

1.

$$\frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 2 \cos x$$

(a) Find $\frac{d^3 y}{dx^3}$ in terms of x , $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$.

(3)

At $x = 0$, $y = 1$ and $\frac{dy}{dx} = 3$

(b) Find the value of $\frac{d^3 y}{dx^3}$ at $x = 0$

(1)

(c) Express y as a series in ascending powers of x , up to and including the term in x^3 .

(3)

$$a) \frac{d}{dx} \left(\frac{d^2 y}{dx^2} \right) + \frac{d}{dx} \left(x \frac{dy}{dx} \right) = \frac{d}{dx} (2 \cos x) \Rightarrow \frac{d^3 y}{dx^3} + x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = -2 \sin x$$

$$\therefore \frac{d^3 y}{dx^3} = -x \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2 \sin x \quad y''' = -x y'' - y' - 2 \sin x$$

b) $x=0 \quad y_0=1 \quad y'_0=3$

$$y'' = -x y' + 2 \cos x \Rightarrow y''_0 = 0 + 2 \cos(0) = 2 \quad \therefore y''_0 = 2$$

$$y''' = -x y'' - y' - 2 \sin x \Rightarrow y'''_0 = 0 - 3 - 2 \sin(0) \quad \therefore y'''_0 = -3$$

c) $y = y_0 + y'_0 x + \frac{y''_0 x^2}{2} + \frac{y'''_0 x^3}{6}$

$$y = 1 + 3x + x^2 - \frac{1}{2} x^3$$

✓

2. (a) Sketch, on the same axes,

(i) $y = |2x - 3|$

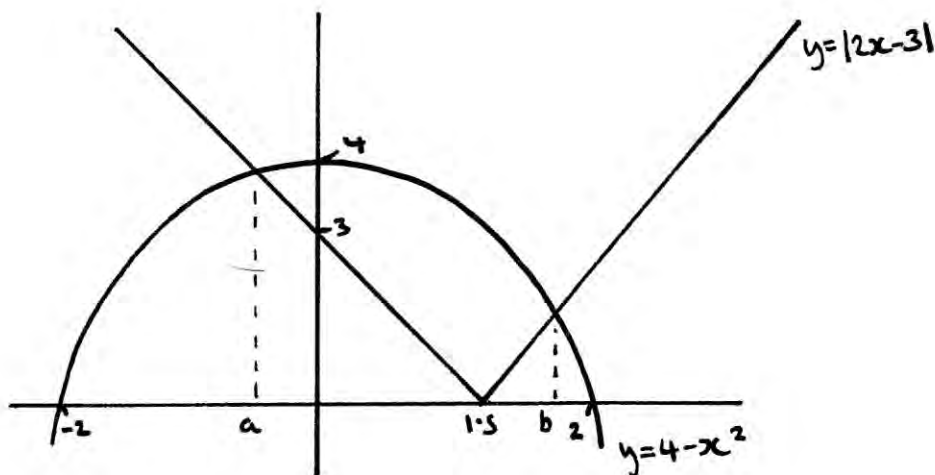
(ii) $y = 4 - x^2$

(3)

(b) Find the set of values of x for which

$$4 - x^2 > |2x - 3|$$

(6)



$$|2x - 3| = 4 - x^2$$

$$2x - 3 = 4 - x^2$$

$$x^2 + 2x - 7 = 0$$

$$(x+1)^2 = 8$$

$$x = -1 \pm 2\sqrt{2}$$

$$b = \underline{1.83}, -3.83$$

$$2x - 3 = x^2 - 4$$

$$x^2 - 2x - 1 = 0$$

$$(x-1)^2 = 2$$

$$x - 1 = \pm\sqrt{2}$$

$$x = 1 \pm \sqrt{2}$$

$$2.41, -0.41 = a$$

$$a = 1 - \sqrt{2}$$

$$b = -1 + 2\sqrt{2}$$

$$4 - x^2 > |2x - 3|$$

↪ smaller

$$\text{when } 1 - \sqrt{2} < x < -1 + 2\sqrt{2}$$

3.

$$f(x) = \ln(1 + \sin kx)$$

where k is a constant, $x \in \mathbb{R}$ and $-\frac{\pi}{2} < kx < \frac{3\pi}{2}$

(a) Find $f'(x)$.

(2)

(b) Show that $f''(x) = \frac{-k^2}{1 + \sin kx}$

(3)

(c) Find the Maclaurin series of $f(x)$, in ascending powers of x , up to and including the term in x^3 .

(4)

$$f(x) = \ln(1 + \sin kx)$$

$$f'(x) = \frac{k \cosh kx}{1 + \sin kx}$$

$$\begin{aligned} u &= k \cosh kx & v &= 1 + \sin kx \\ u' &= -k^2 \sin kx & v' &= k \cosh kx \end{aligned}$$

$$f''(x) = \frac{(1 + \sin kx)(-k^2 \sin kx) - (k \cosh kx)(k \cosh kx)}{(1 + \sin kx)^2}$$

$$= \frac{-k^2 \sin kx - k^2 \sin^2 kx - k^2 \cosh^2 kx}{(1 + \sin kx)^2}$$

$$= \frac{-k^2 (\sin kx + \sin^2 kx + \cosh^2 kx)}{(1 + \sin kx)^2} = \frac{-k^2 (\cancel{\sin kx} + 1)}{(1 + \sin kx)^2}$$

$$\therefore f''(x) = \frac{-k^2}{1 + \sin kx} \quad \text{tr}$$

$$\begin{aligned} \text{c) } f''(x) &= -k^2 (1 + \sin kx)^{-1} \Rightarrow f'''(x) = k^2 (1 + \sin kx)^{-2} \times k \cosh kx \\ &= \frac{k^3 \cosh kx}{(1 + \sin kx)^2} \end{aligned}$$

$$f(0) = \ln 1 = 0$$

$$f'(0) = k$$

$$f''(0) = -k^2$$

$$f'''(0) = k^3$$

$$\therefore f(x) = kx - \frac{k^2 x^2}{2} + \frac{k^3 x^3}{6} \dots$$

4. Find the general solution of the differential equation

$$x \frac{dy}{dx} + (1 + x \cot x)y = \sin x, \quad 0 < x < \pi$$

giving your answer in the form $y = f(x)$.

(9)

$$\frac{dy}{dx} + \left(\frac{1+x \cot x}{x} \right) y = \frac{\sin x}{x}$$

$$\text{IF} \Rightarrow f(x) = e^{\int \frac{1+x \cot x}{x} dx} = e^{\int \frac{1}{x} + \cot x dx}$$

$$= e^{\ln x + \ln |\sin x|} = e^{\ln x} \times e^{\ln |\sin x|} = x \sin x$$

$$\Rightarrow x \sin x \frac{dy}{dx} + x \sin x \left(\frac{1+x \cot x}{x} \right) y = \cancel{x} \sin x \times \frac{\sin x}{\cancel{x}}$$

$$\Rightarrow \frac{d}{dx} (x \sin x \cdot y) = \sin^2 x \quad \Rightarrow x \sin x \cdot y = \int \frac{1}{2} - \frac{1}{2} \cos 2x dx$$

$$\cos 2x = 1 - 2 \sin^2 x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\Rightarrow x \sin x \cdot y = \frac{1}{2} x - \frac{1}{4} \sin 2x + C$$

$$\therefore y = \frac{1}{2 \sin x} - \frac{\sin 2x}{4x \sin x} + \frac{C}{x \sin x}$$

$$\Rightarrow y = \frac{1}{2} \operatorname{cosec} x - \frac{\cos x}{2x} + \frac{C}{x \sin x}$$

2

5. (a) Express $\frac{2}{r(r+1)(r+2)}$ in partial fractions.

(3)

(b) Using your answer to part (a) and the method of differences, show that

$$\sum_{r=1}^n \frac{2}{r(r+1)(r+2)} = \frac{n(n+3)}{2(n+1)(n+2)} \quad (4)$$

$$a) \frac{2}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2} \Rightarrow 2 = A(r+1)(r+2) + B(r)(r+2) + C(r)(r+1)$$

$$r=0 \Rightarrow 2=2A \therefore \underline{A=1} \quad r=-1 \quad 2=-B \therefore \underline{B=-2} \quad r=-2 \Rightarrow 2=+2C \therefore \underline{C=+1}$$

$$\therefore = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$$

$$b) \begin{array}{l} r=1 \quad \cancel{\frac{1}{1}} - \cancel{\frac{2}{2}} + \frac{1}{3} \\ r=2 \quad \frac{1}{2} - \cancel{\frac{2}{3}} + \cancel{\frac{1}{4}} \\ r=3 \quad \cancel{\frac{1}{3}} - \cancel{\frac{2}{4}} + \cancel{\frac{1}{5}} \\ r=4 \quad \cancel{\frac{1}{4}} - \cancel{\frac{2}{5}} + \cancel{\frac{1}{6}} \\ \vdots \\ r=n-2 \quad \cancel{\frac{1}{n-2}} - \cancel{\frac{2}{n-1}} + \cancel{\frac{1}{n}} \\ r=n-1 \quad \cancel{\frac{1}{n-1}} - \cancel{\frac{2}{n}} + \frac{1}{n+1} \\ r=n \quad \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} \end{array}$$

$$\begin{aligned} & \sum_{r=1}^n \frac{2}{r(r+1)(r+2)} \\ &= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2} \\ &= \frac{(n+1)(n+2) - (2)(n+2) + 1(2)(n+1)}{2(n+1)(n+2)} \\ &= \frac{n^2+3n+2-2n-4+2n+2}{2(n+1)(n+2)} \\ &= \frac{n^2+3n}{2(n+1)(n+2)} \\ &= \frac{n(n+3)}{2(n+1)(n+2)} \quad \# \end{aligned}$$

6. Solve the equation

$$z^5 = -16\sqrt{3} + 16i$$

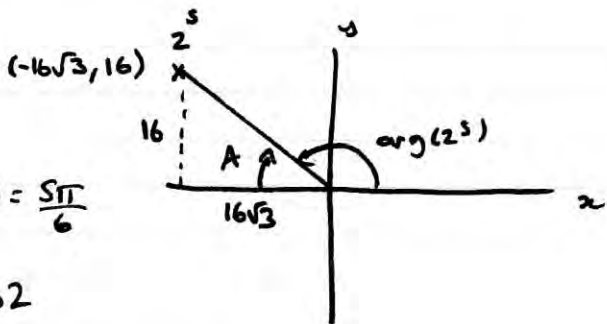
giving your answers in the form $r(\cos \theta + i \sin \theta)$, where $r > 0$ and $-\pi < \theta < \pi$.

(8)

$$\tan A = \frac{16}{16\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\therefore A = \frac{\pi}{6} \quad \therefore \arg(z^5) = \frac{5\pi}{6}$$

$$r = \sqrt{16^2 + (16\sqrt{3})^2} = 32$$



$$\therefore z^5 = 32 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$$

$$\Rightarrow z = 32^{\frac{1}{5}} \left(\cos \left(\frac{5\pi}{6} + 2k\pi \right) + i \sin \left(\frac{5\pi}{6} + 2k\pi \right) \right)^{\frac{1}{5}}$$

$$\Rightarrow z = 2 \left[\cos \left(\frac{12k+5}{6} \pi \right) + i \sin \left(\frac{12k+5}{6} \pi \right) \right]^{\frac{1}{5}}$$

$$\Rightarrow z = 2 \left[\cos \left(\frac{12k+5}{30} \pi \right) + i \sin \left(\frac{12k+5}{30} \pi \right) \right]$$

$$k = -2 \quad z = 2 \left(\cos \frac{-19\pi}{30} + i \sin \frac{-19\pi}{30} \right)$$

$$k = -1 \quad z = 2 \left(\cos \frac{-7\pi}{30} + i \sin \frac{-7\pi}{30} \right)$$

$$k = 0 \quad z = 2 \left(\cos \frac{5\pi}{30} + i \sin \frac{5\pi}{30} \right)$$

$$k = 1 \quad z = 2 \left(\cos \frac{17\pi}{30} + i \sin \frac{17\pi}{30} \right)$$

$$k = 2 \quad z = 2 \left(\cos \frac{29\pi}{30} + i \sin \frac{29\pi}{30} \right)$$

7. (a) Find the value of the constant λ for which $y = \lambda x e^{2x}$ is a particular integral of the differential equation

$$\frac{d^2 y}{dx^2} - 4y = 6e^{2x}$$

(4)

- (b) Hence, or otherwise, find the general solution of the differential equation

$$\frac{d^2 y}{dx^2} - 4y = 6e^{2x}$$

(3)

a) $y = \lambda x e^{2x}$

$$y' = 2\lambda x e^{2x} + \lambda e^{2x} = (2\lambda x + \lambda) e^{2x}$$

$$y'' = 2(2\lambda x + \lambda) e^{2x} + 2\lambda e^{2x} = (4\lambda x + 4\lambda) e^{2x}$$

$$y'' - 4y = 6e^{2x} \Rightarrow (4\cancel{\lambda x} + 4\lambda - 4\cancel{\lambda x}) e^{2x} = 6e^{2x}$$

$$\therefore 4\lambda = 6 \quad \lambda = \frac{3}{2}$$

$$y_{PI} = \frac{3}{2} x e^{2x}$$

b) $y = A e^{mt} \Rightarrow A m^2 e^{mt} - 4 A e^{mt} = 0$
 $y' = A m e^{mt}$
 $y'' = A m^2 e^{mt}$
 $A e^{mt} (m^2 - 4) = 0$
 $\neq 0 \Rightarrow m = \pm 2$

$$\therefore y_{ct} = A e^{2x} + B e^{-2x}$$

$$\therefore y_{as} = A e^{2x} + B e^{-2x} + \frac{3}{2} x e^{2x}$$

$$= (A + \frac{3}{2} x) e^{2x} + B e^{-2x}$$

8. A complex number z is represented by the point P on an Argand diagram.

(a) Given that $|z| = 1$, sketch the locus of P .

(1)

The transformation T from the z -plane to the w -plane is given by

$$w = \frac{z + 7i}{z - 2i}$$

(b) Show that T maps $|z| = 1$ onto a circle in the w -plane.

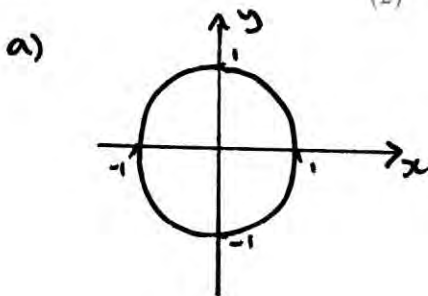
(5)

(c) Show that this circle has its centre at $w = -5$ and find its radius.

(2)

b)

$$\begin{aligned}wz - 2wi &= z + 7i \\wz - z &= 7i + 2wi \\z(w - 1) &= 7i + 2wi \\|z||w - 1| &= |7i + 2wi| \\|w - 1| &= |7 + 2wi| \\|(u-1) + iv| &= |(2u+7) + 2iv| \\(u-1)^2 + v^2 &= (2u+7)^2 + 4v^2 \\u^2 - 2u + 1 + v^2 &= 4u^2 + 28u + 49 + 4v^2\end{aligned}$$



$$3u^2 + 30u + 3v^2 = -48$$

$$\div 3 \quad u^2 + 10u + v^2 = -16$$

$$(u + 5)^2 + v^2 = -16 + 25 = 9$$

$$\therefore (u + 5)^2 + v^2 = 3^2$$

T maps $|z| = 1$ to a circle

$C(-5, 0)$ $r = 3$ in the w -plane

9.

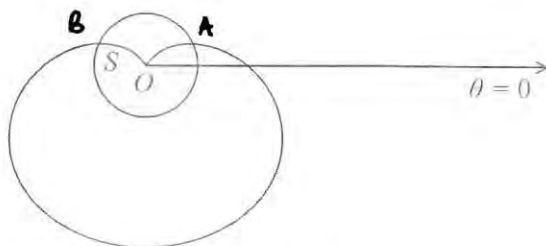


Figure 1

Figure 1 shows a sketch of the curves given by the polar equations

$$r = 1 \text{ and } r = 2 - 2 \sin \theta$$

(a) Find the coordinates of the points where the curves intersect.

(3)

The region S , between the curves, for which $r < 1$ and for which $r < 2 - 2 \sin \theta$, is shown shaded in Figure 1.

(b) Find, by integration, the area of the shaded region S , giving your answer in the form $a\pi + b\sqrt{3}$, where a and b are rational numbers.

(8)

$$1 = 2 - 2 \sin \theta \Rightarrow 2 \sin \theta = 1 \Rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A(1, \frac{\pi}{6}) \quad B(1, \frac{5\pi}{6})$$

b)

$$\text{Area} = 2 \times \left[\frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2 - 2 \sin \theta)^2 d\theta + \frac{1}{2} r^2 \left(\frac{2\pi}{3} \right) \right]$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 4 - 8 \sin \theta + 4 \sin^2 \theta d\theta + \frac{2\pi}{3}$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 6 - 8 \sin \theta - 2 \cos 2\theta d\theta + \frac{2\pi}{3}$$

$$= \left[6\theta + 8 \cos \theta - \sin(2\theta) \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} + \frac{2\pi}{3} = [(3\pi + 0 + 0) - (\pi + 4\sqrt{3} - \frac{\sqrt{3}}{2})] + \frac{2\pi}{3}$$

$$= \left(2\pi - \frac{7\sqrt{3}}{2} \right) + \frac{2\pi}{3} = \frac{8\pi}{3} - \frac{7\sqrt{3}}{2}$$