AQA

ASSESSMENT and QUALIFICATIONS ALLIANCE

Not Formulae and Statistical Tables for GCE Mathematics and GCE Statistics

The Not-Formula Book

Mechanics 2

Everything you need to remember that the formula book won't tell you

The Not-Formula Book for M2

Everything you need to know for Mechanics 2 that *won't* be in the formula book

Examination Board: AQA

Brief

This document is intended as an aid for revision. Although it includes some examples and explanation, it is primarily not for learning content, but for becoming familiar with the requirements of the course as regards formulae and results. It cannot replace the use of a text book, and nothing produces competence and familiarity with mathematical techniques like practice. This document was produced as an addition to classroom teaching and textbook questions, to provide a summary of key points and, in particular, any formulae or results you are expected to know and use in this module.

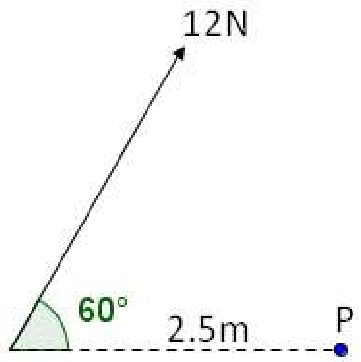
Contents

Chapter 1 – Moments and equilibrium

The **moment** of a force about a point is defined to be the **magnitude of the force** multiplied by the **perpendicular distance** from the point to the line of action of the force.

This can be thought of equivalently as the distance to the force multiplied by the perpendicular component of the force.

Eg:



The moment of the $12N$ force acting on point P has a magnitude equal to:

$$2.5 \times 12sin60 = 25.98Nm\ to\ 2d.p.$$

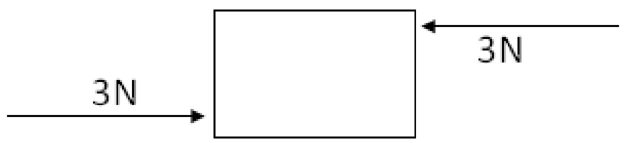
A moment can have either a **clockwise** or an **anticlockwise** direction.
Anticlockwise is defined as **positive**, and **clockwise** as **negative**.

Eg: For the example above, while the *magnitude* of the moment is $25.98Nm$, the actual value would be $-25.98Nm$.

This allows us to sum moments in order to calculate a resultant turning force in one direction or another.

A *particle* is in equilibrium if the **resultant force** acting on it is **zero**.
A *rigid body* is in equilibrium if both the **resultant force** and the **moment** are **zero**.

Eg:



In this diagram, the resultant force is zero, since the only forces acting are balanced. However, the body is not in equilibrium because the moment is not zero.

Chapter 2 – Centres of mass

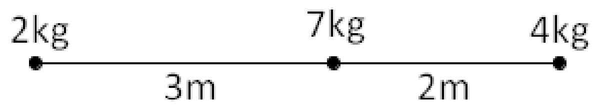
The **centre of mass** of a system is the point at which the system would be **perfectly balanced**. That is, the resultant moment about this point is equal to zero.

The position of the centre of mass can be calculated using the following formula:

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

This will give the distance of the centre of mass from any particular point by dividing the sum of the moments about that point by the total mass of the system.

Eg.



$$\bar{x} = \frac{2 \times 0 + 7 \times 3 + 4 \times 5}{2 + 7 + 4} = \frac{41}{13} = 3.15 \; to \; 2d.p.$$

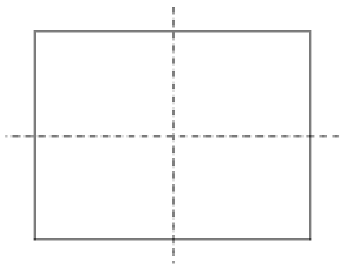
So the centre of mass is 3.15m from the left.

When a body is suspended (with no force other than weight acting on it), it will hang so that the centre of mass is directly below the point of suspension.

This means that, given the properties of a particular body and a particular point it is to be suspended from, the angle between, for instance, the side of the body and the vertical, can be calculated once the centre of mass is known.

Uniform lamina with one or more lines of symmetry will always have their centre of mass lie on all lines of symmetry.

Eg:



The centre of mass of a rectangular lamina with width x and height y will be at $\left(\frac{x}{2}, \frac{y}{2}\right)$

Note: The formula book includes a list of centre of mass formulae for different 2D and 3D shapes.

Chapter 3 – Energy

Kinetic Energy is the energy possessed by any **moving body**. It is proportional to the mass and proportional to the square of the velocity:

$$KE = \frac{1}{2}mv^2$$

Eg. The kinetic energy of a $7500kg$ lorry travelling at $20ms^{-1}$ would be:

$$KE = \frac{1}{2} \times 7500 \times 20^2 = 1500000\, Joules = \mathbf{1.5\ MegaJoules}$$

Work Done is the change in kinetic energy of a body, brought about by a force.
It can be calculated using this formula:

$$Work\ Done = Fs$$

Where F is the force acting to produce the change, and s is the distance moved in the direction of the force.

Note: The amount of kinetic energy a body possesses can be thought of as the amount of work the body could do, due to being in motion, before coming to rest.

Gravitational Potential Energy is defined as the work done by the force of gravity. Since the force acting is weight, which is equal to mg, and the distance travelled in the direction of the force is commonly referred to as height, or h, the formula for Gravitational Potential Energy is:

$$GPE = mgh$$

Note: The amount of gravitational potential energy a body possesses can be thought of as the amount of work the body could do, due to being raised a certain height from a defined base point (usually the earth's surface), before this height is reduced to zero.

Note that the work done by gravity will be the same for both a free-falling body and a body sliding down a smooth slope, provided they have the same mass and drop the same distance.

Eg:
A 500g ball is thrown upwards from an initial height of $2m$ at an initial velocity of $5ms^{-1}$. Calculate the total energy of the ball and, assuming negligible resistive forces, the maximum height reached.

$$KE + GPE = \frac{1}{2} \times 0.5 \times 5^2 + 0.5 \times 9.8 \times 2 = 6.25 + 9.8 = \mathbf{16.05J}$$
$$At \max height, kinetic\ energy\ is\ 0\ since\ v = 0, so\ total\ energy = GPE$$
$$16.05 = (0.5 \times 9.8)h \Longrightarrow \boldsymbol{h = 3.28m\ to\ 3s.f.}$$

Hooke's Law states that the **tension in a spring**, T, is directly proportional to the **extension of the spring**, x, and inversely proportional to the **natural length of the spring**, l. The constant of proportionality, λ, is referred to as the **modulus of elasticity.** The formula is:

$$T = \frac{\lambda x}{l}$$

Note: Hooke's law also applies to a spring in compression – in this case it calculates the **thrust** rather than the **tension** in the spring.

Where the force applied to a body is variable, the work done can be calculated using the following result:

$$Work\ Done = \int_{x_1}^{x_2} F(x).\,dx$$

Where x_1 is the initial position of the body, x_2 is the final position, and $F(x)$ is the force applied as a function of the position, x.

Note: To use this result, the force F must be expressed as a function of the displacement, x.

Since **Work Done is equal to the Change in Kinetic Energy**, we can say:

$$\int_{x_1}^{x_2} F(x)\ dx = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

Where u is the initial velocity and v is the final velocity for a body of mass m.

Using Hooke's law, we can define the force in a spring as $F = \frac{\lambda x}{l}$ and use the integration result to find the energy stored by a spring with an extension e:

$$\int_0^e \frac{\lambda x}{l}\ dx = \left[\frac{\lambda x^2}{2l}\right]_0^e = \frac{\lambda e^2}{2l}$$

Note: This proof of the elastic potential energy formula is a result that must be learned, since it is frequently required in exam questions.

Elastic Potential Energy is the energy possessed by a body attached to a spring (or elastic string, etc). As shown above, the integration result yields this result for Elastic Potential Energy:

$$EPE = \frac{\lambda e^2}{2l}$$

Where e is the extension in the spring, l is the natural length and λ is the modulus of elasticity.

Eg:
A bungee jumper of mass 80kg uses a rope with a modulus of elasticity 3000N and natural length 50m. He is initially at rest. How far below his initial position will he be before being at rest once more?

$$GPE\ converted\ to\ EPE = mgh = (80 \times 9.8)(e + 50) = 784(e + 50)$$

$$EPE = \frac{\lambda e^2}{2l} = \frac{3000e^2}{100} = 30e^2$$

$$784(e + 50) = 30e^2 \Rightarrow 15e^2 - 392e - 19600 = 0$$

$$\Rightarrow e = -25.37\ or\ e = 51.50 \quad so \quad e = 51.50 \Rightarrow \boldsymbol{h = 101.50m}$$

Power is most easily thought of as the **rate of doing work**. This leads to the simple definition:

$$Power = \frac{Work\ Done}{Time\ Taken}$$

Since Work Done can itself be thought of as displacement in the direction of a given force, the rate at which work is done (power) can also be described using the useful formula:

$$Power = Force \times Velocity$$

Eg:
A car of mass $800kg$ experiences resistive forces totalling $8000N$, is currently travelling at $25ms^{-1}$ and accelerating at a rate of $2ms^{-2}$. Calculate its power output.

$$Resultant\ force = Motive\ force - Resistive\ force \quad \Rightarrow \quad F = F_m - 8000$$

$$F = ma \quad \Rightarrow \quad F_m - 8000 = 800 \times 2 = 1600 \quad \Rightarrow \quad F_m = 9600$$

$$Power = Force \times Velocity = 9600 \times 25 = 240000W = 240kW$$

Note: The force in the formula $F = ma$ is the resultant force acting on the vehicle, while the force in the formula $P = Fv$ is the motive force, sometimes F_m - the force exerted, in this case, by the car.

Chapter 4 – Kinematics and variable acceleration

Where acceleration is constant, the standard kinematics equations can be used ('SUVAT equations'), but it is necessary to use calculus when dealing with variable acceleration. If displacement, velocity or acceleration can be given as either functions of time or of each other, we can use differentiation or integration to convert between them.

Velocity is the rate of change of displacement, and **acceleration is the rate of change of velocity**, so we can use:

$$v = \frac{dx}{dt} \qquad a = \frac{dv}{dt} \qquad a = \frac{d^2x}{dt^2}$$

And the equivalent, working backwards:

$$v = \int a\, dt \qquad x = \int v\, dt$$

Note: since information is lost through differentiation, there will be a constant of integration to be accounted for whenever working backwards.

Eg:
A particle travels with a velocity $v = 20t + 3$. Find the displacement at time $t = 5$, given an initial displacement of $12m$.

$$x = \int v\, dt = \int 20t + 3\, dt = 10t^2 + 3t + C$$

$$x_0 = 12 \quad so \quad 12 = 10(0^2) + 3(0) + C \Longrightarrow C = 12$$

$$x = 10t^2 + 3t + 12 \quad and\ at\ t = 5, \quad x = 10(5^2) + 3(5) + 12 = \mathbf{277m}$$

These results can be applied to both 2D and 3D motion simply by extending the results and using vector notation (note, displacement, identified above as x, may be variously denoted s or, more usually when in vector form, r):

$$\boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} \qquad \boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \begin{bmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \\ \frac{d^2z}{dt^2} \end{bmatrix}$$

Note: Equivalent forms can be easily produced for working backwards, by integration.

Chapter 5 – Circular motion

Angular speed is defined as the rate at which an object turns in a circle, in terms of revolutions rather than as a distance travelled. The most common units are rpm (revolutions per second) and $rad\ s^{-1}$ (radians per second).

Angular Speed is denoted by the letter ω (omega), is usually measured in $rad\ s^{-1}$ and the units can be converted using:

$$1\ rpm = \frac{2\pi}{60}\ rad\ s^{-1}$$

The period (time taken for one complete revolution), T, can be calculated from ω using:

$$T = \frac{2\pi}{\omega} \quad or \quad \omega = \frac{2\pi}{T}$$

Angular Speed can be converted into **Velocity** by considering the distance travelled around the arc of a circle with radius r. Velocity will be directed tangentially to the circle, and the formula is:

$$v = r\omega$$

Eg:
A car's wheels are turning at a rate of $1000rpm$. The radius of each wheel is 30cm. How fast is the car moving?

$$1000rpm = 1000 \times 2\pi \div 60\ rad\ s^{-1} = 104.72\ rad\ s^{-1}\ to\ 2d.p. \quad so \quad \omega = 104.72$$

$$r = 0.3 \quad so \quad v = r\omega = 0.3 \times 104.72 = \boldsymbol{31.42ms^{-1}\ to\ 2d.p.}\ (approx. 70mph)$$

It can be shown that the **magnitude of acceleration** of a particle moving with constant circular motion can be calculated using:

$$a = r\omega^2$$

Replacing ω with $\frac{v}{r}$ gives the other useful result:

$$a = \frac{v^2}{r}$$

Note: The **direction** of acceleration is **towards the centre** of the circle. Only the direction, not the magnitude, of velocity is changing. An acceleration is acting at right angles to the direction of motion. Motion would naturally be tangential, so acceleration acts towards the centre.

The **Force** which induces the acceleration towards the centre is also directed towards the centre, and is of magnitude:

$$F = ma \quad \Longrightarrow \quad F = mr\omega^2 \quad or \quad F = \frac{mv^2}{r}$$

Note: When dealing with constant circular motion in a horizontal circle, the resultant of vertical forces will be zero, and the resultant of radial forces can be analysed using the formulae above.

Eg:
A jar of pickle is placed on a 'Lazy Susan' turntable, which is then spun at a rate of $50rpm$. The jar weighs $300g$ and is initially at a constant distance of $10cm$ from the centre of the turntable. Find the least possible value for the coefficient of friction which would prevent the jar from slipping.

$$Vertically: \quad R = 0.3g = 2.94N$$

$$Radially: \quad F = mr\omega^2 = 0.3 \times 0.1 \times (50 \times 2\pi \div 60)^2 = 0.822\ to\ 3s.f.$$

$$F = Fr\ (the\ force\ causing\ circular\ motion\ of\ the\ jar\ is\ friction)$$

$$Fr \leq \mu R \quad \Longrightarrow \quad 0.822 \leq 2.94\mu \quad \Longrightarrow \quad \boldsymbol{\mu \geq 0.280\ to\ 3s.f.}$$

Note: A force is required to induce an object to move through a circular path – in the case of an object being spun on a string, this force is tension in the string; in the case of a planet in orbit, the force is gravitational attraction; in the case of a car driving around a corner, this force is friction between the tyres and the road.

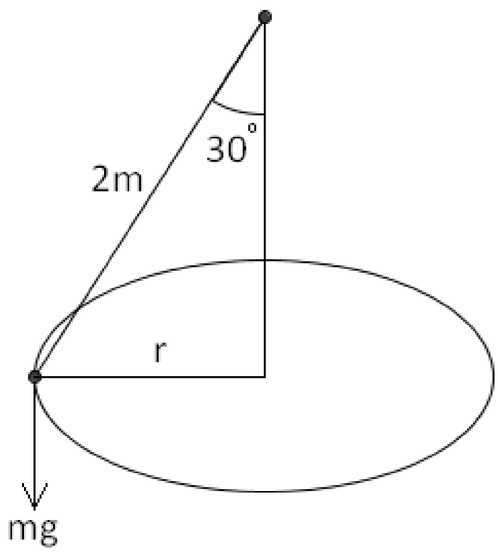
Often horizontal circular motion is not just induced by a straightforward radial force; it may well be induced by a force which has a radial component. This can be dealt with in just the same way.

A force F acting at an angle of $\theta°$ to the vertical can be considered separately as both vertical and radial components:

$$F\cos\theta = mg \quad and \quad F\sin\theta = mr\omega^2 = \frac{mv^2}{r}$$

Note: This means that although the forces involved are affected by the mass of the object, the angle is independent of it, and depends only on the required velocity and radius of the circle.

Eg:
A ball is suspended from a string of length $2m$ angled at 30° from vertical, and moves in a horizontal circle at a constant speed. Find the speed at which the ball travels.



Using right-angled trigonometry:

$$r = 2\sin 30 = 1$$

Resolving vertically:

$$T\cos 30 = mg$$

Resolving horizontally:

$$T\sin 30 = F$$

Applying circular motion formulae:

$$T\sin 30 = \frac{mv^2}{r}$$

$$\frac{T\sin 30}{T\cos 30} = \frac{mv^2}{rmg} = \frac{v^2}{rg}$$

$$\tan 30 = \frac{v^2}{g} \quad \Rightarrow \quad v = \sqrt{g\tan 30} = \boldsymbol{2.38ms^{-1}\ to\ 3s.f.}$$

Chapter 6 – Circular motion with variable speed

When a body moves in a circle with variable speed, while the radial component of acceleration is the same as in the previous section, there is now a tangential component we need to take into account:

The radial component of acceleration is:

$$a = \frac{v^2}{r}$$

The tangential component of acceleration is:

$$\frac{dv}{dt}$$

Eg:
A car accelerates around a corner so that over 4 seconds its speed increases from $10ms^{-1}$ to $20ms^{-1}$. The radius of the curve is $30m$. Calculate the magnitude of acceleration at the end of the 4 seconds.

$$Radial\ acceleration = \frac{v^2}{r} = \frac{20^2}{30} = 13.333 \qquad Tangential\ acceleration = \frac{20-10}{4} = 2.5$$

$$Overall\ acceleration = \sqrt{13.333^2 + 2.5^2} = 13.57ms^{-2}$$

Note: In vertical circular motion, often the only forces acting are the weight (always pulling vertically downwards) and whatever force pulls towards the centre to cause circular motion. This means that **there is no tangential force** as such, but there will be a (variable) component of the weight which acts tangentially.

The easiest way to calculate the speed of a particle at any given point in a vertical circle is to make use of previous results on the **conservation of energy**, **kinetic energy** and **gravitational potential**:

Work done is equal to the change in kinetic energy

$$GPE = mgh \qquad KE = \frac{1}{2}mv^2$$

In addition to this, resolve radially and set the resultant force towards the centre equal to the centripetal force given by:

$$F = \frac{mv^2}{r}$$

Eg:

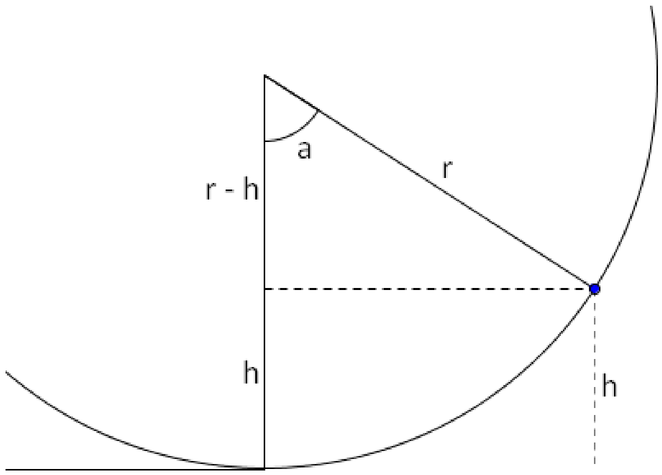
A toy car is travelling along a smooth straight track at a constant speed of $3ms^{-1}$. At a certain point the track describes a vertical circle of radius $80cm$. How far along the circular section of track will the car travel before coming to rest?

$$Initial\ KE = \frac{1}{2}mv^2 = 4.5m \qquad Initial\ GPE = 0$$

$$Final\ KE = 0 \qquad Final\ GPE = mgh$$

$$Since\ the\ track\ is\ smooth, and\ energy\ is\ conserved:$$

$$mgh = 4.5m \quad \Rightarrow \quad gh = 4.5 \quad \Rightarrow \quad h = \frac{4.5}{g} = 0.459m\ to\ 3s.f.$$



$$Since\ we\ know\ r\ and\ h, the\ angle\ at\ the\ centre\ can\ be\ calculated:$$

$$\cos a = \frac{r-h}{r}$$

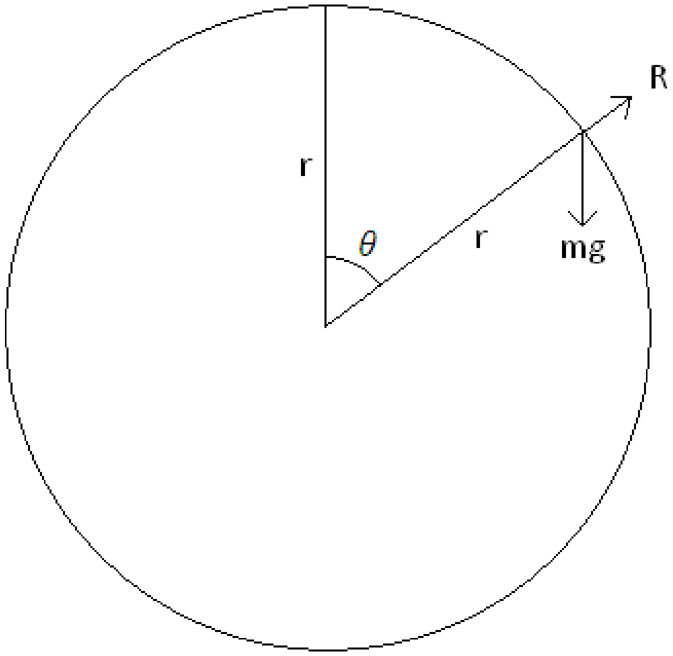
$$a = \cos^{-1}\frac{0.341}{0.8} = 64.8°$$

$$Distance\ travelled\ along\ track:$$

$$\frac{64.8}{360} \times 2\pi(0.8) = \boldsymbol{0.905m\ to\ 3s.f.}$$

Eg:

A ball-bearing is released from rest at the top of a large sphere. By modelling the ball-bearing as a particle, calculate the angle with the vertical at the point when it leaves the sphere.



For the ball-bearing to leave the sphere, the normal reaction must reach 0.

Resolving radially:

$$mg\cos\theta - R = \frac{mv^2}{r} \quad \Rightarrow \quad \boldsymbol{v^2 = gr\cos\theta}$$

Using conservation of energy:

$$mg(2r) = \frac{1}{2}mv^2 + mgr(1+\cos\theta)$$

$$\Rightarrow \quad \boldsymbol{2gr(1-\cos\theta) = v^2}$$

Equating the two expressions for v^2 gives:

$$gr\cos\theta = 2gr(1-\cos\theta) \quad \Rightarrow \quad 3gr\cos\theta = 2gr \quad \Rightarrow \quad \boldsymbol{\cos\theta = \frac{2}{3}}$$

$$\theta = \cos^{-1}\frac{2}{3} = \boldsymbol{48.2^\circ\ to\ 3\ s.f.}$$

Chapter 7 – Application of differential equations in mechanics

If **acceleration** is given as a function of time, this function can be **integrated** to give velocity, and integrated again to give displacement:

$$a = f(t) \quad \Rightarrow \quad v = \int a\,dt + C \quad \Rightarrow \quad x = \int v\,dt + K$$

Note: Since integrating produces an arbitrary constant, it is necessary to have initial conditions to find the specific solution.

If **acceleration** is given as a function of **velocity**, since $a = \frac{dv}{dt}$, this would generate a differential equation in v and t, which can generally be solved using the **separation of variables** method.

Eg:
A particle, of mass m, moves in a straight line on a smooth horizontal surface. As it moves it experiences a resistance force of magnitude kv^2, where k is a constant and v is the speed of the particle, at time t. The particle moves with speed U at time $t = 0$. Find an expression for v.

$$F = ma \quad \Rightarrow \quad -kv^2 = ma$$

This is our differential equation. To see how, we replace a with $\frac{dv}{dt}$*:*

$$m\frac{dv}{dt} = -kv^2$$

$$\Rightarrow \quad \int \frac{m}{v^2}\,dv = \int -k dt \quad \Rightarrow \quad \int mv^{-2}dv = \int -k dt \quad \Rightarrow \quad -mv^{-1} = -kt + C$$

Now we use the initial conditions given to find the value of C:

$$-\frac{m}{U} = -k(0) + C \quad \Rightarrow \quad C = -\frac{m}{U}$$

Substitute into the original solution and rearrange:

$$-\frac{m}{v} = -kt - \frac{m}{U} \quad \Rightarrow \quad -mU = -Uktv - mv \quad \Rightarrow \quad mU = v(Ukt + m)$$

$$\Rightarrow \quad \boldsymbol{v = \frac{mU}{Ukt + m}}$$