OCR Maths FP2

Past Paper Pack

2006-2014

Jan 2006

1 **(i)** Write down and simplify the first three non-zero terms of the Maclaurin series for $\ln(1+3x)$. [3]

(ii) Hence find the first three non-zero terms of the Maclaurin series for

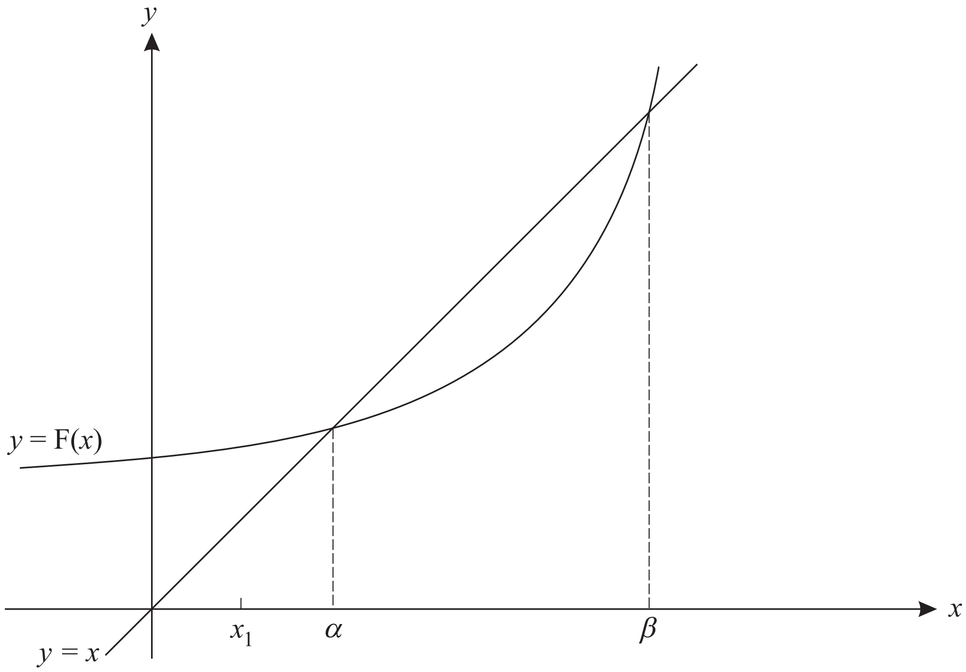
$$e^x \ln(1+3x),$$

simplifying the coefficients. [3]

2 Use the Newton-Raphson method to find the root of the equation $e^{-x} = x$ which is close to $x = 0.5$. Give the root correct to 3 decimal places. [5]

3 Express $\dfrac{x+6}{x(x^2+2)}$ in partial fractions. [5]

4 **Answer the whole of this question on the insert provided.**



The sketch shows the curve with equation $y = \mathrm{F}(x)$ and the line $y = x$. The equation $x = \mathrm{F}(x)$ has roots $x = \alpha$ and $x = \beta$ as shown.

(i) Use the copy of the sketch on the insert to show how an iteration of the form $x_{n+1} = \mathrm{F}(x_n)$, with starting value x_1 such that $0 < x_1 < \alpha$ as shown, converges to the root $x = \alpha$. [3]

(ii) State what happens in the iteration in the following two cases.

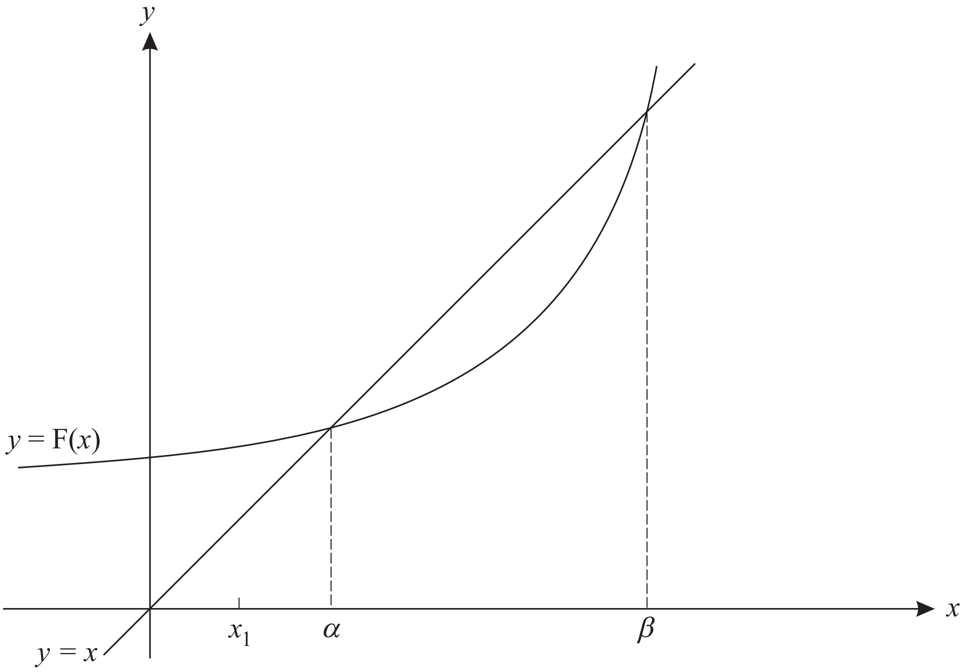
(a) x_1 is chosen such that $\alpha < x_1 < \beta$.

(b) x_1 is chosen such that $x_1 > \beta$.

[3]

Jan 2006

4 **(i)**



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(ii) **(a)** ....

(b) ....

Jan 2006

5 **(i)** Find the equations of the asymptotes of the curve with equation

$$y = \frac{x^2 + 3x + 3}{x + 2}.$$ [3]

(ii) Show that y cannot take values between -3 and 1. [5]

6 **(i)** It is given that, for non-negative integers n,

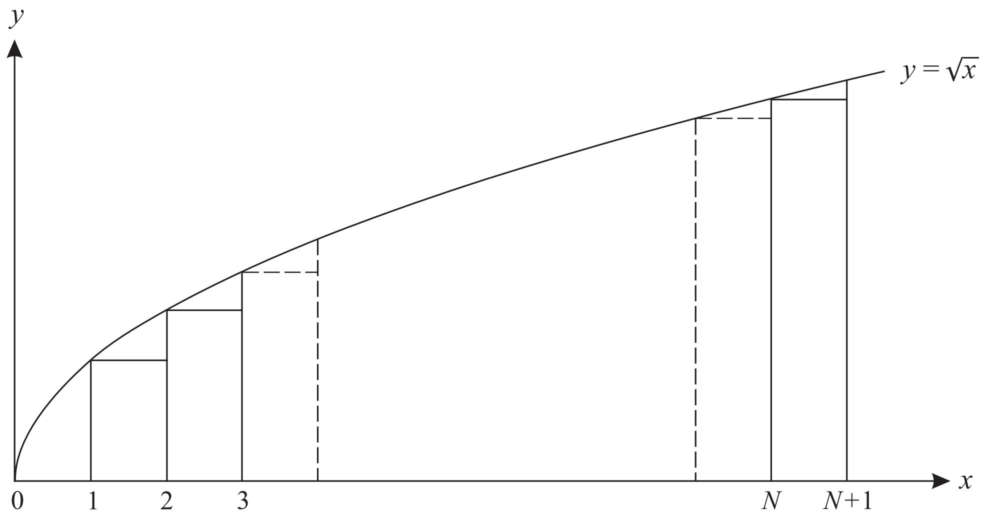
$$I_n = \int_0^1 e^{-x} x^n \, dx.$$

Prove that, for $n \geqslant 1$,

$$I_n = nI_{n-1} - e^{-1}.$$ [4]

(ii) Evaluate I_3, giving the answer in terms of e. [4]

7



The diagram shows the curve with equation $y = \sqrt{x}$. A set of N rectangles of unit width is drawn, starting at $x = 1$ and ending at $x = N + 1$, where N is an integer (see diagram).

(i) By considering the areas of these rectangles, explain why

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{N} < \int_1^{N+1} \sqrt{x} \, dx.$$ [3]

(ii) By considering the areas of another set of rectangles, explain why

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{N} > \int_0^{N} \sqrt{x} \, dx.$$ [3]

(iii) Hence find, in terms of N, limits between which $\sum_{r=1}^{N} \sqrt{r}$ lies. [3]

Jan 2006

8 The equation of a curve, in polar coordinates, is

$$r = 1 + \cos 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

(i) State the greatest value of r and the corresponding values of θ. [2]

(ii) Find the equations of the tangents at the pole. [2]

(iii) Find the exact area enclosed by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{2}\pi$. [5]

(iv) Find, in simplified form, the cartesian equation of the curve. [4]

9 **(i)** Using the definitions of $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}, prove that

$$\sinh 2x = 2 \sinh x \cosh x.$$ [4]

(ii) Show that the curve with equation

$$y = \cosh 2x - 6 \sinh x$$

has just one stationary point, and find its x-coordinate in logarithmic form. Determine the nature of the stationary point. [8]

June 2006

1 Find the first three non-zero terms of the Maclaurin series for

$$(1+x)\sin x,$$

simplifying the coefficients. [3]

2 **(i)** Given that $y = \tan^{-1} x$, prove that $\frac{dy}{dx} = \frac{1}{1+x^2}$. [3]

(ii) Verify that $y = \tan^{-1} x$ satisfies the equation

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0.$$ [3]

3 The equation of a curve is $y = \frac{x+1}{x^2+3}$.

(i) State the equation of the asymptote of the curve. [1]

(ii) Show that $-\frac{1}{6} \leqslant y \leqslant \frac{1}{2}$. [5]

4 **(i)** Using the definition of $\cosh x$ in terms of e^x and e^{-x}, prove that

$$\cosh 2x = 2\cosh^2 x - 1.$$ [3]

(ii) Hence solve the equation

$$\cosh 2x - 7\cosh x = 3,$$

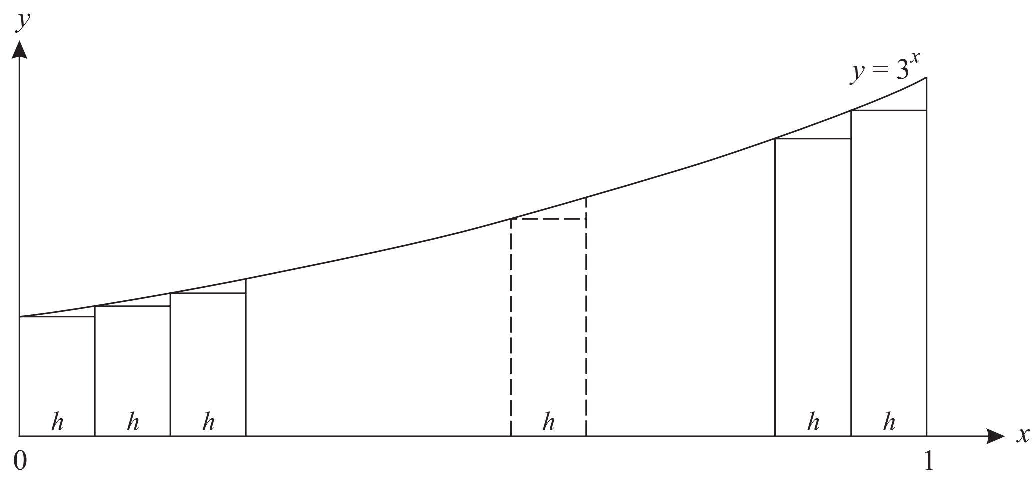
giving your answer in logarithmic form. [4]

5 **(i)** Express $t^2 + t + 1$ in the form $(t+a)^2 + b$. [1]

(ii) By using the substitution $\tan \frac{1}{2}x = t$, show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{2+\sin x}\, dx = \frac{\sqrt{3}}{9}\pi.$$ [6]

6



The diagram shows the curve with equation $y = 3^x$ for $0 \leqslant x \leqslant 1$. The area A under the curve between these limits is divided into n strips, each of width h where $nh = 1$.

(i) By using the set of rectangles indicated on the diagram, show that $A > \dfrac{2h}{3^h - 1}$. [3]

(ii) By considering another set of rectangles, show that $A < \dfrac{(2h)3^h}{3^h - 1}$. [3]

(iii) Given that $h = 0.001$, use these inequalities to find values between which A lies. [2]

7 The equation of a curve, in polar coordinates, is

$$r = \sqrt{3} + \tan\theta, \quad \text{for } -\tfrac{1}{3}\pi \leqslant \theta \leqslant \tfrac{1}{4}\pi.$$

(i) Find the equation of the tangent at the pole. [2]

(ii) State the greatest value of r and the corresponding value of θ. [2]

(iii) Sketch the curve. [2]

(iv) Find the exact area of the region enclosed by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{4}\pi$. [5]

8 The curve with equation $y = \dfrac{\sinh x}{x^2}$, for $x > 0$, has one turning point.

(i) Show that the x-coordinate of the turning point satisfies the equation $x - 2\tanh x = 0$. [3]

(ii) Use the Newton-Raphson method, with a first approximation $x_1 = 2$, to find the next two approximations, x_2 and x_3, to the positive root of $x - 2\tanh x = 0$. [5]

(iii) By considering the approximate errors in x_1 and x_2, estimate the error in x_3. (You are not expected to evaluate x_4.) [3]

[Question 9 is printed overleaf.]

June 2006

9 **(i)** Given that $y = \sinh^{-1} x$, prove that $y = \ln\left(x + \sqrt{x^2 + 1}\right)$. [3]

(ii) It is given that, for non-negative integers n,

$$I_n = \int_0^{\alpha} \sinh^n \theta \, \mathrm{d}\theta,$$

where $\alpha = \sinh^{-1} 1$. Show that

$$nI_n = \sqrt{2} - (n-1)I_{n-2}, \quad \text{for } n \geqslant 2.$$ [6]

(iii) Evaluate I_4, giving your answer in terms of $\sqrt{2}$ and logarithms. [4]

Jan 2007

1 It is given that $\mathrm{f}(x) = \ln(3 + x)$.

(i) Find the exact values of $\mathrm{f}(0)$ and $\mathrm{f}'(0)$, and show that $\mathrm{f}''(0) = -\frac{1}{9}$. [3]

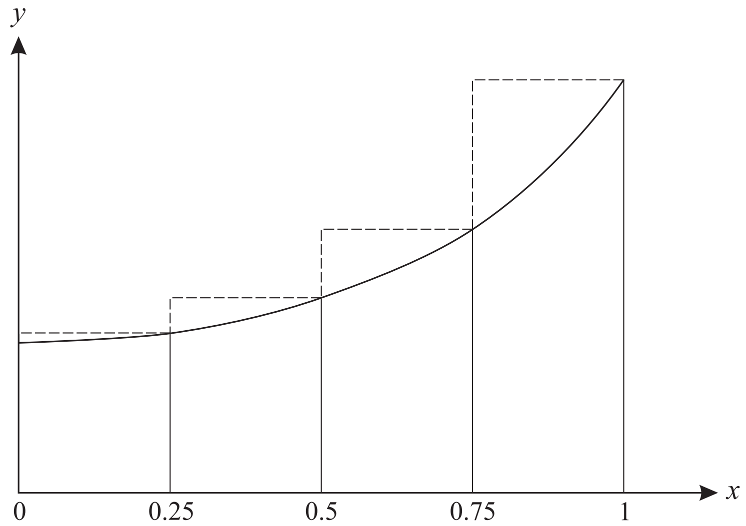
(ii) Hence write down the first three terms of the Maclaurin series for $\mathrm{f}(x)$, given that $-3 < x \leqslant 3$. [2]

2 It is given that $\mathrm{f}(x) = x^2 - \tan^{-1} x$.

(i) Show by calculation that the equation $\mathrm{f}(x) = 0$ has a root in the interval $0.8 < x < 0.9$. [2]

(ii) Use the Newton-Raphson method, with a first approximation 0.8, to find the next approximation to this root. Give your answer correct to 3 decimal places. [4]

3



The diagram shows the curve with equation $y = \mathrm{e}^{x^2}$, for $0 \leqslant x \leqslant 1$. The region under the curve between these limits is divided into four strips of equal width. The area of this region under the curve is A.

(i) By considering the set of rectangles indicated in the diagram, show that an upper bound for A is 1.71. [3]

(ii) By considering an appropriate set of four rectangles, find a lower bound for A. [3]

4 **(i)** On separate diagrams, sketch the graphs of $y = \sinh x$ and $y = \operatorname{cosech} x$. [3]

(ii) Show that $\operatorname{cosech} x = \dfrac{2\mathrm{e}^x}{\mathrm{e}^{2x} - 1}$, and hence, using the substitution $u = \mathrm{e}^x$, find $\displaystyle\int \operatorname{cosech} x \, \mathrm{d}x$. [6]

Jan 2007

5 It is given that, for non-negative integers n,

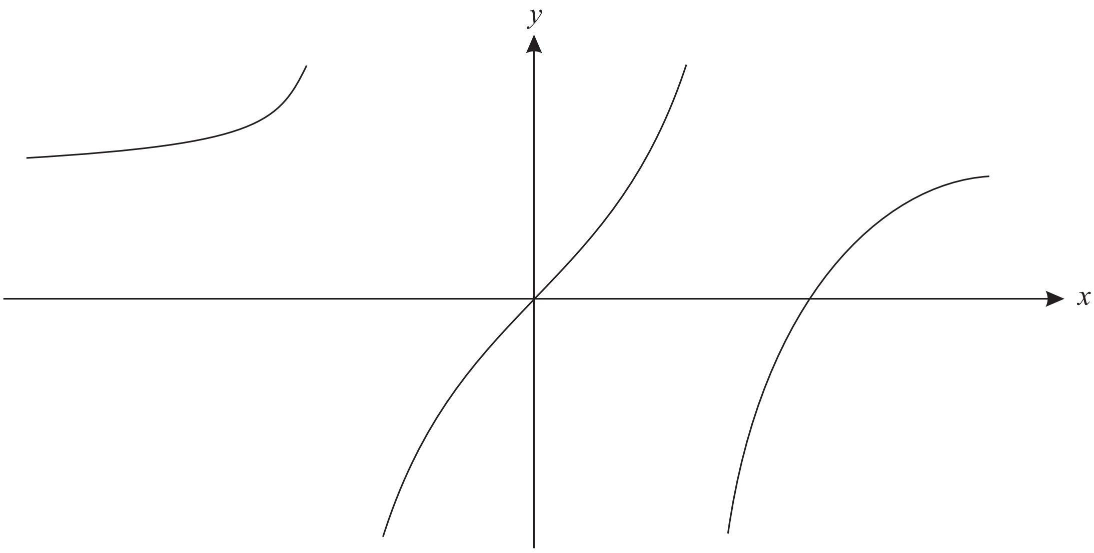
$$I_n = \int_0^{\frac{1}{2}\pi} x^n \cos x \, dx.$$

(i) Prove that, for $n \geqslant 2$,

$$I_n = \left(\tfrac{1}{2}\pi\right)^n - n(n-1)I_{n-2}.$$ [5]

(ii) Find I_4 in terms of π. [4]

6



The diagram shows the curve with equation $y = \dfrac{2x^2 - 3ax}{x^2 - a^2}$, where a is a positive constant.

(i) Find the equations of the asymptotes of the curve. [3]

(ii) Sketch the curve with equation

$$y^2 = \frac{2x^2 - 3ax}{x^2 - a^2}.$$

State the coordinates of any points where the curve crosses the axes, and give the equations of any asymptotes. [5]

7 **(i)** Express $\dfrac{1-t^2}{t^2(1+t^2)}$ in partial fractions. [4]

(ii) Use the substitution $t = \tan\frac{1}{2}x$ to show that

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \frac{\cos x}{1 - \cos x} \, dx = \sqrt{3} - 1 - \tfrac{1}{6}\pi.$$ [5]

Jan 2007

8 **(i)** Define $\tanh y$ in terms of e^y and e^{-y}. [1]

(ii) Given that $y = \tanh^{-1} x$, where $-1 < x < 1$, prove that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. [3]

(iii) Find the exact solution of the equation $3\cosh x = 4\sinh x$, giving the answer in terms of a logarithm. [2]

(iv) Solve the equation

$$\tanh^{-1} x + \ln(1-x) = \ln\left(\tfrac{4}{5}\right).$$ [3]

9 The equation of a curve, in polar coordinates, is

$$r = \sec\theta + \tan\theta, \quad \text{for } 0 \leqslant \theta \leqslant \tfrac{1}{3}\pi.$$

(i) Sketch the curve. [2]

(ii) Find the exact area of the region bounded by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{3}\pi$. [6]

(iii) Find a cartesian equation of the curve. [3]

June 2007

1 The equation of a curve, in polar coordinates, is

$$r = 2\sin 3\theta, \quad \text{for } 0 \leqslant \theta \leqslant \tfrac{1}{3}\pi.$$

Find the exact area of the region enclosed by the curve between $\theta = 0$ and $\theta = \frac{1}{3}\pi$. [4]

2 **(i)** Given that $\mathrm{f}(x) = \sin(2x + \frac{1}{4}\pi)$, show that $\mathrm{f}(x) = \frac{1}{2}\sqrt{2}(\sin 2x + \cos 2x)$. [2]

(ii) Hence find the first four terms of the Maclaurin series for $\mathrm{f}(x)$. [You may use appropriate results given in the List of Formulae.] [3]

3 It is given that $\mathrm{f}(x) = \dfrac{x^2 + 9x}{(x-1)(x^2+9)}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [4]

(ii) Hence find $\displaystyle\int \mathrm{f}(x)\,\mathrm{d}x$. [2]

4 **(i)** Given that

$$y = x\sqrt{1-x^2} - \cos^{-1} x,$$

find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in a simplified form. [4]

(ii) Hence, or otherwise, find the exact value of $\displaystyle\int_0^1 2\sqrt{1-x^2}\,\mathrm{d}x$. [3]

5 It is given that, for non-negative integers n,

$$I_n = \int_1^{\mathrm{e}} (\ln x)^n\,\mathrm{d}x.$$

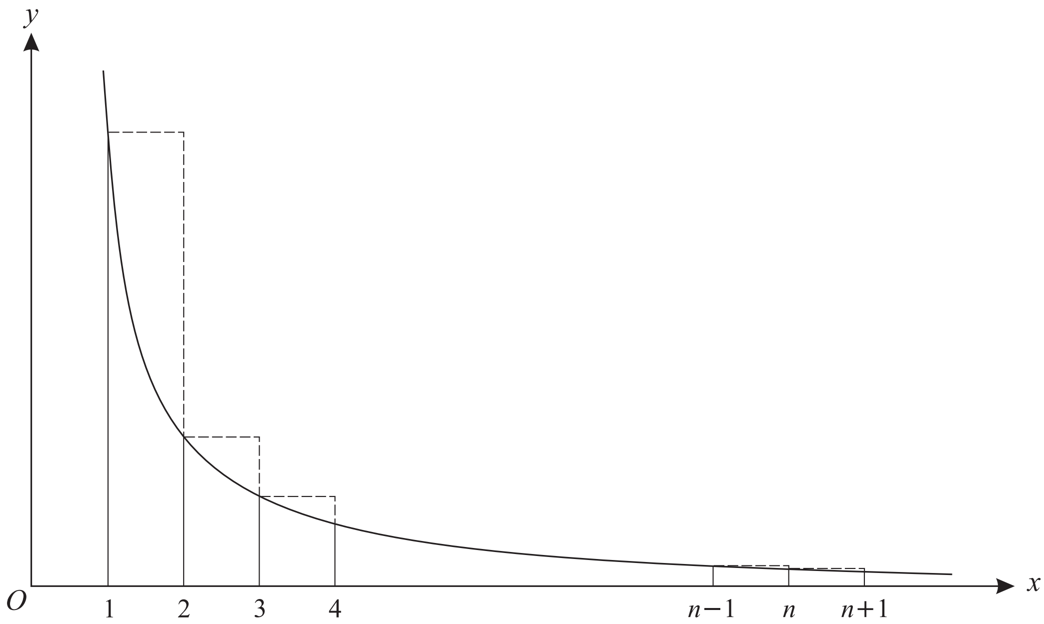
(i) Show that, for $n \geqslant 1$,

$$I_n = \mathrm{e} - nI_{n-1}.$$ [4]

(ii) Find I_3 in terms of e. [4]

June 2007

6



The diagram shows the curve with equation $y = \frac{1}{x^2}$ for $x > 0$, together with a set of n rectangles of unit width, starting at $x = 1$.

(i) By considering the areas of these rectangles, explain why

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} > \int_1^{n+1} \frac{1}{x^2}\,\mathrm{d}x. \qquad [2]$$

(ii) By considering the areas of another set of rectangles, explain why

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2} < \int_1^{n} \frac{1}{x^2}\,\mathrm{d}x. \qquad [3]$$

(iii) Hence show that

$$1 - \frac{1}{n+1} < \sum_{r=1}^{n} \frac{1}{r^2} < 2 - \frac{1}{n}. \qquad [4]$$

(iv) Hence give bounds between which $\sum_{r=1}^{\infty} \frac{1}{r^2}$ lies. [2]

7 **(i)** Using the definitions of hyperbolic functions in terms of exponentials, prove that

$$\cosh x \cosh y - \sinh x \sinh y = \cosh(x - y). \qquad [4]$$

(ii) Given that $\cosh x \cosh y = 9$ and $\sinh x \sinh y = 8$, show that $x = y$. [2]

(iii) Hence find the values of x and y which satisfy the equations given in part **(ii)**, giving the answers in logarithmic form. [4]

June 2007

8 The iteration $x_{n+1} = \dfrac{1}{(x_n + 2)^2}$, with $x_1 = 0.3$, is to be used to find the real root, α, of the equation $x(x+2)^2 = 1$.

(i) Find the value of α, correct to 4 decimal places. You should show the result of each step of the iteration process. [4]

(ii) Given that $\mathrm{f}(x) = \dfrac{1}{(x+2)^2}$, show that $\mathrm{f}'(\alpha) \neq 0$. [2]

(iii) The difference, δ_r, between successive approximations is given by $\delta_r = x_{r+1} - x_r$. Find δ_3. [1]

(iv) Given that $\delta_{r+1} \approx \mathrm{f}'(\alpha)\delta_r$, find an estimate for δ_{10}. [3]

9 It is given that the equation of a curve is

$$y = \frac{x^2 - 2ax}{x - a},$$

where a is a positive constant.

(i) Find the equations of the asymptotes of the curve. [4]

(ii) Show that y takes all real values. [4]

(iii) Sketch the curve $y = \dfrac{x^2 - 2ax}{x - a}$. [3]

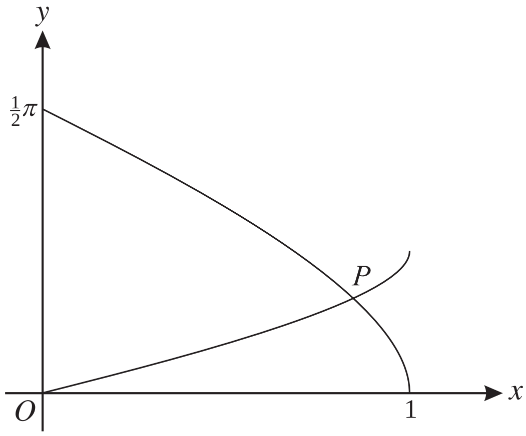
Jan 2008

1 It is given that $\mathrm{f}(x) = \ln(1 + \cos x)$.

(i) Find the exact values of $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. [4]

(ii) Hence find the first two non-zero terms of the Maclaurin series for $\mathrm{f}(x)$. [2]

2

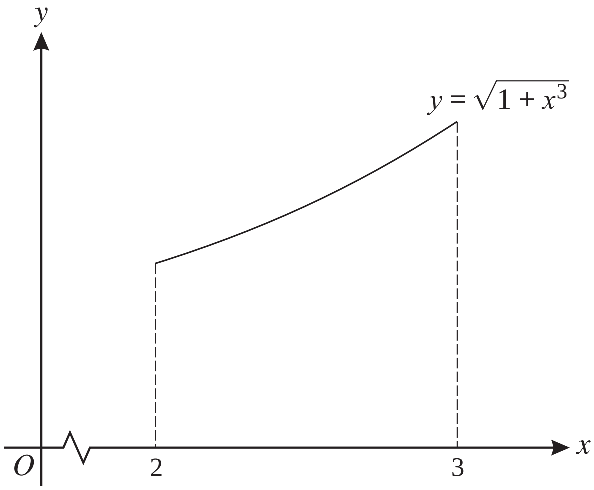


The diagram shows parts of the curves with equations $y = \cos^{-1} x$ and $y = \frac{1}{2}\sin^{-1} x$, and their point of intersection P.

(i) Verify that the coordinates of P are $\left(\frac{1}{2}\sqrt{3}, \frac{1}{6}\pi\right)$. [2]

(ii) Find the gradient of each curve at P. [3]

3



The diagram shows the curve with equation $y = \sqrt{1 + x^3}$, for $2 \leqslant x \leqslant 3$. The region under the curve between these limits has area A.

(i) Explain why $3 < A < \sqrt{28}$. [2]

(ii) The region is divided into 5 strips, each of width 0.2. By using suitable rectangles, find improved lower and upper bounds between which A lies. Give your answers correct to 3 significant figures. [4]

Jan 2008

4 The equation of a curve, in polar coordinates, is

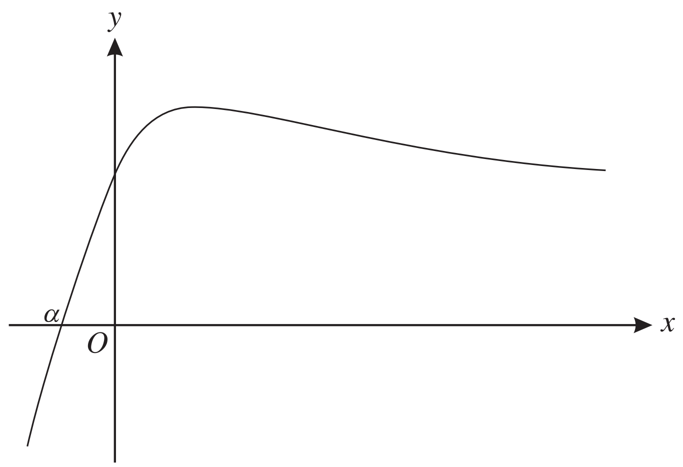
$$r = 1 + 2\sec\theta, \quad \text{for } -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi.$$

(i) Find the exact area of the region bounded by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{6}\pi$. [5]

[The result $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta|$ may be assumed.]

(ii) Show that a cartesian equation of the curve is $(x-2)\sqrt{x^2+y^2} = x$. [3]

5



The diagram shows the curve with equation $y = xe^{-x} + 1$. The curve crosses the x-axis at $x = \alpha$.

(i) Use differentiation to show that the x-coordinate of the stationary point is 1. [2]

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

(ii) Explain why this method will not converge to α if an initial approximation x_1 is chosen such that $x_1 > 1$. [2]

(iii) Use this method, with a first approximation $x_1 = 0$, to find the next three approximations x_2, x_3 and x_4. Find α, correct to 3 decimal places. [5]

6 The equation of a curve is $y = \dfrac{2x^2 - 11x - 6}{x - 1}$.

(i) Find the equations of the asymptotes of the curve. [3]

(ii) Show that y takes all real values. [5]

Jan 2008

7 It is given that, for integers $n \geqslant 1$,

$$I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx.$$

(i) Use integration by parts to show that $I_n = 2^{-n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$. [3]

(ii) Show that $2nI_{n+1} = 2^{-n} + (2n-1)I_n$. [3]

(iii) Find I_2 in terms of π. [3]

8 **(i)** By using the definition of $\sinh x$ in terms of e^x and e^{-x}, show that

$$\sinh^3 x = \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x. \quad [4]$$

(ii) Find the range of values of the constant k for which the equation

$$\sinh 3x = k \sinh x$$

has real solutions other than $x = 0$. [3]

(iii) Given that $k = 4$, solve the equation in part **(ii)**, giving the non-zero answers in logarithmic form. [3]

9 **(i)** Prove that $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$. [3]

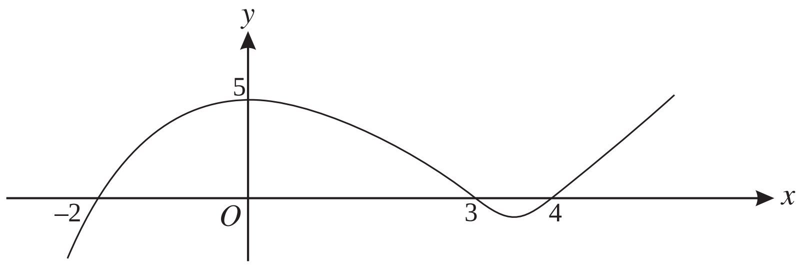
(ii) Hence, or otherwise, find $\int \frac{1}{\sqrt{4x^2-1}}\,dx$. [2]

(iii) By means of a suitable substitution, find $\int \sqrt{4x^2-1}\,dx$. [6]

June 2008

1 It is given that $f(x) = \dfrac{2ax}{(x-2a)(x^2+a^2)}$, where a is a non-zero constant. Express $f(x)$ in partial fractions. [5]

2



The diagram shows the curve $y = f(x)$. The curve has a maximum point at (0, 5) and crosses the x-axis at (–2, 0), (3, 0) and (4, 0). Sketch the curve $y^2 = f(x)$, showing clearly the coordinates of any turning points and of any points where this curve crosses the axes. [5]

3 By using the substitution $t = \tan \frac{1}{2}x$, find the exact value of

$$\int_0^{\frac{1}{2}\pi} \frac{1}{2-\cos x}\,dx,$$

giving the answer in terms of π. [6]

4 **(i)** Sketch, on the same diagram, the curves with equations $y = \operatorname{sech} x$ and $y = x^2$. [3]

(ii) By using the definition of $\operatorname{sech} x$ in terms of e^x and e^{-x}, show that the x-coordinates of the points at which these curves meet are solutions of the equation

$$x^2 = \frac{2e^x}{e^{2x}+1}.$$ [3]

(iii) The iteration

$$x_{n+1} = \sqrt{\frac{2e^{x_n}}{e^{2x_n}+1}}$$

can be used to find the positive root of the equation in part **(ii)**. With initial value $x_1 = 1$, the approximations $x_2 = 0.8050$, $x_3 = 0.8633$, $x_4 = 0.8463$ and $x_5 = 0.8513$ are obtained, correct to 4 decimal places. State with a reason whether, in this case, the iteration produces a 'staircase' or a 'cobweb' diagram. [2]

5 It is given that, for $n \geqslant 0$,

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^n x\,dx.$$

(i) By considering $I_n + I_{n-2}$, or otherwise, show that, for $n \geqslant 2$,

$$(n-1)(I_n + I_{n-2}) = 1.$$ [4]

(ii) Find I_4 in terms of π. [4]

June 2008

6 It is given that $\mathrm{f}(x) = 1 - \dfrac{7}{x^2}$.

(i) Use the Newton-Raphson method, with a first approximation $x_1 = 2.5$, to find the next approximations x_2 and x_3 to a root of $\mathrm{f}(x) = 0$. Give the answers correct to 6 decimal places. [3]

(ii) The root of $\mathrm{f}(x) = 0$ for which x_1, x_2 and x_3 are approximations is denoted by α. Write down the exact value of α. [1]

(iii) The error e_n is defined by $e_n = \alpha - x_n$. Find e_1, e_2 and e_3, giving your answers correct to 5 decimal places. Verify that $e_3 \approx \dfrac{e_2^3}{e_1^2}$. [3]

7 It is given that $\mathrm{f}(x) = \tanh^{-1}\left(\dfrac{1-x}{2+x}\right)$, for $x > -\frac{1}{2}$.

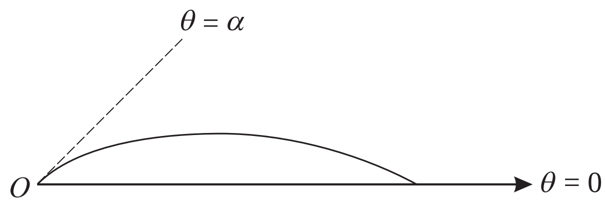
(i) Show that $\mathrm{f}'(x) = -\dfrac{1}{1+2x}$, and find $\mathrm{f}''(x)$. [6]

(ii) Show that the first three terms of the Maclaurin series for $\mathrm{f}(x)$ can be written as $\ln a + bx + cx^2$, for constants a, b and c to be found. [4]

8 The equation of a curve, in polar coordinates, is

$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

(i)



The diagram shows the part of the curve for which $0 \leqslant \theta \leqslant \alpha$, where $\theta = \alpha$ is the equation of the tangent to the curve at O. Find α in terms of π. [2]

(ii) **(a)** If $\mathrm{f}(\theta) = 1 - \sin 2\theta$, show that $\mathrm{f}\left(\frac{1}{2}(2k+1)\pi - \theta\right) = \mathrm{f}(\theta)$ for all θ, where k is an integer. [3]

(b) Hence state the equations of the lines of symmetry of the curve

$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi. \qquad [2]$$

(iii) Sketch the curve with equation

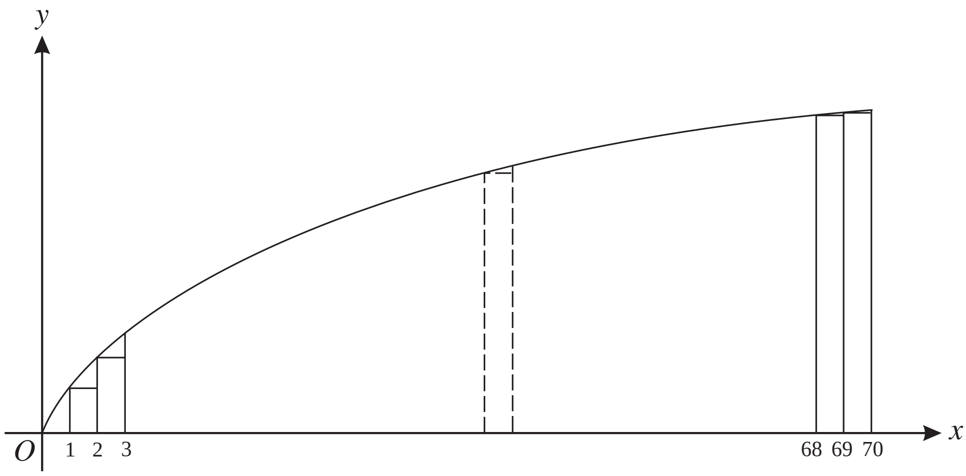
$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

State the maximum value of r and the corresponding values of θ. [4]

June 2008

9 **(i)** Prove that $\int_0^N \ln(1+x)\,dx = (N+1)\ln(N+1) - N$, where N is a positive constant. [4]

(ii)



The diagram shows the curve $y = \ln(1+x)$, for $0 \leqslant x \leqslant 70$, together with a set of rectangles of unit width.

(a) By considering the areas of these rectangles, explain why

$$\ln 2 + \ln 3 + \ln 4 + \ldots + \ln 70 < \int_0^{70} \ln(1+x)\,dx.$$ [2]

(b) By considering the areas of another set of rectangles, show that

$$\ln 2 + \ln 3 + \ln 4 + \ldots + \ln 70 > \int_0^{69} \ln(1+x)\,dx.$$ [3]

(c) Hence find bounds between which $\ln(70!)$ lies. Give the answers correct to 1 decimal place. [3]

Jan 2009

1 **(i)** Write down and simplify the first three terms of the Maclaurin series for e^{2x}. **[2]**

(ii) Hence show that the Maclaurin series for

$$\ln(e^{2x} + e^{-2x})$$

begins $\ln a + bx^2$, where a and b are constants to be found. **[4]**

2 It is given that α is the only real root of the equation $x^5 + 2x - 28 = 0$ and that $1.8 < \alpha < 2$.

(i) The iteration $x_{n+1} = \sqrt[5]{28 - 2x_n}$, with $x_1 = 1.9$, is to be used to find α. Find the values of x_2, x_3 and x_4, giving the answers correct to 7 decimal places. **[3]**

(ii) The error e_n is defined by $e_n = \alpha - x_n$. Given that $\alpha = 1.891\,574\,9$, correct to 7 decimal places, evaluate $\frac{e_3}{e_2}$ and $\frac{e_4}{e_3}$. Comment on these values in relation to the gradient of the curve with equation $y = \sqrt[5]{28 - 2x}$ at $x = \alpha$. **[3]**

3 **(i)** Prove that the derivative of $\sin^{-1} x$ is $\frac{1}{\sqrt{1 - x^2}}$. **[3]**

(ii) Given that

$$\sin^{-1} 2x + \sin^{-1} y = \tfrac{1}{2}\pi,$$

find the exact value of $\frac{dy}{dx}$ when $x = \frac{1}{4}$. **[4]**

4 **(i)** By means of a suitable substitution, show that

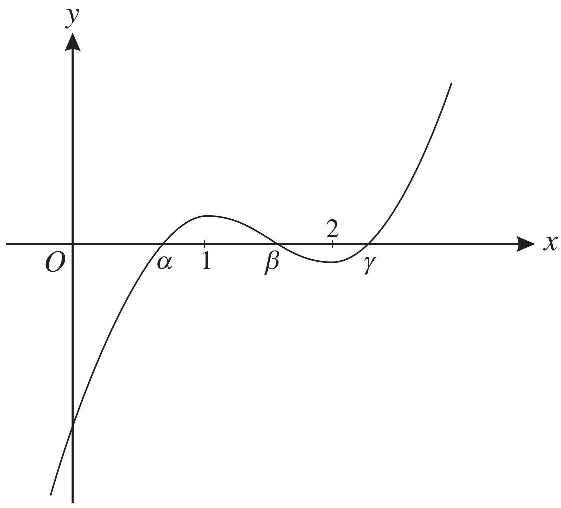
$$\int \frac{x^2}{\sqrt{x^2 - 1}}\,dx$$

can be transformed to $\int \cosh^2 \theta \, d\theta$. **[2]**

(ii) Hence show that $\int \frac{x^2}{\sqrt{x^2 - 1}}\,dx = \frac{1}{2}x\sqrt{x^2 - 1} + \frac{1}{2}\cosh^{-1} x + c$. **[4]**

Jan 2009

5



The diagram shows the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2x^3 - 9x^2 + 12x - 4.36.$$

The curve has turning points at $x = 1$ and $x = 2$ and crosses the x-axis at $x = \alpha$, $x = \beta$ and $x = \gamma$, where $0 < \alpha < \beta < \gamma$.

(i) The Newton-Raphson method is to be used to find the roots of the equation $\mathrm{f}(x) = 0$, with $x_1 = k$.

(a) To which root, if any, would successive approximations converge in each of the cases $k < 0$ and $k = 1$? **[2]**

(b) What happens if $1 < k < 2$? **[2]**

(ii) Sketch the curve with equation $y^2 = \mathrm{f}(x)$. State the coordinates of the points where the curve crosses the x-axis and the coordinates of any turning points. **[4]**

6 **(i)** Using the definitions of $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}, show that

$$1 + 2\sinh^2 x \equiv \cosh 2x.$$ **[3]**

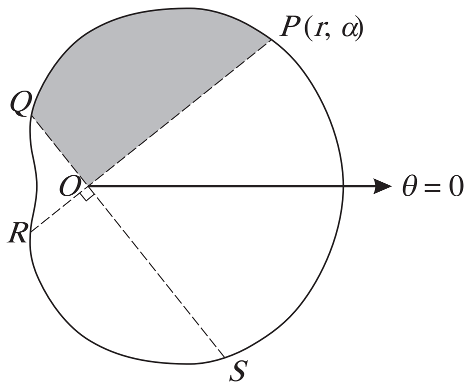
(ii) Solve the equation

$$\cosh 2x - 5\sinh x = 4,$$

giving your answers in logarithmic form. **[5]**

Jan 2009

7



The diagram shows the curve with equation, in polar coordinates,

$$r = 3 + 2\cos\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

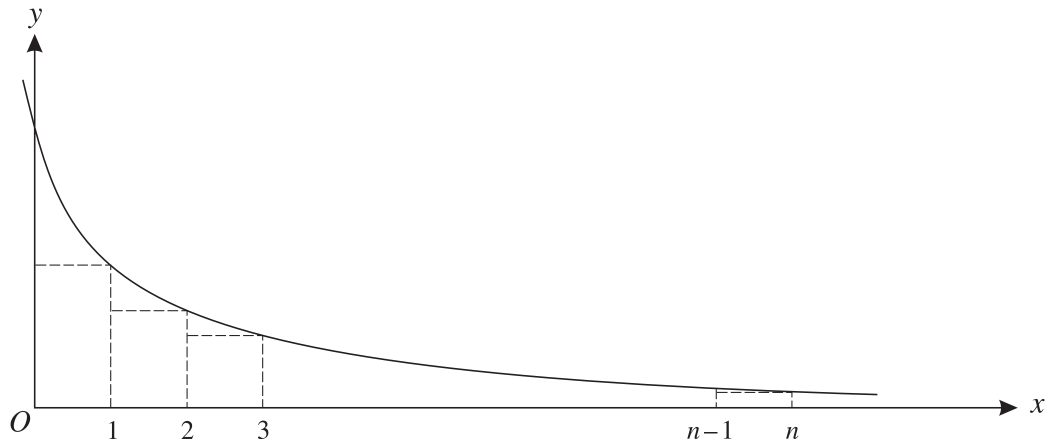
The points P, Q, R and S on the curve are such that the straight lines POR and QOS are perpendicular, where O is the pole. The point P has polar coordinates (r, α).

(i) Show that $OP + OQ + OR + OS = k$, where k is a constant to be found. **[3]**

(ii) Given that $\alpha = \frac{1}{4}\pi$, find the exact area bounded by the curve and the lines OP and OQ (shaded in the diagram). **[5]**

Jan 2009

8



The diagram shows the curve with equation $y = \dfrac{1}{x+1}$. A set of n rectangles of unit width is drawn, starting at $x = 0$ and ending at $x = n$, where n is an integer.

(i) By considering the areas of these rectangles, explain why

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1} < \ln(n+1).$$ **[5]**

(ii) By considering the areas of another set of rectangles, show that

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} > \ln(n+1).$$ **[2]**

(iii) Hence show that

$$\ln(n+1) + \frac{1}{n+1} < \sum_{r=1}^{n+1} \frac{1}{r} < \ln(n+1) + 1.$$ **[2]**

(iv) State, with a reason, whether $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r}$ is convergent. **[2]**

9 A curve has equation

$$y = \frac{4x - 3a}{2(x^2 + a^2)},$$

where a is a positive constant.

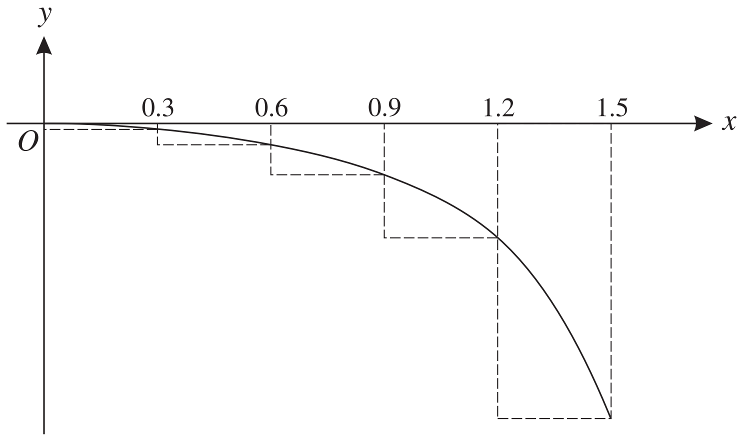
(i) Explain why the curve has no asymptotes parallel to the y-axis. **[2]**

(ii) Find, in terms of a, the set of values of y for which there are no points on the curve. **[5]**

(iii) Find the exact value of $\displaystyle\int_a^{2a} \frac{4x - 3a}{2(x^2 + a^2)}\,\mathrm{d}x$, showing that it is independent of a. **[5]**

June 2009

1



The diagram shows the curve with equation $y = \ln(\cos x)$, for $0 \leqslant x \leqslant 1.5$. The region bounded by the curve, the x-axis and the line $x = 1.5$ has area A. The region is divided into five strips, each of width 0.3.

(i) By considering the set of rectangles indicated in the diagram, find an upper bound for A. Give the answer correct to 3 decimal places. **[2]**

(ii) By considering another set of five suitable rectangles, find a lower bound for A. Give the answer correct to 3 decimal places. **[2]**

(iii) How could you reduce the difference between the upper and lower bounds for A? **[1]**

2 Given that $y = \dfrac{x^2 + x + 1}{(x-1)^2}$, prove that $y \geqslant \frac{1}{4}$ for all $x \neq 1$. **[4]**

3 (i) Given that $\mathrm{f}(x) = \mathrm{e}^{\sin x}$, find $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. **[4]**

(ii) Hence find the first three terms of the Maclaurin series for $\mathrm{f}(x)$. **[2]**

4 Express $\dfrac{x^3}{(x-2)(x^2+4)}$ in partial fractions. **[6]**

5 It is given that $I = \displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos\theta}{1+\cos\theta}\,\mathrm{d}\theta$.

(i) By using the substitution $t = \tan\frac{1}{2}\theta$, show that $I = \displaystyle\int_0^1 \left(\frac{2}{1+t^2} - 1\right)\mathrm{d}t$. **[5]**

(ii) Hence find I in terms of π. **[2]**

June 2009

6 Given that

$$\int_0^1 \frac{1}{\sqrt{16+9x^2}}\,dx + \int_0^2 \frac{1}{\sqrt{9+4x^2}}\,dx = \ln a,$$

find the exact value of a. **[6]**

7 **(i)** Sketch the graph of $y = \coth x$, and give the equations of any asymptotes. **[3]**

(ii) It is given that $f(x) = x\tanh x - 2$. Use the Newton-Raphson method, with a first approximation $x_1 = 2$, to find the next three approximations x_2, x_3 and x_4 to a root of $f(x) = 0$. Give the answers correct to 4 decimal places. **[4]**

(iii) If $f(x) = 0$, show that $\coth x = \frac{1}{2}x$. Hence write down the roots of $f(x) = 0$, correct to 4 decimal places. **[3]**

8 **(i)** Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}, show that

(a) $\cosh(\ln a) \equiv \dfrac{a^2+1}{2a}$, where $a > 0$, **[3]**

(b) $\cosh x \cosh y - \sinh x \sinh y \equiv \cosh(x - y)$. **[3]**

(ii) Use part **(i)(b)** to show that $\cosh^2 x - \sinh^2 x \equiv 1$. **[1]**

(iii) Given that $R > 0$ and $a > 1$, find R and a such that

$$13\cosh x - 5\sinh x \equiv R\cosh(x - \ln a).$$ **[5]**

(iv) Hence write down the coordinates of the minimum point on the curve with equation $y = 13\cosh x - 5\sinh x$. **[2]**

9 **(i)** It is given that, for non-negative integers n,

$$I_n = \int_0^{\frac{1}{2}\pi} \sin^n\theta\,d\theta.$$

Show that, for $n \geqslant 2$,

$$nI_n = (n-1)I_{n-2}.$$ **[4]**

(ii) The equation of a curve, in polar coordinates, is

$$r = \sin^3\theta, \quad \text{for } 0 \leqslant \theta \leqslant \pi.$$

(a) Find the equations of the tangents at the pole and sketch the curve. **[4]**

(b) Find the exact area of the region enclosed by the curve. **[6]**

1 It is given that $\mathrm{f}(x) = x^2 - \sin x$.

(i) The iteration $x_{n+1} = \sqrt{\sin x_n}$, with $x_1 = 0.875$, is to be used to find a real root, α, of the equation $\mathrm{f}(x) = 0$. Find x_2, x_3 and x_4, giving the answers correct to 6 decimal places. **[2]**

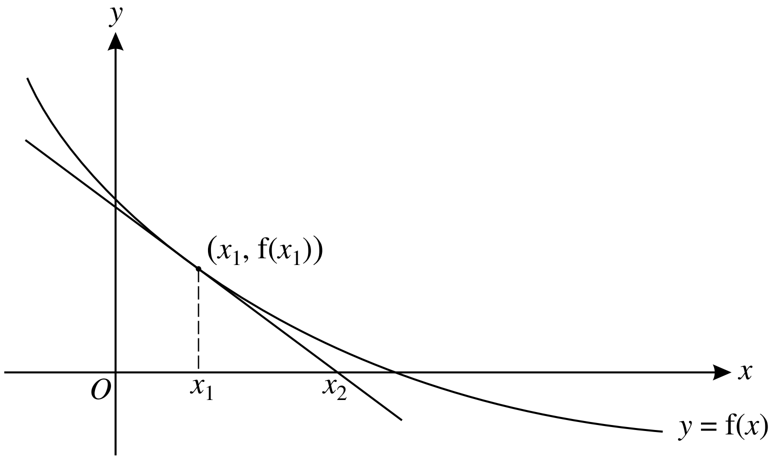
(ii) The error e_n is defined by $e_n = \alpha - x_n$. Given that $\alpha = 0.876\,726$, correct to 6 decimal places, find e_3 and e_4. Given that $\mathrm{g}(x) = \sqrt{\sin x}$, use e_3 and e_4 to estimate $\mathrm{g}'(\alpha)$. **[3]**

2 It is given that $\mathrm{f}(x) = \tan^{-1}(1 + x)$.

(i) Find $\mathrm{f}(0)$ and $\mathrm{f}'(0)$, and show that $\mathrm{f}''(0) = -\frac{1}{2}$. **[4]**

(ii) Hence find the Maclaurin series for $\mathrm{f}(x)$ up to and including the term in x^2. **[2]**

3



A curve with no stationary points has equation $y = \mathrm{f}(x)$. The equation $\mathrm{f}(x) = 0$ has one real root α, and the Newton-Raphson method is to be used to find α. The tangent to the curve at the point $(x_1, \mathrm{f}(x_1))$ meets the x-axis where $x = x_2$ (see diagram).

(i) Show that $x_2 = x_1 - \dfrac{\mathrm{f}(x_1)}{\mathrm{f}'(x_1)}$. **[3]**

(ii) Describe briefly, with the help of a sketch, how the Newton-Raphson method, using an initial approximation $x = x_1$, gives a sequence of approximations approaching α. **[2]**

(iii) Use the Newton-Raphson method, with a first approximation of 1, to find a second approximation to the root of $x^2 - 2\sinh x + 2 = 0$. **[2]**

4 The equation of a curve, in polar coordinates, is

$$r = \mathrm{e}^{-2\theta}, \quad \text{for } 0 \leqslant \theta \leqslant \pi.$$

(i) Sketch the curve, stating the polar coordinates of the point at which r takes its greatest value. **[2]**

(ii) The pole is O and points P and Q, with polar coordinates (r_1, θ_1) and (r_2, θ_2) respectively, lie on the curve. Given that $\theta_2 > \theta_1$, show that the area of the region enclosed by the curve and the lines OP and OQ can be expressed as $k\left(r_1^2 - r_2^2\right)$, where k is a constant to be found. **[5]**

Jan 2010

5 **(i)** Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}, show that

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

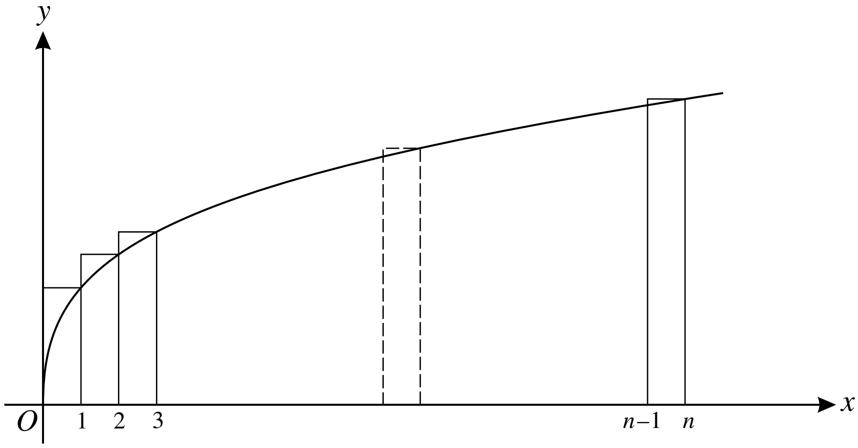
Deduce that $1 - \tanh^2 x \equiv \operatorname{sech}^2 x$. **[4]**

(ii) Solve the equation $2\tanh^2 x - \operatorname{sech} x = 1$, giving your answer(s) in logarithmic form. **[4]**

6 **(i)** Express $\dfrac{4}{(1-x)(1+x)(1+x^2)}$ in partial fractions. **[5]**

(ii) Show that $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{4}{1-x^4}\,dx = \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + \tfrac{1}{3}\pi$. **[4]**

7



The diagram shows the curve with equation $y = \sqrt[3]{x}$, together with a set of n rectangles of unit width.

(i) By considering the areas of these rectangles, explain why

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n} > \int_0^n \sqrt[3]{x}\,dx.$$ **[2]**

(ii) By drawing another set of rectangles and considering their areas, show that

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n} < \int_1^{n+1} \sqrt[3]{x}\,dx.$$ **[3]**

(iii) Hence find an approximation to $\displaystyle\sum_{n=1}^{100} \sqrt[3]{n}$, giving your answer correct to 2 significant figures. **[3]**

[Questions 8 and 9 are printed overleaf.]

Jan 2010

8 The equation of a curve is

$$y = \frac{kx}{(x-1)^2},$$

where k is a positive constant.

(i) Write down the equations of the asymptotes of the curve. **[2]**

(ii) Show that $y \geqslant -\frac{1}{4}k$. **[4]**

(iii) Show that the x-coordinate of the stationary point of the curve is independent of k, and sketch the curve. **[4]**

9 **(i)** Given that $y = \tanh^{-1} x$, for $-1 < x < 1$, prove that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. **[3]**

(ii) It is given that $\mathrm{f}(x) = a\cosh x - b\sinh x$, where a and b are positive constants.

(a) Given that $b \geqslant a$, show that the curve with equation $y = \mathrm{f}(x)$ has no stationary points. **[3]**

(b) In the case where $a > 1$ and $b = 1$, show that $\mathrm{f}(x)$ has a minimum value of $\sqrt{a^2 - 1}$. **[6]**

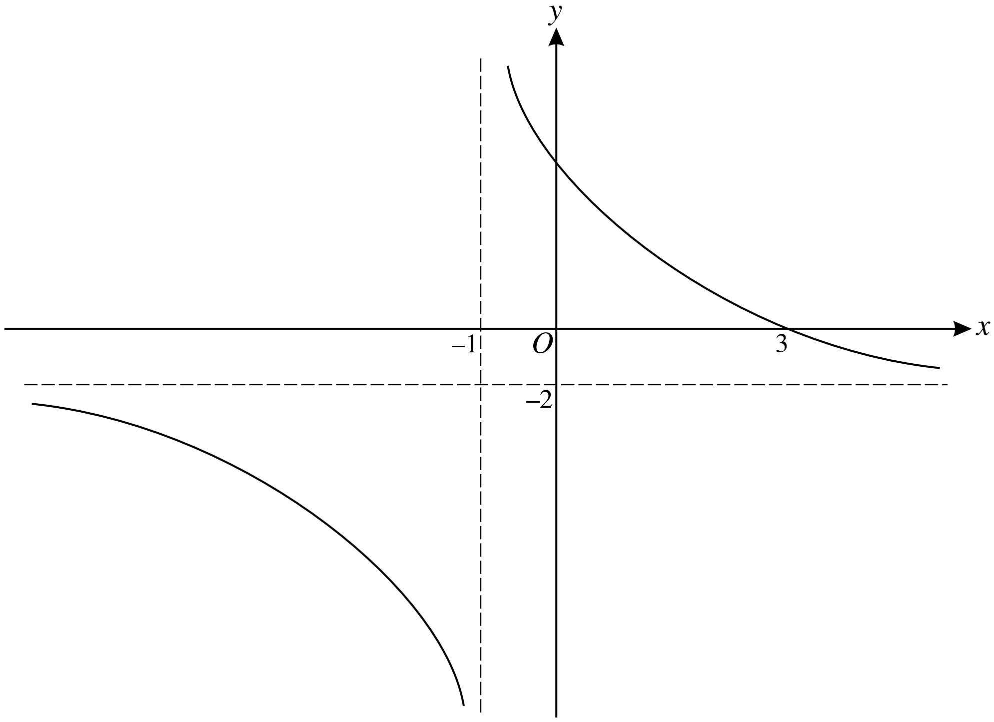
1 It is given that $f(x) = \tan^{-1} 2x$ and $g(x) = p\tan^{-1} x$, where p is a constant. Find the value of p for which $f'(\frac{1}{2}) = g'(\frac{1}{2})$. **[4]**

2 Given that the first three terms of the Maclaurin series for $(1 + \sin x)e^{2x}$ are identical to the first three terms of the binomial series for $(1 + ax)^n$, find the values of the constants a and n. (You may use appropriate results given in the List of Formulae (MF1).) **[6]**

3 Use the substitution $t = \tan\frac{1}{2}x$ to show that

$$\int_0^{\frac{1}{3}\pi} \frac{1}{1 - \sin x}\, dx = 1 + \sqrt{3}.$$ **[6]**

4



The diagram shows the curve with equation

$$y = \frac{ax + b}{x + c},$$

where a, b and c are constants.

(i) Given that the asymptotes of the curve are $x = -1$ and $y = -2$ and that the curve passes through $(3, 0)$, find the values of a, b and c. **[3]**

(ii) Sketch the curve with equation

$$y^2 = \frac{ax + b}{x + c},$$

for the values of a, b and c found in part **(i)**. State the coordinates of any points where the curve crosses the axes, and give the equations of any asymptotes. **[4]**

5 It is given that, for $n \geqslant 0$,

$$I_n = \int_0^{\frac{1}{2}} (1-2x)^n e^x \, dx.$$

(i) Prove that, for $n \geqslant 1$,

$$I_n = 2nI_{n-1} - 1.$$ **[4]**

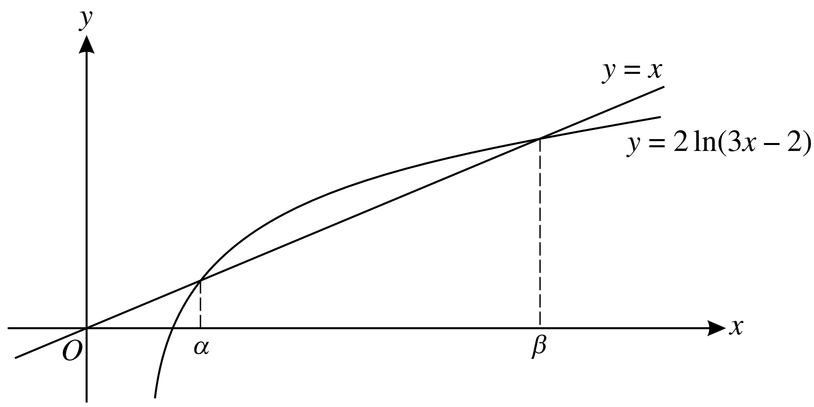
(ii) Find the exact value of I_3. **[4]**

6 **(i)** Show that $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$. **[2]**

(ii) Given that $y = \cosh(a \sinh^{-1} x)$, where a is a constant, show that

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - a^2 y = 0.$$ **[5]**

7



The line $y = x$ and the curve $y = 2\ln(3x-2)$ meet where $x = \alpha$ and $x = \beta$, as shown in the diagram.

(i) Use the iteration $x_{n+1} = 2\ln(3x_n - 2)$, with initial value $x_1 = 5.25$, to find the value of β correct to 2 decimal places. Show all your working. **[2]**

(ii) With the help of a 'staircase' diagram, explain why this iteration will not converge to α, whatever value of x_1 (other than α) is used. **[3]**

(iii) Show that the equation $x = 2\ln(3x-2)$ can be rewritten as $x = \frac{1}{3}\left(e^{\frac{1}{2}x} + 2\right)$. Use the Newton-Raphson method, with $f(x) = \frac{1}{3}\left(e^{\frac{1}{2}x} + 2\right) - x$ and $x_1 = 1.2$, to find α correct to 2 decimal places. Show all your working. **[4]**

(iv) Given that $x_1 = \ln 36$, explain why the Newton-Raphson method would not converge to a root of $f(x) = 0$. **[2]**

[Questions 8 and 9 are printed overleaf.]

June 2010

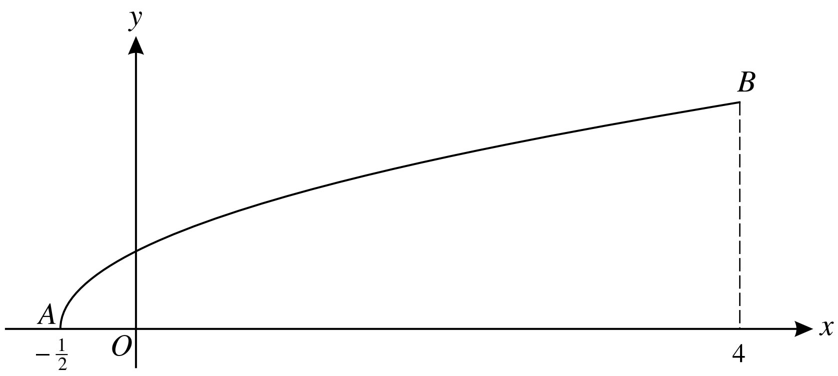
8 **(i)** Using the definition of $\cosh x$ in terms of e^x and e^{-x}, show that

$$4\cosh^3 x - 3\cosh x \equiv \cosh 3x.$$ **[4]**

(ii) Use the substitution $u = \cosh x$ to find, in terms of $5^{\frac{1}{3}}$, the real root of the equation

$$20u^3 - 15u - 13 = 0.$$ **[6]**

9



The diagram shows the curve with equation $y = \sqrt{2x+1}$ between the points $A\,(-\frac{1}{2}, 0)$ and $B\,(4, 3)$.

(i) Find the area of the region bounded by the curve, the x-axis and the line $x = 4$. Hence find the area of the region bounded by the curve and the lines OA and OB, where O is the origin. **[4]**

(ii) Show that the curve between B and A can be expressed in polar coordinates as

$$r = \frac{1}{1-\cos\theta}, \quad \text{where } \tan^{-1}\left(\tfrac{3}{4}\right) \leqslant \theta \leqslant \pi.$$ **[5]**

(iii) Deduce from parts **(i)** and **(ii)** that $\displaystyle\int_{\tan^{-1}\left(\frac{3}{4}\right)}^{\pi} \operatorname{cosec}^4\left(\tfrac{1}{2}\theta\right)\,d\theta = 24.$ **[4]**

Jan 2011

1 Use the substitution $t = \tan \frac{1}{2}x$ to find $\displaystyle\int \frac{1}{1 + \sin x + \cos x}\,dx$. **[5]**

2 It is given that $f(x) = \tanh^{-1} x$.

(i) Show that $f'''(x) = \dfrac{2(1 + 3x^2)}{(1 - x^2)^3}$. **[5]**

(ii) Hence find the Maclaurin series for $f(x)$, up to and including the term in x^3. **[3]**

3 The function f is defined by $f(x) = \dfrac{5ax}{x^2 + a^2}$, for $x \in \mathbb{R}$ and $a > 0$.

(i) For the curve with equation $y = f(x)$,

(a) write down the equation of the asymptote, **[1]**

(b) find the range of values that y can take. **[4]**

(ii) For the curve with equation $y^2 = f(x)$, write down

(a) the equation of the line of symmetry, **[1]**

(b) the maximum and minimum values of y, **[2]**

(c) the set of values of x for which the curve is defined. **[1]**

4 **(i)** Use the definitions of hyperbolic functions in terms of exponentials to prove that

$$8 \sinh^4 x \equiv \cosh 4x - 4 \cosh 2x + 3.$$ **[4]**

(ii) Solve the equation

$$\cosh 4x - 3 \cosh 2x + 1 = 0,$$

giving your answer(s) in logarithmic form. **[5]**

5 The equation

$$x^3 - 5x + 3 = 0 \qquad \text{(A)}$$

may be solved by the Newton-Raphson method. Successive approximations to a root are denoted by $x_1, x_2, \ldots, x_n, \ldots$.

(i) Show that the Newton-Raphson formula can be written in the form $x_{n+1} = F(x_n)$, where

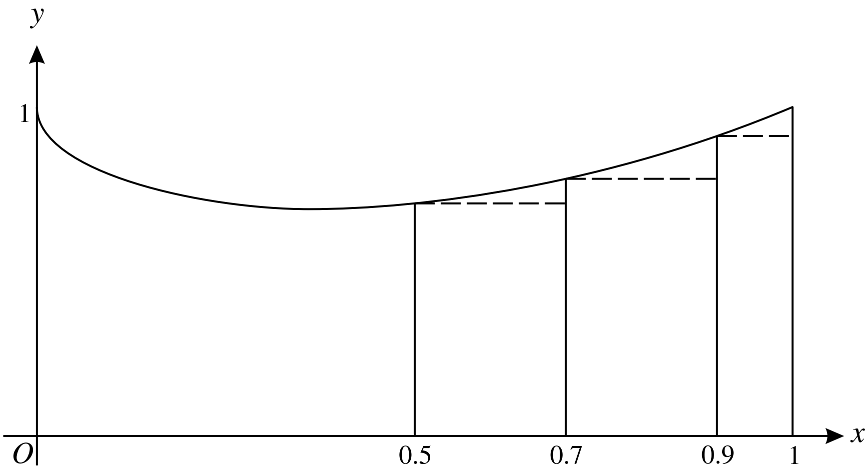
$$F(x) = \frac{2x^3 - 3}{3x^2 - 5}.$$ **[3]**

(ii) Find $F'(x)$ and hence verify that $F'(\alpha) = 0$, where α is any one of the roots of equation (A). **[3]**

(iii) Use the Newton-Raphson method to find the root of equation (A) which is close to 2. Write down sufficient approximations to find the root correct to 4 decimal places. **[3]**

Jan 2011

6



The diagram shows the curve $y = \mathrm{f}(x)$, defined by

$$\mathrm{f}(x) = \begin{cases} x^x & \text{for } 0 < x \leqslant 1, \\ 1 & \text{for } x = 0. \end{cases}$$

(i) By first taking logarithms, show that the curve has a stationary point at $x = \mathrm{e}^{-1}$. **[3]**

The area under the curve from $x = 0.5$ to $x = 1$ is denoted by A.

(ii) By considering the set of three rectangles shown in the diagram, show that a lower bound for A is 0.388. **[2]**

(iii) By considering another set of three rectangles, find an upper bound for A, giving 3 decimal places in your answer. **[2]**

The area under the curve from $x = 0$ to $x = 0.5$ is denoted by B.

(iv) Draw a diagram to show rectangles which could be used to find lower and upper bounds for B, using not more than three rectangles for each bound. (You are not required to find the bounds.) **[3]**

7 A curve has polar equation $r = 1 + \cos 3\theta$, for $-\pi < \theta \leqslant \pi$.

(i) Show that the line $\theta = 0$ is a line of symmetry. **[2]**

(ii) Find the equations of the tangents at the pole. **[3]**

(iii) Find the exact value of the area of the region enclosed by the curve between $\theta = -\frac{1}{3}\pi$ and $\theta = \frac{1}{3}\pi$. **[5]**

8 **(i)** Without using a calculator, show that $\sinh(\cosh^{-1} 2) = \sqrt{3}$. **[2]**

(ii) It is given that, for non-negative integers n,

$$I_n = \int_0^\beta \cosh^n x \, \mathrm{d}x, \quad \text{where } \beta = \cosh^{-1} 2.$$

Show that $nI_n = 2^{n-1}\sqrt{3} + (n-1)I_{n-2}$, for $n \geqslant 2$. **[6]**

(iii) Evaluate I_5, giving your answer in the form $k\sqrt{3}$. **[4]**

June 2011

1 Express $\dfrac{2x+3}{(x+3)(x^2+9)}$ in partial fractions. **[5]**

2 A curve has equation $y = \dfrac{x^2-6x-5}{x-2}$.

(i) Find the equations of the asymptotes. **[3]**

(ii) Show that y can take all real values. **[4]**

3 It is given that $\mathrm{F}(x) = 2 + \ln x$. The iteration $x_{n+1} = \mathrm{F}(x_n)$ is to be used to find a root, α, of the equation $x = 2 + \ln x$.

(i) Taking $x_1 = 3.1$, find x_2 and x_3, giving your answers correct to 5 decimal places. **[2]**

(ii) The error e_n is defined by $e_n = \alpha - x_n$. Given that $\alpha = 3.146\,19$, correct to 5 decimal places, use the values of e_2 and e_3 to make an estimate of $\mathrm{F}'(\alpha)$ correct to 3 decimal places. State the true value of $\mathrm{F}'(\alpha)$ correct to 4 decimal places. **[3]**

(iii) Illustrate the iteration by drawing a sketch of $y = x$ and $y = \mathrm{F}(x)$, showing how the values of x_n approach α. State whether the convergence is of the 'staircase' or 'cobweb' type. **[3]**

4 A curve C has the cartesian equation $x^3 + y^3 = axy$, where $x \geqslant 0$, $y \geqslant 0$ and $a > 0$.

(i) Express the polar equation of C in the form $r = \mathrm{f}(\theta)$ and state the limits between which θ lies. **[3]**

The line $\theta = \alpha$ is a line of symmetry of C.

(ii) Find and simplify an expression for $\mathrm{f}\left(\frac{1}{2}\pi - \theta\right)$ and hence explain why $\alpha = \frac{1}{4}\pi$. **[3]**

(iii) Find the value of r when $\theta = \frac{1}{4}\pi$. **[1]**

(iv) Sketch the curve C. **[2]**

5 **(i)** Prove that, if $y = \sin^{-1} x$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{1-x^2}}$. **[3]**

(ii) Find the Maclaurin series for $\sin^{-1} x$, up to and including the term in x^3. **[5]**

(iii) Use the result of part **(ii)** and the Maclaurin series for $\ln(1+x)$ to find the Maclaurin series for $(\sin^{-1} x)\ln(1+x)$, up to and including the term in x^4. **[4]**

6 It is given that $I_n = \displaystyle\int_0^1 x^n(1-x)^{\frac{3}{2}}\,\mathrm{d}x$, for $n \geqslant 0$.

(i) Show that $I_n = \dfrac{2n}{2n+5} I_{n-1}$, for $n \geqslant 1$. **[6]**

(ii) Hence find the exact value of I_3. **[4]**

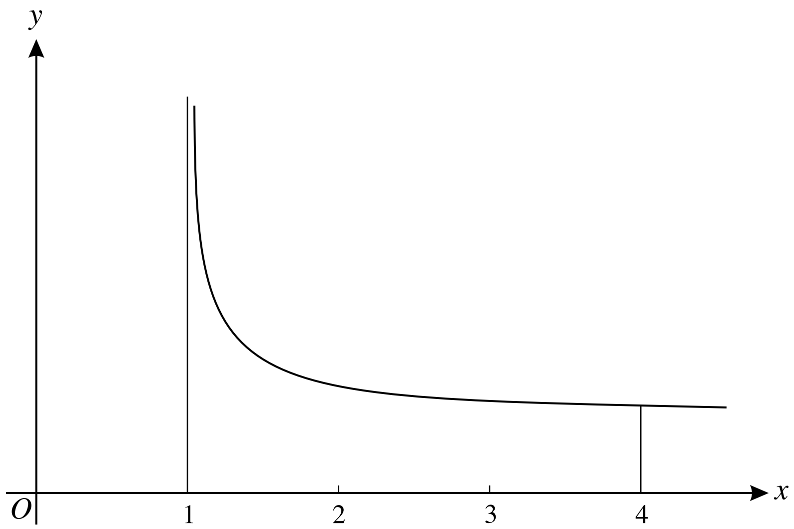
June 2011

7 **(i)** Sketch the graph of $y = \tanh x$ and state the value of the gradient when $x = 0$. On the same axes, sketch the graph of $y = \tanh^{-1} x$. Label each curve and give the equations of the asymptotes. **[4]**

(ii) Find $\int_0^k \tanh x \, dx$, where $k > 0$. **[2]**

(iii) Deduce, or show otherwise, that $\int_0^{\tanh k} \tanh^{-1} x \, dx = k \tanh k - \ln(\cosh k)$. **[4]**

8 **(i)** Use the substitution $x = \cosh^2 u$ to find $\int \sqrt{\frac{x}{x-1}} \, dx$, giving your answer in the form $f(x) + \ln(g(x))$. **[7]**



(ii) Hence calculate the exact area of the region between the curve $y = \sqrt{\frac{x}{x-1}}$, the x-axis and the lines $x = 1$ and $x = 4$ (see diagram). **[1]**

(iii) What can you say about the volume of the solid of revolution obtained when the region defined in part **(ii)** is rotated completely about the x-axis? Justify your answer. **[3]**

1 Given that $f(x) = \ln(\cos 3x)$, find $f'(0)$ and $f''(0)$. Hence show that the first term in the Maclaurin series for $f(x)$ is ax^2, where the value of a is to be found. **[4]**

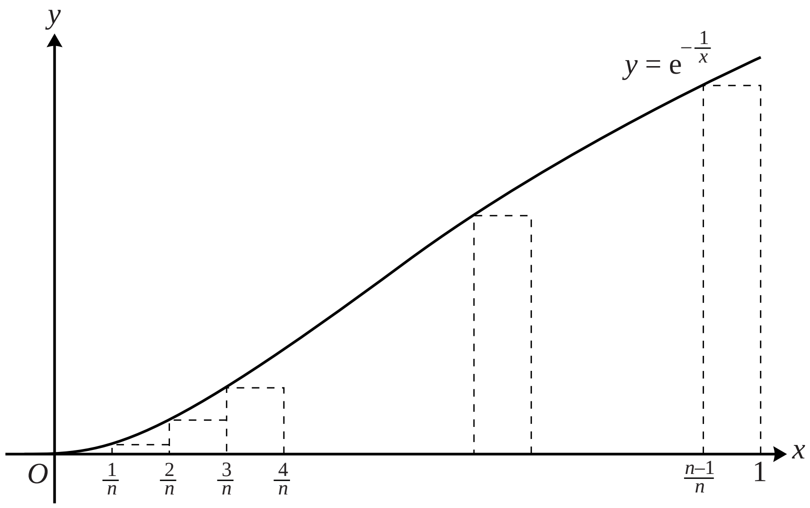
2 By first completing the square in the denominator, find the exact value of

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4x^2 - 4x + 5}\,dx\,.$$

[5]

3 Express $\dfrac{2x^3 + x + 12}{(2x-1)(x^2+4)}$ in partial fractions. **[7]**

4



The diagram shows the curve $y = e^{-\frac{1}{x}}$ for $0 < x \leqslant 1$. A set of $(n-1)$ rectangles is drawn under the curve as shown.

(i) Explain why a lower bound for $\int_0^1 e^{-\frac{1}{x}}\,dx$ can be expressed as

$$\frac{1}{n}\left(e^{-n} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + \ldots + e^{-\frac{n}{n-1}}\right).$$

[2]

(ii) Using a set of n rectangles, write down a similar expression for an upper bound for $\int_0^1 e^{-\frac{1}{x}}\,dx$. **[2]**

(iii) Evaluate these bounds in the case $n = 4$, giving your answers correct to 3 significant figures. **[2]**

(iv) When $n \geqslant N$, the difference between the upper and lower bounds is less than 0.001. By expressing this difference in terms of n, find the least possible value of N. **[3]**

Jan 2012

5 It is given that $f(x) = x^3 - k$, where $k > 0$, and that α is the real root of the equation $f(x) = 0$. Successive approximations to α, using the Newton-Raphson method, are denoted by $x_1, x_2, \ldots, x_n, \ldots$.

(i) Show that $x_{n+1} = \dfrac{2x_n^3 + k}{3x_n^2}$. **[2]**

(ii) Sketch the graph of $y = f(x)$, giving the coordinates of the intercepts with the axes. Show on your sketch how it is possible for $|\alpha - x_2|$ to be greater than $|\alpha - x_1|$. **[3]**

It is now given that $k = 100$ and $x_1 = 5$.

(iii) Write down the exact value of α and find x_2 and x_3 correct to 5 decimal places. **[3]**

(iv) The error e_n is defined by $e_n = \alpha - x_n$. By finding e_1, e_2 and e_3, verify that $e_3 \approx \dfrac{e_2^3}{e_1^2}$. **[3]**

6 **(i)** Prove that the derivative of $\cos^{-1}x$ is $-\dfrac{1}{\sqrt{1-x^2}}$. **[3]**

A curve has equation $y = \cos^{-1}(1 - x^2)$, for $0 < x < \sqrt{2}$.

(ii) Find and simplify $\dfrac{dy}{dx}$, and hence show that

$$\left(2 - x^2\right)\frac{d^2y}{dx^2} = x\frac{dy}{dx} .$$

[5]

7 **(i)** Given that $y = \sinh^{-1}x$, prove that $y = \ln\left(x + \sqrt{x^2+1}\right)$. **[3]**

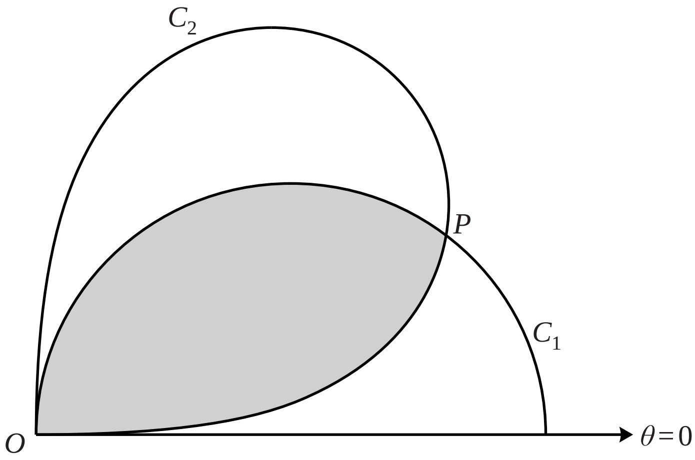
(ii) It is given that x satisfies the equation $\sinh^{-1}x - \cosh^{-1}x = \ln 2$. Use the logarithmic forms for $\sinh^{-1}x$ and $\cosh^{-1}x$ to show that

$$\sqrt{x^2+1} - 2\sqrt{x^2-1} = x .$$

Hence, by squaring this equation, find the exact value of x. **[5]**

[Questions 8 and 9 are printed overleaf.]

8



The diagram shows two curves, C_1 and C_2, which intersect at the pole O and at the point P. The polar equation of C_1 is $r = \sqrt{2}\cos\theta$ and the polar equation of C_2 is $r = \sqrt{2\sin 2\theta}$. For both curves, $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The value of θ at P is α.

(i) Show that $\tan\alpha = \frac{1}{2}$. **[2]**

(ii) Show that the area of the region common to C_1 and C_2, shaded in the diagram, is $\frac{1}{4}\pi - \frac{1}{2}\alpha$. **[7]**

9 **(i)** Show that $\tanh(\ln n) = \dfrac{n^2 - 1}{n^2 + 1}$. **[2]**

It is given that, for non-negative integers n, $I_n = \displaystyle\int_0^{\ln 2} \tanh^n u \, du$.

(ii) Show that $I_n - I_{n-2} = -\dfrac{1}{n-1}\left(\dfrac{3}{5}\right)^{n-1}$, for $n \geqslant 2$. **[3]**

(iii) Find the value of I_3, giving your answer in the form $a + \ln b$, where a and b are constants. **[4]**

(iv) Use the method of differences on the result of part **(ii)** to find the sum of the infinite series

$$\frac{1}{2}\left(\frac{3}{5}\right)^2 + \frac{1}{4}\left(\frac{3}{5}\right)^4 + \frac{1}{6}\left(\frac{3}{5}\right)^6 + \dots .$$

[2]

RECOGNISING ACHIEVEMENT

June 2012

1 Express $\operatorname{sech} 2x$ in terms of exponentials and hence, by using the substitution $u = e^{2x}$, find $\int \operatorname{sech} 2x \, dx$. **[5]**

2 A curve has polar equation $r = \cos\theta \sin 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Find

(i) the equations of the tangents at the pole, **[2]**

(ii) the maximum value of r, **[4]**

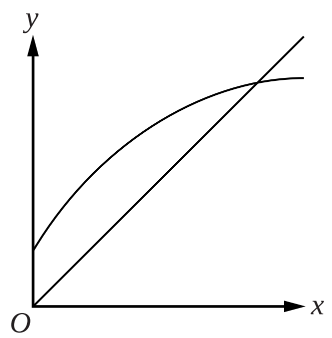
(iii) a cartesian equation of the curve, in a form not involving fractions. **[3]**

3 **(i)** By quoting results given in the List of Formulae (MF1), prove that $\tanh 2x \equiv \dfrac{2\tanh x}{1 + \tanh^2 x}$. **[2]**

(ii) Solve the equation $5\tanh 2x = 1 + 6\tanh x$, giving your answers in logarithmic form. **[6]**

4 It is given that the equation $x^4 - 2x - 1 = 0$ has only one positive root, α, and $1.3 < \alpha < 1.5$.

(i)



The diagram shows a sketch of $y = x$ and $y = \sqrt[4]{2x + 1}$ for $x \geqslant 0$. Use the iteration $x_{n+1} = \sqrt[4]{2x_n + 1}$ with $x_1 = 1.35$ to find x_2 and x_3, correct to 4 decimal places. On the copy of the diagram show how the iteration converges to α. **[3]**

(ii) For the same equation, the iteration $x_{n+1} = \frac{1}{2}\left(x_n^4 - 1\right)$ with $x_1 = 1.35$ gives $x_2 = 1.1608$ and $x_3 = 0.4077$, correct to 4 decimal places. Draw a sketch of $y = x$ and $y = \frac{1}{2}\left(x^4 - 1\right)$ for $x \geqslant 0$, and show how this iteration does not converge to α. **[2]**

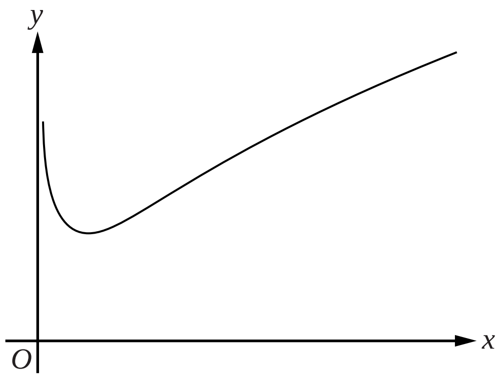
(iii) Find the positive root of the equation $x^4 - 2x - 1 = 0$ by using the Newton-Raphson method with $x_1 = 1.35$, giving the root correct to 4 decimal places. **[4]**

June 2012

5 A function is defined by $\mathrm{f}(x) = \sinh^{-1} x + \sinh^{-1}\left(\frac{1}{x}\right)$, for $x \neq 0$.

(i) When $x > 0$, show that the value of $\mathrm{f}(x)$ for which $\mathrm{f}'(x) = 0$ is $2\ln\left(1 + \sqrt{2}\right)$. **[5]**

(ii)



The diagram shows the graph of $y = \mathrm{f}(x)$ for $x > 0$. Sketch the graph of $y = \mathrm{f}(x)$ for $x < 0$ and state the range of values that $\mathrm{f}(x)$ can take for $x \neq 0$. **[3]**

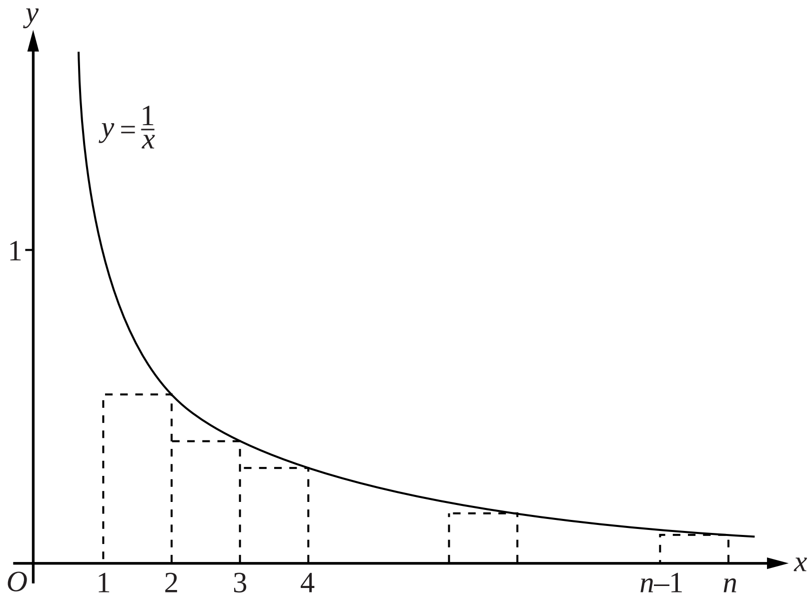
6 It is given that, for non-negative integers n,

$$I_n = \int_0^{\pi} x^n \sin x \, \mathrm{d}x.$$

(i) Prove that, for $n \geqslant 2$, $I_n = \pi^n - n(n-1)I_{n-2}$. **[5]**

(ii) Find I_5 in terms of π. **[4]**

7



The diagram shows the curve $y = \frac{1}{x}$ for $x > 0$ and a set of $(n - 1)$ rectangles of unit width below the curve. These rectangles can be used to obtain an inequality of the form

$$\frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+2} + \dots + \frac{1}{b} < \int_1^n \frac{1}{x}\,dx.$$

Another set of rectangles can be used similarly to obtain

$$\int_1^n \frac{1}{x}\,dx < \frac{1}{c} + \frac{1}{c+1} + \frac{1}{c+2} + \dots + \frac{1}{d}\,.$$

(i) Write down the values of the constants a and c, and express b and d in terms of n. **[3]**

The function f is defined by $f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$, for positive integers n.

(ii) Use your answers to part **(i)** to obtain upper and lower bounds for $f(n)$. **[4]**

(iii) By using the first 2 terms of the Maclaurin series for $\ln(1 + x)$ show that, for large n,

$$f(n+1) - f(n) \approx -\frac{n-1}{2n^2(n+1)}\,.$$ **[5]**

June 2012

8 The curve C_1 has equation $y = \frac{\mathrm{p}(x)}{\mathrm{q}(x)}$, where p($x$) and q($x$) are polynomials of degree 2 and 1 respectively. The asymptotes of the curve are $x = -2$ and $y = \frac{1}{2}x + 1$, and the curve passes through the point $\left(-1, \frac{17}{2}\right)$.

(i) Express the equation of C_1 in the form $y = \frac{\mathrm{p}(x)}{\mathrm{q}(x)}$. **[4]**

(ii) For the curve C_1, find the range of values that y can take. **[4]**

Another curve, C_2, has equation $y^2 = \frac{\mathrm{p}(x)}{\mathrm{q}(x)}$, where p($x$) and q($x$) are the polynomials found in part **(i)**.

(iii) It is given that C_2 intersects the line $y = \frac{1}{2}x + 1$ exactly once. Find the coordinates of the point of intersection. **[4]**

Jan 2013

1 Express $\dfrac{5x}{(x-1)(x^2+4)}$ in partial fractions. **[5]**

2 The equation of a curve is $y = \dfrac{x^2-3}{x-1}$.

(i) Find the equations of the asymptotes of the curve. **[3]**

(ii) Write down the coordinates of the points where the curve cuts the axes. **[1]**

(iii) Show that the curve has no stationary points. **[3]**

(iv) Sketch the curve and the asymptotes. **[3]**

3 By first expressing $\cosh x$ and $\sinh x$ in terms of exponentials, solve the equation

$$3\cosh x - 4\sinh x = 7,$$

giving your answer in an exact logarithmic form. **[6]**

4 You are given that $I_n = \int_0^1 x^n e^{2x}\,dx$ for $n \geqslant 0$.

(i) Show that $I_n = \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}$ for $n \geqslant 1$. **[4]**

(ii) Find I_3 in terms of e. **[4]**

5 You are given that $f(x) = e^{-x}\sin x$.

(i) Find $f(0)$ and $f'(0)$. **[3]**

(ii) Show that $f''(x) = -2f'(x) - 2f(x)$ and hence, or otherwise, find $f''(0)$. **[4]**

(iii) Find a similar expression for $f'''(x)$ and hence, or otherwise, find $f'''(0)$. **[2]**

(iv) Find the Maclaurin series for $f(x)$ up to and including the term in x^3. **[2]**

Jan 2013

6 By first completing the square, find $\int_0^1 \frac{1}{\sqrt{x^2+4x+8}}\,dx$, giving your answer in an exact logarithmic form. **[6]**

7 A curve has polar equation $r = 5\sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Sketch the curve, indicating the line of symmetry and stating the polar coordinates of the point P on the curve which is furthest away from the pole. **[4]**

(ii) Calculate the area enclosed by the curve. **[3]**

(iii) Find the cartesian equation of the tangent to the curve at P. **[3]**

(iv) Show that a cartesian equation of the curve is $(x^2+y^2)^3 = (10xy)^2$. **[3]**

8 It is required to solve the equation $\ln(x-1) - x + 3 = 0$.

You are given that there are two roots, α and β, where $1.1 < \alpha < 1.2$ and $4.1 < \beta < 4.2$.

(i) The root β can be found using the iterative formula

$$x_{n+1} = \ln(x_n - 1) + 3.$$

(a) Using this iterative formula with $x_1 = 4.15$, find β correct to 3 decimal places. Show all your working. **[2]**

(b) Explain with the aid of a sketch why this iterative formula will not converge to α whatever initial value is taken. **[3]**

(ii) (a) Show that the Newton-Raphson iterative formula for this equation can be written in the form

$$x_{n+1} = \frac{3 - 2x_n - (x_n - 1)\ln(x_n - 1)}{2 - x_n}.$$ **[5]**

(b) Use this formula with $x_1 = 1.2$ to find α correct to 3 decimal places. **[3]**

June 2013

1 By using the substitution $t = \tan\frac{1}{2}\theta$, find $\int_0^{\frac{1}{2}\pi} \frac{1}{1+\cos\theta}\,d\theta$. **[5]**

2 **(i)** Using the definitions for $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}, show that $\cosh^2 x - \sinh^2 x \equiv 1$. **[3]**

(ii) Hence solve the equation $\sinh^2 x = 5\cosh x - 7$, giving your answers in logarithmic form. **[5]**

3 It is given that $f(x) = \tanh^{-1}\left(\frac{1-x}{3+x}\right)$ for $x > -1$.

(i) Show that $f''(x) = \frac{1}{2(x+1)^2}$. **[6]**

(ii) Hence find the Maclaurin series for $f(x)$ up to and including the term in x^2. **[4]**

4 It is given that $I_n = \int_0^{\frac{1}{2}\pi} \cos^n x\,dx$ for $n \geqslant 0$.

(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geqslant 2$. **[5]**

(ii) Hence find I_{11} as an exact fraction. **[3]**

OCR
RECOGNISING ACHIEVEMENT

June 2013

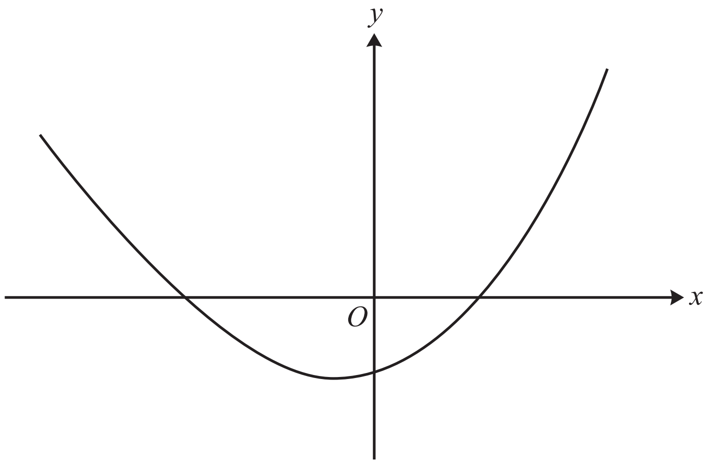
5 You are given that the equation $x^3 + 4x^2 + x - 1 = 0$ has a root, α, where $-1 < \alpha < 0$.

(i) Show that the Newton-Raphson iterative formula for this equation can be written in the form

$$x_{n+1} = \frac{2x_n^{\ 3} + 4x_n^{\ 2} + 1}{3x_n^{\ 2} + 8x_n + 1}.$$ **[3]**

(ii) Using the initial value $x_1 = -0.7$, find x_2 and x_3 and find α correct to 5 decimal places. **[3]**

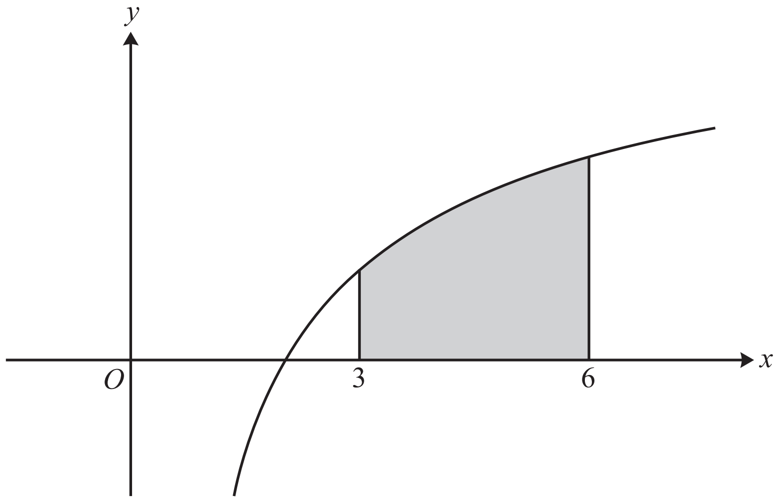
(iii) The diagram shows a sketch of the curve $y = x^3 + 4x^2 + x - 1$ for $-1.5 \leqslant x \leqslant 1$.



Using the copy of the diagram in your answer book, explain why the initial value $x_1 = 0$ will fail to find α. **[2]**

[Questions 6, 7 and 8 are printed overleaf.]

6



The diagram shows part of the curve $y = \ln(\ln(x))$. The region between the curve and the x-axis for $3 \leqslant x \leqslant 6$ is shaded.

(i) By considering n rectangles of equal width, show that a lower bound, L, for the area of the shaded region is $\frac{3}{n}\sum_{r=0}^{n-1} \ln\left(\ln\left(3 + \frac{3r}{n}\right)\right)$. **[3]**

(ii) By considering another set of n rectangles of equal width, find a similar expression for an upper bound, U, for the area of the shaded region. **[1]**

(iii) Find the least value of n for which $U - L < 0.001$. **[4]**

7 The equation of a curve is $y = \dfrac{x^2 + 1}{(x+1)(x-7)}$.

(i) Write down the equations of the asymptotes. **[3]**

(ii) Find the coordinates of the stationary points on the curve. **[5]**

(iii) Find the coordinates of the point where the curve meets one of its asymptotes. **[3]**

(iv) Sketch the curve. **[3]**

8 The equation of a curve is $x^2 + y^2 - x = \sqrt{x^2 + y^2}$.

(i) Find the polar equation of this curve in the form $r = \mathrm{f}(\theta)$. **[3]**

(ii) Sketch the curve. **[2]**

(iii) The line $x + 2y = 2$ divides the region enclosed by the curve into two parts. Find the ratio of the two areas. **[6]**

1 Find $\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx$, giving your answer exactly in logarithmic form. **[3]**

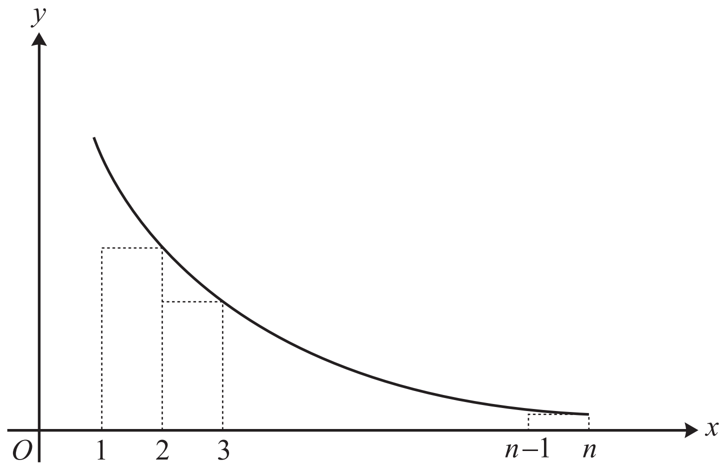
2 It is given that $f(x) = \ln(1+x^2)$.

(i) Using the standard Maclaurin expansion for $\ln(1+x)$, write down the first four terms in the expansion of $f(x)$, stating the set of values of x for which the expansion is valid. **[3]**

(ii) Hence find the exact value of

$$1 - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^4 - \frac{1}{4}\left(\frac{1}{2}\right)^6 + \ldots .$$ **[2]**

3 The diagram shows the curve $y = \frac{1}{x^3}$ for $1 \leqslant x \leqslant n$ where n is an integer. A set of $(n-1)$ rectangles of unit width is drawn under the curve.



(i) Write down the sum of the areas of the rectangles. **[2]**

(ii) Hence show that $\sum_{r=1}^{\infty} \frac{1}{r^3} < \frac{3}{2}$. **[5]**

4 The curves $y = \cos^{-1}x$ and $y = \tan^{-1}(\sqrt{2}x)$ intersect at a point A.

(i) Verify that the coordinates of A are $\left(\frac{1}{\sqrt{2}}, \frac{1}{4}\pi\right)$. **[2]**

(ii) Determine whether the tangents to the curves at A are perpendicular. **[4]**

June 2014

5 A curve has equation $y = \dfrac{x^2 - 8}{x - 3}$.

(i) Find the equations of the asymptotes of the curve. **[3]**

(ii) Prove that there are no points on the curve for which $4 < y < 8$. **[4]**

(iii) Sketch the curve. Indicate the asymptotes in your sketch. **[2]**

6 **(i)** Given that $y = \cosh^{-1} x$, show that $y = \ln\left(x + \sqrt{x^2 - 1}\right)$. **[4]**

(ii) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{x^2 - 1}}$. **[2]**

(iii) Solve the equation $\cosh x = 3$, giving your answers in logarithmic form. **[3]**

7 It is given that, for non-negative integers n, $I_n = \int_0^{\frac{1}{2}\pi} \sin^n x \, \mathrm{d}x$.

(i) Show that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geqslant 2$. **[3]**

(ii) Explain why $I_{2n+1} < I_{2n-1}$. **[2]**

(iii) It is given that $I_{2n+1} < I_{2n} < I_{2n-1}$. Take $n = 5$ to find an interval within which the value of π lies. **[6]**

8 A curve has polar equation $r = a(1 + \cos\theta)$, where a is a positive constant and $0 \leqslant \theta < 2\pi$.

(i) Find the equation of the tangent at the pole. **[2]**

(ii) Sketch the curve. **[2]**

(iii) Find the area enclosed by the curve. **[6]**

9 The equation $10x - 8\ln x = 28$ has a root α in the interval [3, 4]. The iteration $x_{n+1} = \mathrm{g}(x_n)$, where $\mathrm{g}(x) = 2.8 + 0.8\ln x$ and $x_1 = 3.8$, is to be used to find α.

(i) Find the value of α correct to 5 decimal places. You should show the result of each step of the iteration to 6 decimal places. **[4]**

(ii) Illustrate this iteration by means of a sketch. **[2]**

(iii) The difference, δ_r, between successive approximations is given by $\delta_r = x_{r+1} - x_r$. Find δ_3. **[2]**

(iv) Given that $\delta_{n+1} \approx \mathrm{g}'(\alpha)\delta_n$, for all positive integers n, estimate the smallest value of n such that $\delta_n < 10^{-6}\delta_1$. **[4]**

END OF QUESTION PAPER