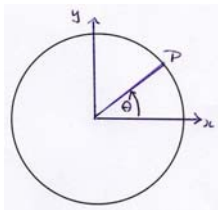
References:
Chapter 1
Pages 1-3

Angular Speed and Velocities

A force is required to keep a body moving in a circle as its velocity is always changing.
If a particle at P is moving in a circle then the angle of OP with any fixed direction changes with time.

Example 1.1
Page 4



$\dfrac{d\theta}{dt}$ is written as $\dot{\theta}$ or ω

Exercise 1A
Q. 3, 8

The position is:

$$x = r\cos\theta,\ y = r\sin\theta.$$

References:
Chapter 1
Pages 7, 8

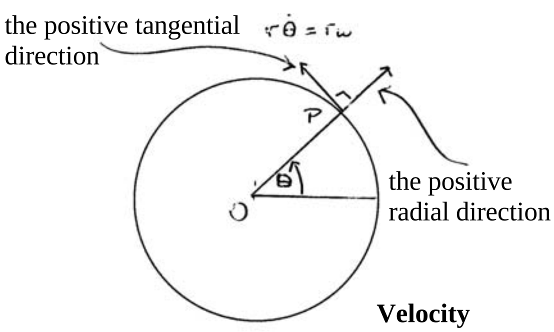
Velocity: tangential component: $v = r\dfrac{d\theta}{dt} = r\dot{\theta} = r\omega$

radial component: 0

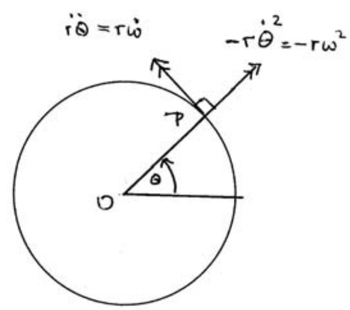
Acceleration: tangential component: $r\ddot{\theta} = r\dot{\omega}$

radial component: $-r\dot{\theta}^2 = -r\omega^2 = -\dfrac{v^2}{r}$

Example 1.2
Page 8



Velocity



Acceleration

E.g. Car A travels round a circle of radius 30 m at 33 m s^{-1}. Car B travels round a circle of radius 25 m at 1.2 rad sec^{-1}. Which car is moving the faster?

If linear speeds are compared,

Car B: $v = r\omega = 1.2 \times 2.5 = 30$ m s^{-1}

so car A has greater linear speed.

If angular speeds are compared,

Car A: $v = 33, r = 30 \Rightarrow \omega = \dfrac{33}{30} = 1.1$ rad s^{-1}

So B has greater angular speed.

E.g. The minute hand of my watch is 2 cm long. What is the speed of the tip of the hand?

$r = 0.02$ m, $\omega = 2\pi$ rads per hour $= \dfrac{2\pi}{3600}$ rads s^{-1}

$\Rightarrow v = \dfrac{0.02 \times 2\pi}{3600}$ m s$^{-1} \approx 3.5 \times 10^{-5}$ m s^{-1} (2 s.f.)

E.g. What is the speed of the second hand, which is 2.1 cm long?

$v = 3.5 \times 10^{-5} \times 60 \times \dfrac{2.1}{2}$ m s$^{-1} \approx 2.2 \times 10^{-3}$ m s^{-1} (2 s.f.)

E.g. A horizontal, circular bend on a railway track has a radius of 500 m and a train of mass m kg negotiates it at 60 mph. What is the lateral force, R N, between its wheels and the rails in terms of m?

60 mph ≈ 26.8 m s^{-1}

$R = ma = m\dfrac{v^2}{r} = \dfrac{26.8^2}{500} = 1.44m$

References:
Chapter 1
Pages 9-13

Forces required for circular motion

Newton's 2nd Law ($F = ma$) may be applied in the tangential and radial directions.

Tangential component is $mr\ddot{\theta}$

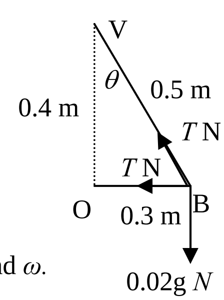
Radial component is $-mr\dot{\theta}^2 = -mr\omega^2 = -\dfrac{mv^2}{r}$

Example 1.4
Page 13

Exercise 1B
Q. 2

Mechanics 3
Version B: page 2
Competence statements r 1, 2, 3, 4, 5, 6, 7, 8

E.g. One end of a light inextensible string of length 0.8 m is attached to a point V. The other end is attached to the point O which is 0.4 m vertically below P. A small smooth bead, B, of mass 0.02 kg is threaded on the string and moves in a horizontal circle, with centre O and radius 0.3 m. OB rotates with constant angular speed ω rad s^{-1}.
Find the tension in the string and ω.



Res vert. $T\cos\theta - mg = 0$

$\Rightarrow T = 0.02g \times \dfrac{5}{4} = 0.025g\ (= 0.245...)$

Res. Hor. $T + T\sin\theta = mr\omega^2 \Rightarrow \dfrac{8}{5}T = mr\omega^2$

$\Rightarrow \omega^2 = \dfrac{8 \times 0.025g}{5 \times 0.02 \times 0.3} = 65.33...$

$\Rightarrow \omega = 8.1$ (2 s.f.)

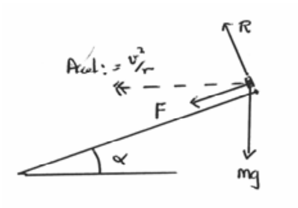
Summary M3 Topic 1: **Circular Motion – 2**

References: Chapter 1 Pages 15-19

Example 1.6 Page 19

Exercise 1B Q. 9

Banked Tracks - for motion in horizontal circle
When there is "perfect banking" there will be no friction or lateral force due to rails, etc., so the only horizontal force acting will be the resolved part of the normal reaction, R. - i.e. $F = 0$



Given that $F = 0$,

$$R\sin\theta = \frac{mv^2}{r}$$

where $R\cos\theta = mg$

$$\Rightarrow g\tan\theta = \frac{v^2}{r}$$

If the angle of the banking is greater or less than that for perfect banking, then the body will stay on the circular track only if there is friction or lateral force. If there is friction (i.e. a road rather than rails) then it may be enough to hold the body on that circular motion or it may move to a circular motion with a greater or lesser radius.

E.g. A car of mass 1200 kg is moving horizontally at constant speed round a banked track which is a circle of radius 50 m. The bank is angled at 12^0.
Find the force on the wheels if the speed is
(i) 10.2 m s^{-1}, (ii) 13 m s^{-1}.
(The diagram is opposite)

No vertical acceleration $\Rightarrow R\cos\theta = mg + F\sin\theta$

Horizontally: $\frac{mv^2}{r} = R\sin\theta + F\cos\theta$

$$\Rightarrow \frac{mv^2}{r} = \frac{(mg + F\sin\theta)\sin\theta}{\cos\theta} + F\cos\theta$$

$$= \frac{mg\sin\theta + F\left(\sin^2\theta + \cos^2\theta\right)}{\cos\theta}$$

$$= \frac{mg\sin\theta + F}{\cos\theta} \Rightarrow F = \frac{mv^2\cos\theta}{r} - mg\sin\theta$$

(i) $v = 10.2, r = 50, \theta = 12$

$\Rightarrow F = m(2.035 - 2.035) = 0$

(ii) $v = 13, r = 50, \theta = 12$

$\Rightarrow F = m(3.306 - 2.037) = 1522$ N

N.B. You do not need to know the positive direction of F before resolving. If F had been taken positive up the plane you would simply have obtained the answer -1522 N.

References: Chapter 1 Pages 26-28

Example 1.7 Page 28

Exercise 1C Q. 3

Circular motion with constant angular acceleration
The equations for motion with constant angular acceleration α are

$\omega = \omega_0 + \alpha t$ $\qquad$ $\omega^2 = \omega_0{}^2 + 2\alpha\theta$

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\qquad$ $\theta = \frac{1}{2}(\omega_0 + \omega)t$

where ω_0 is the initial angular speed.

E.g. A particle is swung in a vertical circle at the end of a light rod of radius r with initial speed u at the bottom of the circle. Find the condition for the particle to complete the circle.

The particle completes the circle if $v > 0$ when $\theta = 180^0$
i.e. $\cos\theta = -1$

Given $v^2 = u^2 - 2rg(1 - \cos\theta)$ then if $v > 0$ and $\cos\theta = -1$,

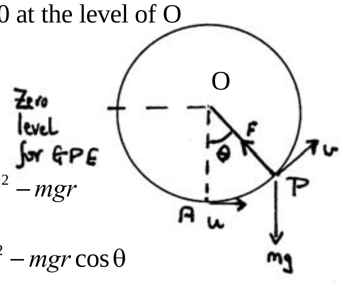
$\frac{u^2}{r} > 2g(1 - \cos\theta) = 4g$ so $u^2 > 4rg$

References: Chapter 1 Pages 30-35

Example 1.8 Page 32

Exercise 1D Q. 4

Motion in a vertical circle
If a particle is constrained to move in a vertical circle (i.e. at the end of a rigid rod or on a wire)
Using conservation of mechanical energy:
KE + GPE = const;
If we take GPE = 0 at the level of O



energy at A: $\frac{1}{2}mu^2 - mgr$

energy at P: $\frac{1}{2}mv^2 - mgr\cos\theta$

Since energy is conserved:

$$\frac{1}{2}mu^2 - mgr = \frac{1}{2}mv^2 - mgr\cos\theta$$

$$\Rightarrow v^2 = u^2 - 2gr(1 - \cos\theta)$$

E.g. A small ring of mass m is threaded onto a smooth vertical circular wire of radius 0.15 m. The ring is projected from the lowest point of the wire with speed 2.1 m s^{-1}. How far above the bottom of the wire does the ring reach?
The ring stops when $v = 0$ and the angle is θ,

Energy at A $= \frac{1}{2}mu^2 - mg \times 0.15$

Energy when ring stops $= 0 - mg \times 0.15 \times \cos\theta$

Referred to the diagram opposite and conserving energy:

$$\frac{1}{2}mu^2 - mg \times 0.15 = -mg \times 0.15\cos\theta$$

$$\Rightarrow \cos\theta = -\frac{1}{2} \times \frac{2.1^2 m}{0.15mg} + 1 = -0.5 \Rightarrow \theta = 120^0$$

So the height reached is 0.15 - 0.15cos120^0 = 0.225 m

Competence statements r 1, 2, 3, 4, 5, 6, 7, 8

Summary M3 Topic 1: **Circular Motion – 3**

References: Chapter 1 Pages 36-37

When vertical circular motion breaks down (i.e. when the particle may leave the circular path)

- a particle on a light, inextensible string; no air resistance

References: Chapter 1 Page 37

N2L towards the centre gives

$T + mg\cos\theta = mr\omega^2$

The particle will remain moving in the circle if $T \geq 0$

$\Rightarrow \cos\theta \leq \frac{r\omega^2}{g}$

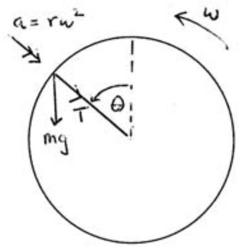


Fig. 1

If this is so when $\cos\theta = 1$

(i.e at the top, with the maximum value for $\cos\theta$)

then $r\omega^2 \geq g$ or $v^2 \geq rg$ where v is the least speed and ω is the least angular velocity.

References: Chapter 1 Page 38

-motion on the inside of a vertical circle; no friction

N2L towards the centre gives

$\text{R} + mg\cos\theta = mr\omega^2$

The particle will remain moving in the circle if $\text{R} \geq 0$

We require $\cos\theta \leq \frac{r\omega^2}{g}$

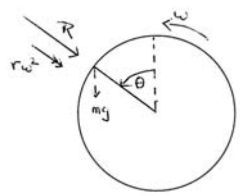


Fig. 2

If this is so when $\cos\theta = 1$

(i.e. at the top, with the maxiumum value for cos θ)

then $r\omega^2 \geq g$ or $v^2 \geq rg$ where v is the least speed and ω is the least angular velocity.

References: Chapter 1 Page 39

Example 1.10 Page 40

-motion on the outside of a circle; no friction

N2L towards the centre gives

$mg\cos\theta - \text{R} = mr\omega^2$

The particle will remain on the surface of the circle if $R \geq 0$

We require $\cos\theta \geq \frac{r\omega^2}{g}$

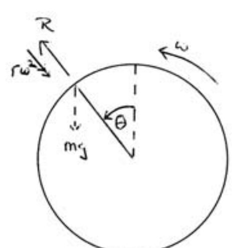


Fig. 3

or $\frac{v^2}{rg} \leq \cos\theta$ where v is the speed when the angle is θ.

Exercise 1D Q. 5, 7

E.g. A particle is attached to the end of a light inextensible string of length r and swung in a vertical circle with no air resistance. Find the least speed at the bottom of the circle for the particle to complete circular motion.
Using the notation of Fig. 1 and the work opposite;
If the speed at the bottom is u and that at the top is v then the conservation of energy gives

$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + 2mgr$ so $v^2 = u^2 - 4rg$

Using the result that $v^2 \geq rg$ derived opposite

we require $u^2 - 4rg \geq rg \Rightarrow u^2 \geq 5rg$

E.g. A particle is projected up the inside of a smooth sphere of radius r. Find the least angular speed at the bottom of the circle for the particle to complete circular motion.
Using the notation of Fig. 2 and the work opposite;
If the angular speed at the bottom is Ω and that at the top is ω then the conservation of energy gives

$\frac{1}{2}mr^2\Omega^2 = \frac{1}{2}mr^2\omega^2 + 2mgr$ so $\omega^2 = \Omega^2 - \frac{4g}{r}$

Using the result that $r\omega^2 \geq g$ derived opposite

we require $r\left(\Omega^2 - \frac{4g}{r}\right) \geq g \Rightarrow \Omega^2 \geq \frac{5g}{r}$

Note that the solution is very similar to that above. All that has been done is to replace u by $r\Omega$ and v with $r\omega$.

E.g. A particle of mass 0.05 kg is gently displaced from rest from the top of a solid smooth sphere of radius 0.5 m. Find where it leaves the surface.
Refer to Fig 3.

N2L towards the centre gives $mg\cos\theta - R = 0.05 \times 0.5 \times \omega^2$

If θ_1 is the angle where the particle leaves the surface

then $R = 0$ and at this point $0.05g\cos\theta_1 = 0.05 \times 0.5 \times \omega^2$

so $\omega^2 = 2g\cos\theta_1$

Loss of GPE is $mgh = mgr(1 - \cos\theta_1)$

Gain in KE is $\frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2$

Conserving energy gives $\frac{1}{2}mr^2\omega^2 = mgr(1 - \cos\theta_1)$

so $\omega^2 = \frac{2gr(1-\cos\theta_1)}{r^2} = 4g(1 - \cos\theta_1)$

But $\omega^2 = 2g\cos\theta_1$ so $4g(1 - \cos\theta_1) = 2g\cos\theta_1$

$\Rightarrow 6\cos\theta_1 = 4 \Rightarrow \cos\theta = \frac{2}{3}$

$\Rightarrow \theta = 48.2^0$ (3 s.f.)

Mechanics 3
Version B: page 4
Competence statements r 1, 2, 3, 4, 5, 6, 7, 8

References:
Chapter 2
Pages 51-56

Example 2.1
Page 55

Strings and springs
We sometimes model a string to be **inextensible**. This is not always the case: strings and springs which stretch are said to be **elastic**.
The length when there is no force attached to it is called the **natural length**. When stretched the increase in length is called the **extension**.
A spring may be **compressed**. A string cannot be compressed.

E.g. A piece of light string is 50 cm long. Experiments have indicated that $\lambda = 80$ N. A mass of 5 kg is hung from the string. Find the extension.

$$T = 5g = \frac{\lambda}{l_0}x \Rightarrow x = \frac{5g \times 0.5}{80} = 0.3062$$

Extension is 30.6 cm (3 s.f.).

Exercise 2A
Q. 2, 4

Example 2.3
Page 58

Exercise 2B
Q. 7

Hooke's Law
The tension in an elastic spring is proportional to the extension. If a spring is compressed the thrust is proportional to the decrease in length of the spring.
If the natural length is l_0 and the extension is x, resulting from a force T, then

$$T = kx.$$

where k is the **stiffness**.

Or we write $T = \frac{\lambda}{l_0}x$

where λ is called the **modulus of elasticity.**

E.g. A light spring of natural length 50 cm is attached to a ceiling. When a mass of 4 g is hung in equilibrium, the extension is 5 cm. Find
(i) the tension in the spring,
(ii) the stiffness of the spring,
(iii) the modulus of elasticity of the spring.
Take $g = 10$ m s^{-2}.

(i) $T = mg = 0.004g = 0.04$ N
(ii) $T = kx \Rightarrow 0.04 = 0.05k \Rightarrow k = 0.8$ N m^{-1}
(iii) $T = \frac{\lambda}{x_0}x \Rightarrow \lambda = T\frac{x_0}{x} = \frac{50}{5} \times 0.04 = 0.4\,\text{N}$

E.g. Find the EPE stored in the example above when the system is in equilibrium..

$$\frac{1}{2}kx^2 = \frac{1}{2} \times 0.8 \times 0.05^2 = 1 \times 10^{-3} = 0.001\ \text{J}$$

References:
Chapter 2
Pages 66-70

Example 2.5
Page 68

Exercise 2C
Q. 4

Work and Energy
The total work done in stretching a string by x from its natural length l_0 is given by

$$\text{Work done} = \frac{1}{2}kx^2 = \frac{\lambda}{2l_0}x^2.$$

This is known as the **elastic potential energy.** (E.P.E.)

E.g. Suppose, in the example above, the mass is pulled down a further 10 cm and released from rest. Does the string go slack?

Total extension is 15 cm. EPE stored is $\frac{1}{2}0.8 \times 0.15^2 = 0.009$ J

GPE gained by mass in being raised 15 cm is

$$0.004 \times 10 \times 0.15 = 0.006\ \text{J}$$

As EPE >GPE the mass is still moving upwards when the string goes slack. Its speed is given by

$$\frac{1}{2} \times 0.004 \times v^2 = 0.009 - 0.006$$

so $v^2 = 1.5$; speed is 1.22 m s^{-1} (3 s.f.)

References:
Chapter 2
Pages 74-75

The Principle of Conservation of Energy
The principle of conservation of energy may be used when the energy changes only involve (some of) KE, GPE and EPE.

A light buffer of natural length 50 cm has a modulus of elasticity 100 kN.
(i) What force will compress it to half its length?
(ii) How much energy is stored in the buffer when it is compressed to 20 cm.

(i) $T = \frac{\lambda}{l}x = \frac{100000 \times 0.25}{0.5} = 50\,\text{kN}$

(ii) EPE $= \frac{1}{2 \times 0.5}10^5\left(0.5 - 0.2\right)^2 = 9000\,\text{J}$

Example 2.6
Page 74

Exercise 2D
Q. 4

Vertical motion
When a particle attached to a spring is released from a point other than the equilibrium position then the principles of conservation of energy may be applied.
Conservation of energy gives:
Increase in KE + increase in GPE + increase in EPE = 0
Care must be taken when an elastic string is raised above its equilibrium position as it will go slack rather than compress.

E.g. A light spring of stiffness 50 N m^{-1} can project a 50 g stone 5 m vertically. How much compression in the spring is required?

$$\frac{1}{2}kx^2 = mgh \Rightarrow x = \sqrt{\frac{2mgh}{k}}$$

$$= \sqrt{\frac{2 \times 0.05 \times 9.8 \times 5}{50}} = 0.313\ \text{m}$$

References:
Chapter 3
Pages 82-87

Example 3.1
Page 87

Exercise 3A
Q. 3

Oscillating motion

- Suppose a particle oscillates about a central position, O, on the x-axis.
- The particle moves between two points, $x = -a$ and $x = +a$. The distance a is called the **amplitude** of the motion.
- The motion repeats itself in a cyclic fashion. The number of cycles per second is called the **frequency**, denoted by ν.
- The motion repeats itself after time T. The time interval, T, is called the **period**: it is the time for a complete cycle of the motion.

The frequency is the reciprocal of the period.

So $\nu = \dfrac{1}{T}$

The S.I. unit for frequency is the Hertz. 1 Hertz is one cycle per second.

E.g. The height, h m, of water in a harbour may be modelled over a short time by simple harmonic motion, with period 12 hours . At high water the depth of water is 6 m greater than the depth at low water.

$\omega = \dfrac{2\pi}{T} = \dfrac{\pi}{6}$ rads per hour.

Amplitude of oscillation = 3 m.

Maximum speed = $a\omega = \dfrac{\pi}{2}$ metres per hour.

$= \dfrac{100\pi}{60\times 2} \approx 2.62$ cm per minute (3.s.f.).

Assuming $h = 0$ when $t = 0$, $h = 3\sin\left(\dfrac{\pi}{6}t\right)$

Checking gives $\dot{h} = \dfrac{\pi}{2}\cos\left(\dfrac{\pi}{6}t\right)$, $\ddot{h} = -\dfrac{\pi^2}{12}\sin\left(\dfrac{\pi}{6}t\right)$

$\Rightarrow \ddot{h} = -\left(\dfrac{\pi}{6}\right)^2 h = -\omega^2 h$ as required.

Simple Harmonic Motion

The acceleration is proportional to the magnitude of the displacement from the centre point of the motion and is directed towards this centre point.

If x is the displacement the SHM equation is

$\ddot{x} = -\omega^2 x$ which gives $v^2 = \omega^2(a^2 - x^2)$

In addition:

$\ddot{x}$ is directed towards O, so the force causing the action must also be directed towards O.

$\ddot{x}$ (and therefore the force) = 0 when $x = 0$.

$|\ddot{x}|$ has a maximum value at the extremes $x = \pm a$.

The maximum velocity is when $x = 0$ and is $\pm a\omega$.

The period, $T = \dfrac{2\pi}{\omega}$.

E.g. An oscillating weight on the end of a light spring completes 48 cycles per minute. The amplitude is estimated as 5 cm. If the oscillation begins at the equilibrium point, write an equation of motion and find the greatest speed.

Let x be the displacement from the equilibrium point.

$\nu = 48$ cpm $= \dfrac{48}{60}$ Hz $\Rightarrow T = \dfrac{60}{48}$

$\omega = \dfrac{2\pi}{T} = \dfrac{2\pi\times 48}{60} = \dfrac{8\pi}{5}\ (= 5.0265)$

$x = 0.05\sin\dfrac{2\pi\times 48}{60}t = 0.05\sin 1.6\pi t$

Max $v = a\omega = 0.251$ m s^{-1} (3.s.f.)

Equation of motion: $\ddot{x} = -\omega^2 x = -25.27x$ (4 s.f.)

References:
Chapter 3
Pages 89-95

Example 3.2
Page 89

Exercise 3B
Q. 4

SHM as a function of time.

Any motion of the form $x = a\sin\omega t$ is a solution of the SHM equation since

$x = a\sin\omega t \Rightarrow v = \dot{x} = a\omega\cos\omega t$

$\Rightarrow \ddot{x} = -a\omega^2\sin\omega t = -\omega^2 x$

and so is a solution of the SHM equation

The solution above is a particular solution of the more general form

$$x = C\sin(\omega t + \varepsilon)$$

(See next page for a list of expressions for SHM.)

Also, given $v = \dot{x} = a\omega\cos\omega t$, $v^2 = a^2\omega^2\cos^2\omega t$

$\Rightarrow v^2 = a^2\left(1 - \sin^2\omega t\right) = a^2 - x^2$

So $v^2 = a^2 - x^2$

E.g. If $t = 0$ at the middle of the motion then when $t = 0$, $x = 0$.

Using $x = C\sin(\omega t + \varepsilon)$ we have $0 = a\sin\varepsilon \Rightarrow \varepsilon = 0$

giving $x = a\sin\omega t$

If $t = 0$ at one end of the motion then when $t = 0$, $x = \pm a$.

Again, using $x = C\sin(\omega t + \varepsilon)$

Taking $x = a$ when $t = 0$ $\Rightarrow a = a\sin\varepsilon \Rightarrow \varepsilon = \dfrac{\pi}{2}$

giving $x = a\sin\left(\omega t + \dfrac{\pi}{2}\right) = a\cos\omega t$

Taking $x = -a$ when $t = 0 \Rightarrow x = -a\cos\omega t$

References:
Chapter 3
Pages 99-105

Exercise 3C
Q. 2, 11

Alternative forms of the solution of the SHM equation

Motion in a straight line of the form

$x = A\sin\omega t + B\cos\omega t$

or $x = C\sin(\omega t + \varepsilon)$

or $x = C\cos(\omega t + \varepsilon')$

satisfy the equation $\ddot{x} = -\omega^2 x$

and the motion is SHM.

References:
Chapter 3
Pages 107-109

Example 3.6
Page 107

Exercise 3C
Q. 8

Circular motion and SHM

If a particle is moving at constant angular speed ω round a circle of radius a whose centre is O and the x-axis is a diameter, then the x coordinate of its position is given by

$x = a\cos\theta$ where $\theta = \omega t$

and ω is the angular speed.

$\Rightarrow x = a\cos\omega t$ and $\dot{x} = -a\omega\sin\omega t$

So $\ddot{x} = -a\omega^2\cos\omega t = -\omega^2 x$

Use $x = a\sin\omega t$ when $t = 0$ at the centre of motion.

Use $x = a\cos\omega t$ when $t = 0$ at the extreme of the motion in the positive direction.

References:
Chapter 3
Pages 115-119

Example 3.7
Page 117

Exercise 3D
Q. 3

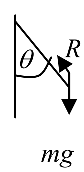
The simple pendulum

In tangential direction: $ml\ddot{\theta} = -mg\sin\theta$

For small $\theta, \sin\theta \approx \theta$ giving $\ddot{\theta} \approx -\dfrac{g\theta}{l}$

This equation of motion is the SHM equation.

The solution has period $T = 2\pi\sqrt{\dfrac{l}{g}}$.



The solution of the equation may be written as $\theta = \alpha\sin\left(\dfrac{g}{l}t + \varepsilon\right)$ where α is the maximum amplitude. Note that the equation for motion is the angular displacement of the pendulum.

References:
Chapter 3
Pages 119-124

Exercise 3D
Q. 3

Oscillating Springs

If a particle is suspended from a perfectly elastic light spring of natural length l_0, and modulus of elasticity λ, then the equation of motion of oscillation is given by

$$\ddot{x} = -\frac{\lambda}{ml_0}x$$

where x is the displacement from the equilibrium position. This is the standard equation for SHM with

$\omega^2 = \dfrac{\lambda}{ml_0}$; This gives a period of $2\pi\sqrt{\dfrac{ml_0}{\lambda}}$

Consider the motion of a particle in a straight line where the displacement, x, is given by

$x = 2\sin\left(\dfrac{\pi}{4}t\right) + 3\cos\left(\dfrac{\pi}{4}t\right)$.

$\dot{x} = \dfrac{\pi}{2}\cos\left(\dfrac{\pi}{4}t\right) - \dfrac{3\pi}{4}\sin\left(\dfrac{\pi}{4}t\right)$

$\ddot{x} = -\dfrac{\pi^2}{8}\sin\left(\dfrac{\pi}{4}t\right) - \dfrac{3\pi^2}{16}\cos\left(\dfrac{\pi}{4}t\right)$

$= -\dfrac{\pi^2}{16}\left(2\sin\left(\dfrac{\pi}{4}t\right) + 3\cos\left(\dfrac{\pi}{4}t\right)\right) = -\dfrac{\pi^2}{16}x$

So the motion is SHM.

Also $x = 2\sin\left(\dfrac{\pi}{4}t\right) + 3\cos\left(\dfrac{\pi}{4}t\right)$

$\Rightarrow x = \lambda\left(\sin\left(\dfrac{\pi}{4}t\right)\cos\varepsilon + \cos\left(\dfrac{\pi}{4}t\right)\sin\varepsilon\right)$

where $\lambda\cos\varepsilon = 2$ and $\lambda\sin\varepsilon = 3$

$\Rightarrow \lambda^2 = 13$ and $\cos\varepsilon = \dfrac{2}{\sqrt{13}} \Rightarrow \varepsilon = 0.983$ (3 s.f.)

$\Rightarrow x = \sqrt{13}\sin\left(\dfrac{\pi}{4}t + 0.983\right)$ (3 s.f.)

E.g. A simple pendulum is in the form of a light rod of length 1.2 m with a heavy particle at the end. It is pulled to one side through an angle of 20^0 from the vertical and then released from rest. Find
(i) the period,
(ii) the time to reach the lowest point,
(iii) the angular speed at the lowest point.
(Take g = 9.8 m s^{-2}.)

(i) $\omega^2 = \dfrac{g}{l} = 8.166... \Rightarrow \omega = 2.858...$

$\Rightarrow$ Period $= \dfrac{2\pi}{\omega} \approx 2.20$ s (3 s.f.)

(ii) Motion is modelled by $\theta = \alpha\cos\omega t$

where $\alpha = \dfrac{20\times\pi}{180} \approx 0.349...$ rad

At lowest point $\theta = 0 \Rightarrow \cos\omega t = 0 \Rightarrow t = \dfrac{\pi}{2\omega} \approx 0.5496...$ sec.

(iii) Speed $= a\omega = 0.3490\times 2.85773 = 0.9975 \approx 1$ rad s^{-1}.

E.g. How long should be a simple pendulum for a 1 second "tick".
2 second period so $\dfrac{T^2 g}{4\pi^2} = l = \dfrac{4g}{4\pi^2} \approx 0.993$ m

Volumes of revolution and centres of mass by integration

References:
Chapter 4
Pages 132-138

Example 4.1
Page 135

Exercise 4A
Q. 1(v),4

Volume of revolution about the x-axis

If the curve $y = \mathrm{f}(x)$ between the ordinates $x = a$ and $x = b$ is rotated through 360^0 about the x-axis then the volume of the resulting solid is given by:

$$V = \pi\int_a^b y^2 \; \mathrm{d}x$$

Note that y must be replaced by $\mathrm{f}(x)$ before integrating.

Volume of revolution about the y-axis

If the curve $x = \mathrm{f}(y)$ between $y = c$ and $y = d$ is rotated through 360^0 about the y-axis then the volume of the resulting solid is given by:

$$V = \pi\int_c^d x^2 \; \mathrm{d}y$$

Note that x must be replaced by $\mathrm{f}(y)$ before integrating.

E.g.1. The line $y = x + 1$ from $x = 0$ to $x = 2$ is rotated through 360^0 about the x-axis. Find the volume of the solid formed.

$$V = \pi\int_0^2 y^2 \,\mathrm{d}x = \pi\int_0^2 (x+1)^2 \,\mathrm{d}x = \pi\int_0^2 \left(x^2+2x+1\right)\mathrm{d}x$$

$$= \pi\left[\frac{x^3}{3}+x^2+x\right]_0^2 = \pi\left(\frac{8}{3}+4+2\right) = \frac{26\pi}{3}$$

E.g.2. The curve $y = x^2 - 1$ from $y = 1$ to $y = 3$ is rotated through 360^0 about the y-axis. Find the volume of the solid formed.

$$V = \pi\int_1^3 x^2 \,\mathrm{d}y = \pi\int_1^3 (y+1)\,\mathrm{d}y$$

$$= \pi\left[\frac{y^2}{2}+y\right]_1^3 = \pi\left(\frac{9}{2}+3\right) - \pi\left(\frac{1}{2}+1\right) = 6\pi$$

References:
Chapter 4
Pages 144-151

Example 4.5
Page 149

Exercise 4B
Q. 3, 6

Centres of mass (c.o.m)

For a volume of revolution about the x-axis:

$$\bar{x}\int_a^b y^2 \,\mathrm{d}x = \int_a^b xy^2 \,\mathrm{d}x, \quad \bar{y} = 0$$

For a volume of revolution about the y-axis:

$$\bar{y}\int_c^d x^2 \,\mathrm{d}y = \int_c^d yx^2 \,\mathrm{d}y, \quad \bar{x} = 0.$$

Centres of mass of standard shapes

Solid shape	Volume	Height of c.o.m. above plane base
Solid hemisphere, radius r	$\frac{2}{3}\pi r^3$	$\frac{3}{8}r$
Solid cone, height h, radius r	$\frac{1}{3}\pi r^2 h$	$\frac{1}{4}h$

Hollow shape	Curved surface area	Height of c.o.m. above base
Hollow hemisphere, radius r	$2\pi r^2$	$\frac{1}{2}r$
Hollow cone, height h, radius r	πrl	$\frac{1}{3}h$

Composite bodies may be made by combining simpler shapes in some way.

References:
Chapter 4
Pages 156-159

Exercise 4B
Q. 8

Centres of mass of plane regions

For the lamina which is the area between the curve $y = \mathrm{f}(x)$, the x-axis, $x = a$ and $x = b$.

$$A\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\int_a^b xy \,\mathrm{d}x\\ \int_a^b \frac{y^2}{2}\,\mathrm{d}x\end{pmatrix} \quad \text{where } A = \int_a^b y \,\mathrm{d}x$$

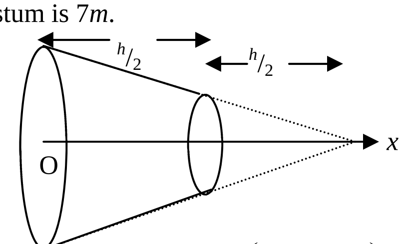
References:
Chapter 4
Page 160

Exercise 4C
Q. 4

For the lamina which is the area between the curve $x = \mathrm{f}(y)$, the y-axis, $y = c$ and $y = d$.

$$A\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\int_c^d \frac{x^2}{2}\,\mathrm{d}y\\ \int_c^d xy \,\mathrm{d}y\end{pmatrix} \quad \text{where } A = \int_c^d x \,\mathrm{d}y$$

E.g. The c.o.m. of a frustum which is the lower half of a solid cone, height h.
The height of the frustum is $^h/_2$.
Let the mass of cone be $8m$.
Then mass of cone cut off is m and mass of frustum is $7m$.



$$8m\frac{h}{4} = 7m\bar{x} + m\left(\frac{h}{2}+\frac{h}{2}\times\frac{1}{4}\right)$$

$$\Rightarrow 2h = 7\bar{x} + \frac{5}{8}h \Rightarrow 7\bar{x} = \frac{11}{8}h \Rightarrow \bar{x} = \frac{11}{56}h$$

References:
Chapter 5
Pages 168-171

The three fundamental dimensions

Mass	Length	Time
M	L	T

E.g. KE = $\frac{1}{2}mv^2$

$\Rightarrow [KE] = \left[\frac{1}{2}mv^2\right] = \mathrm{M}\left(\mathrm{LT}^{-1}\right)^2 = \mathrm{ML}^2\mathrm{T}^{-2}$

Example 5.1
Page 170

Other dimensions

[Area] means the dimensions of area which is L^2.

[Area] = L^2 [Volume] = L^3

[Speed] = LT^{-1}

[Acceleration] = LT^{-2}

[Force] = MLT^{-2}

In any equation or formula with units there must be dimensional consistency.

E.g. $v = u + at$

$[v] = [u + at]$

Dimensions of L.H.S. are LT^{-1}

Dimensions of R.H.S. are LT^{-1} and $\mathrm{LT}^{-2}.\mathrm{T} = \mathrm{LT}^{-1}$

So the formula has dimensional consistency.

References:
Chapter 5
Page 171

Dimensionless Quantities

All numbers, including π and e, angles and trigonometrical ratios have no dimensions.

Exercise 5A

E.g. Justify the dimensional consistency of the formula $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mgh\sin\theta$

where the variables have the usual meaning.

$\left[\frac{1}{2}mv^2\right] = \left[\frac{1}{2}mu^2\right] = \mathrm{M}.\left(\frac{\mathrm{L}}{\mathrm{T}}\right)^2 = \frac{\mathrm{ML}^2}{\mathrm{T}^2}$

$[mgh\sin\theta] = \mathrm{M}.\frac{\mathrm{L}}{\mathrm{T}^2}.\mathrm{L} = \frac{\mathrm{ML}^2}{\mathrm{T}^2}$

References:
Chapter 5
Pages 172

Using dimensions to check relationships

When all quantities are given in algebraic form, dimensional consistency can be used to check answers to problems.

Example 5.3
Page 173

The dimensions of a quantity help to determine which units are appropriate.

E.g. Given that pressure = force per unit area, find the dimensions of pressure.

$\text{Pressure} = \frac{\text{Force}}{\text{Area}} \Rightarrow [\text{Pressure}] = \frac{[\text{Force}]}{[\text{Area}]}$

$= \frac{\mathrm{MLT}^{-2}}{\mathrm{L}^2} = \mathrm{ML}^{-1}\mathrm{T}^{-2}$

Exercise 5B
Q. 1

References:
Chapter 5
Pages 174-175

Finding the form of a relationship

It is sometimes possible to determine the form of a relationship just by looking at the dimensions of the quantities

Example 5.5
Page 175

1. There are three fundamental quantities - Mass, Length and Time.
2. Modelling assumptions are made in the usual way.
3. The method can only be used when a quantity can be written as a product of powers of other quantities. There are restrictions on the number of quantities involved.

Exercise 5B
Q. 2

E.g. the frequency, *f*, of the note emitted by an organ pipe of length *a* when the air pressure is *p* and the air density is *d* is believed to obey a law of the form

$f = ka^\alpha p^\beta d^\gamma$

where *k* is a dimensionless constant.

Find the values of α, β and γ.

$f = ka^\alpha p^\beta d^\gamma \Rightarrow [f] = [a^\alpha p^\beta d^\gamma] = [a^\alpha][p^\beta][d^\gamma]$

$\mathrm{T}^{-1} = \mathrm{L}^\alpha\left(\mathrm{ML}^{-1}\mathrm{T}^{-2}\right)^\beta\left(\mathrm{ML}^{-3}\right)^\gamma$

Equating:

For T: $-1 = -2\beta$

For L: $0 = \alpha - \beta - 3\gamma$

For M: $0 = \beta + \gamma$

Solving gives $\beta = \frac{1}{2}$, $\gamma = -\frac{1}{2}$, $\alpha = -1$

$\Rightarrow f = ka^{-1}p^{1/2}d^{-1/2} = \frac{k}{a}\sqrt{\frac{p}{d}}.$