Pure
Further Mathematics 1

Revision Notes

June 2016

Further Pure 1

1 Complex Numbers

Definitions and arithmetical operations

$i = \sqrt{-1}$, so $\sqrt{-16} = 4i$, $\sqrt{-11} = \sqrt{11}\,i$, etc.
These are called *imaginary* numbers

Complex numbers are written as $z = a + bi$, where a and $b \in \mathbb{R}$.
a is the *real part* and b is the *imaginary part*.

+, –, × are defined in the 'sensible' way; division is more complicated.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$
$$(a + bi) \times (c + di) = ac + bdi^2 + adi + bci$$
$$= (ac - bd) + (ad + bc)i \qquad \text{since } i^2 = -1$$

So $(3 + 4i) - (7 - 3i) = -4 + 7i$
and $(4 + 3i)(2 - 5i) = 23 - 14i$

Division – this is just rationalising the denominator.

$$\frac{3+4i}{5+2i} = \frac{3+4i}{5+2i} \times \frac{5-2i}{5-2i} \qquad \text{multiply top and bottom by the complex conjugate}$$

$$= \frac{23+14i}{25+4} = \frac{23}{29} + \frac{14}{29}i$$

Complex conjugate

$z = a + bi$
The *complex conjugate* of z is $z^* = \overline{z} = a - bi$

Properties

If $z = a + bi$ and $w = c + di$, then

(i) $\{(a + bi) + (c + di)\}^* = \{(a + c) + (b + d)i\}^*$

$$= \{(a + c) - (b + d)i\}$$

$$= (a - bi) + (c - di)$$

$\Leftrightarrow \quad (z + w)^* = z^* + w^*$

(ii) $\{(a + bi)(c + di)\}^*$ $= \{(ac - bd) + (ad + bc)i\}^*$

$= \{(ac - bd) - (ad + bc)i\}$

$= (a - bi)(c - di)$

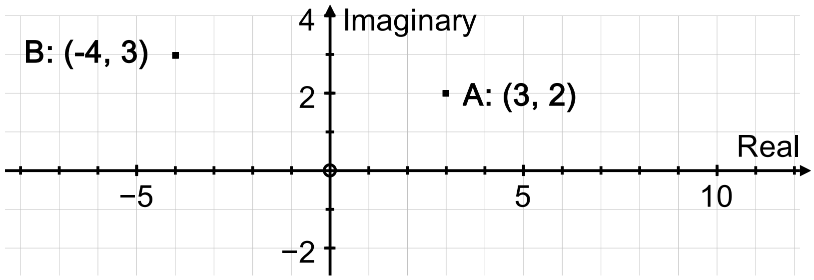
$= (a + bi)^*(c + di)^*$

$\Leftrightarrow$ $(zw)^* = z^* w^*$

Complex number plane, or Argand diagram

We can represent complex numbers as points on the complex number plane:

$3 + 2i$ as the point $A\,(3, 2)$, and $-4 + 3i$ as the point $(-4, 3)$.



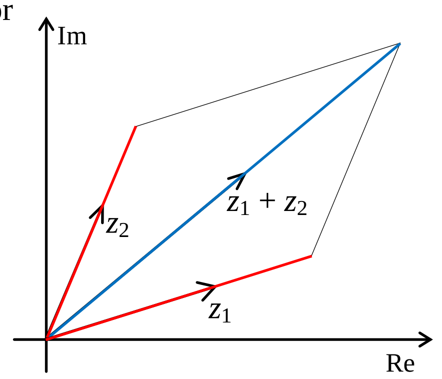
Complex numbers and vectors

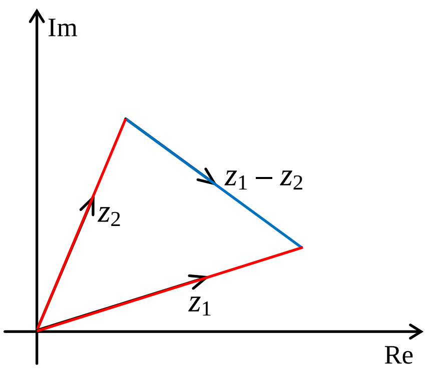
Complex numbers under addition (or subtraction) behave just like vectors under addition (or subtraction). We can show complex numbers on the Argand diagram as **either** points **or** vectors.

$(a + bi) + (c + di) = (a + c) + (b + d)\,i \quad \Leftrightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$

$(a + bi) - (c + di) = (a - c) + (b - d)\,i \quad \Leftrightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a - c \\ b - d \end{pmatrix}$

or



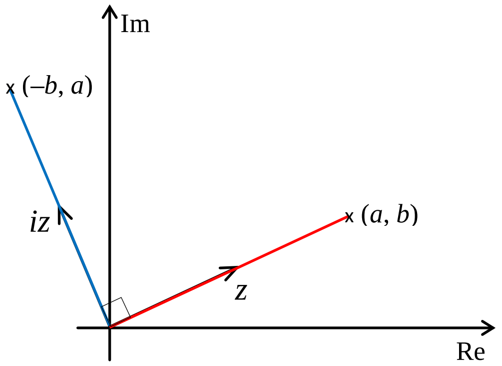


Multiplication by *i*

$i(3 + 4i) = -4 + 3i$ – on an Argand diagram this would have the effect of a positive quarter turn about the origin.

In general;

$i(a + bi) = -b + ai$

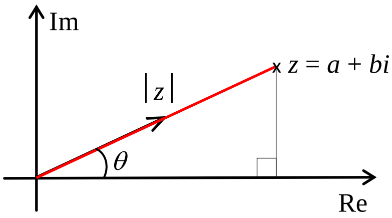


Modulus of a complex number

This is just like polar co-ordinates.

The modulus of z is $|z|$ and

is the length of the complex number



$$|z| = \sqrt{a^2 + b^2}.$$

$$z\,z^* = (a + bi)(a - bi) = a^2 + b^2$$

$$\Rightarrow \quad z\,z^* = |z|^2.$$

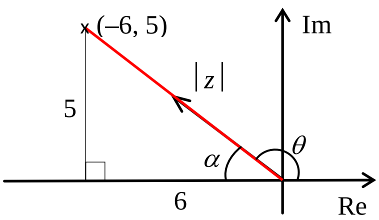
Argument of a complex number

The argument of z is $\arg z$ = the angle made by the complex number with the positive x-axis.
By convention, $-\pi < \arg z \le \pi$.

N.B. **Always draw a diagram when finding arg *z*.**

Example: Find the modulus and argument of $z = -6 + 5i$.

Solution: First sketch a diagram (it is easy to get the argument wrong if you don't).



$$|z| = \sqrt{6^2 + 5^2} = \sqrt{61}$$

and $\tan\alpha = \frac{5}{6} \Rightarrow \alpha = 0{\cdot}694738276$

$\Rightarrow \quad \arg z = \theta = \pi - \alpha = 2.45$ to 3 S.F.

Equality of complex numbers

$$a + bi = c + di \quad \Rightarrow \quad a - c = (d - b)i$$

$\Rightarrow \quad (a - c)^2 = (d - b)^2 i^2 = -(d - b)^2$ squaring both sides

But $(a - c)^2 \geq 0$ and $-(d - b)^2 \leq 0$

$\Rightarrow \quad (a - c)^2 = -(d - b)^2 = 0$

$\Rightarrow \quad a = c$ and $b = d$

Thus $a + bi = c + di$

$\Rightarrow$ real parts are equal $(a = c)$, and imaginary parts are equal $(b = d)$.

Square roots

Example: Find the square roots of $5 + 12i$, in the form $a + bi$, $a, b \in \mathbb{R}$.

Solution: Let $\sqrt{5+12i} = a + bi$

$\Rightarrow \quad 5 + 12i = (a + bi)^2 = a^2 - b^2 + 2abi$

Equating real parts $\Rightarrow \quad a^2 - b^2 = 5$, **I**

equating imaginary parts $\Rightarrow \quad 2ab = 12 \quad \Rightarrow \quad a = \frac{6}{b}$

Substitute in **I** $\Rightarrow \quad \left(\frac{6}{b}\right)^2 - b^2 = 5$

$\Rightarrow \quad 36 - b^4 = 5b^2 \quad \Rightarrow \quad b^4 + 5b^2 - 36 = 0$

$\Rightarrow \quad (b^2 - 4)(b^2 + 9) = 0 \quad \Rightarrow \quad b^2 = 4$

$\Rightarrow \quad b = \pm 2$, and $a = \pm 3$

$\Rightarrow \quad \sqrt{5+12i} = 3 + 2i$ or $-3 - 2i$.

Roots of equations

(*a*) **Any** polynomial equation with complex coefficients has a complex solution.

The is *The Fundamental Theorem of Algebra*, and is too difficult to prove at this stage.

Corollary: Any complex polynomial can be factorised into linear factors over the complex numbers.

(b) **If** $z = a + bi$ is a root of $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$,
and if all the α_i are **real**,
then the conjugate, $z^* = a - bi$ is also a root.

The proof of this result is in the appendix.

(c) For any polynomial with zeros $a + bi$, $a - bi$,
$(z - (a + bi))(z - (a - bi)) = z^2 - 2az + a^2 - b^2$ will be a quadratic factor in which the coefficients are all **real**.

(d) Using (a), (b), (c) we can see that any polynomial with **real** coefficients can be factorised into a mixture of linear and quadratic factors, all of which have **real** coefficients.

Example: Show that $3 - 2i$ is a root of the equation $z^3 - 8z^2 + 25z - 26 = 0$.
Find the other two roots.

Solution: Put $z = 3 - 2i$ in $z^3 - 8z^2 + 25z - 26$

$= (3 - 2i)^3 - 8(3 - 2i)^2 + 25(3 - 2i) - 26$

$= 27 - 54i + 36i^2 - 8i^3 - 8(9 - 12i + 4i^2) + 75 - 50i - 26$

$= 27 - 54i - 36 + 8i - 72 + 96i + 32 + 75 - 50i - 26$

$= 27 - 36 - 72 + 32 + 75 - 26 + (-54 + 8 + 96 - 50)i$

$= 0 + 0i$

$\Rightarrow$ $3 - 2i$ is a root

$\Rightarrow$ the conjugate, $3 + 2i$, is also a root — since all coefficients are **real**

$\Rightarrow$ $(z - (3 + 2i))(z - (3 - 2i)) = z^2 - 6z + 13$ is a factor.

Factorising, by inspection,

$$z^3 - 8z^2 + 25z - 26 = (z^2 - 6z + 13)(z - 2) = 0$$

$\Rightarrow$ roots are $z = 3 \pm 2i$, or 2

2 Numerical solutions of equations

Accuracy of solution

When asked to show that a solution is accurate to n D.P., you must look at the value of $f(x)$ 'half' below and 'half' above, and conclude that

there is a **change of sign** in the **interval**, and the function is **continuous**, therefore there is a **solution in the interval correct to *n* D.P.**

Example: Show that $\alpha = 2{\cdot}0946$ is a root of the equation $f(x) = x^3 - 2x - 5 = 0$, accurate to 4 D.P.

Solution:

$f(2.09455) = -0{\cdot}0000165\ldots$, and $f(2.09465) = +0{\cdot}00997$

There is a **change of sign** and f is **continuous**

$\Rightarrow$ there is a **root** in **[2·09455, 2·09465]** $\Rightarrow$ **root is** $\alpha = 2{\cdot}0946$ **to 4 D.P.**

Interval bisection

(i) Find an interval $[a, b]$ which contains the root of an equation $f(x) = 0$.

(ii) $x = \frac{a+b}{2}$ is the mid-point of the interval $[a, b]$

Find $f\left(\frac{a+b}{2}\right)$ to decide whether the root lies in $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$.

(iii) Continue finding the mid-point of each subsequent interval to narrow the interval which contains the root.

Example: (i) Show that there is a root of the equation $f(x) = x^3 - 2x - 7 = 0$ in the interval [2, 3].

(ii) Find an interval of width 0·25 which contains the root.

Solution: (i) $f(2) = 8 - 4 - 7 = -3$, and $f(3) = 27 - 6 - 7 = 14$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in [2, 3].

(ii) Mid-point of [2, 3] is $x = 2{\cdot}5$, and $f(2{\cdot}5) = 15{\cdot}625 - 5 - 7 = 3{\cdot}625$

$\Rightarrow$ change of sign between $x = 2$ and $x = 2{\cdot}5$

$\Rightarrow$ root in $[2, 2{\cdot}5]$

Mid-point of [2, 2·5] is $x = 2{\cdot}25$,
and $f(2{\cdot}25) = 11{\cdot}390625 - 4{\cdot}5 - 7 = -0{\cdot}109375$

$\Rightarrow$ change of sign between $x = 2{\cdot}25$ and $x = 2{\cdot}5$

$\Rightarrow$ root in [2·25, 2·5], which is an interval of width 0·25

Linear interpolation

To solve an equation $f(x)$ using linear interpolation.

First, find an interval which contains a root,

second, assume that the curve is a straight line and use similar triangles to find where the line crosses the x-axis,

third, repeat the process as often as necessary.

Example: (i) Show that there is a root, α, of the equation

$$f(x) = x^3 - 2x - 9 = 0 \text{ in the interval } [2, 3].$$

(ii) Use linear interpolation once to find an approximate value of α.
Give your answer to 3 D.P.

Solution: (i) $f(2) = 8 - 4 - 9 = -5$, and $f(3) = 27 - 6 - 9 = 12$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in [2, 3].

(ii) From (i), curve passes through (2, –5) and (3, 12), and we assume that the curve is a straight line between these two points.

Let the line cross the x-axis at $(\alpha, 0)$

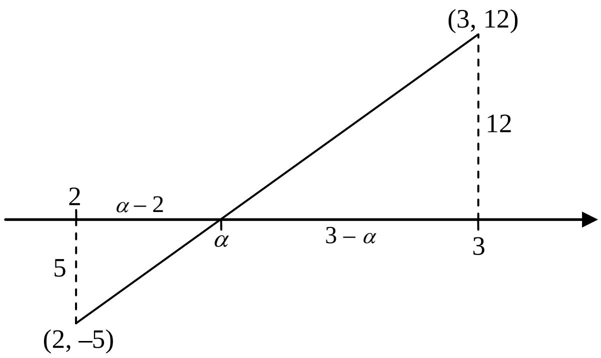
Using similar triangles

$$\frac{3-\alpha}{\alpha-2} = \frac{12}{5}$$

$\Rightarrow$ $15 - 5\alpha = 12\alpha - 24$

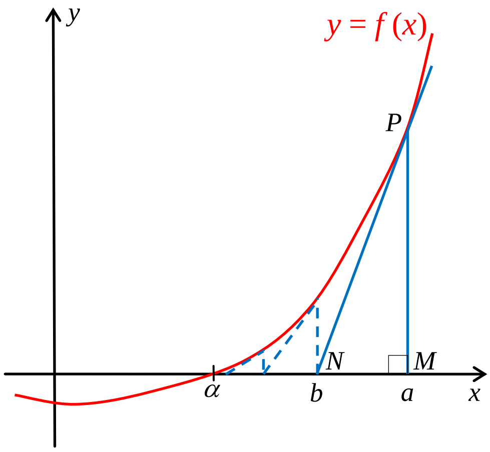
$\Rightarrow$ $\alpha = \frac{39}{17} = 2\frac{5}{17}$

$\Rightarrow$ $\alpha = 2{\cdot}294$ to 3 D.P.



Repeating the process will improve accuracy.

Newton-Raphson



Suppose that the equation $f(x) = 0$ has a root at $x = \alpha$, $\Rightarrow f(\alpha) = 0$

To find an approximation for this root, we first find a value $x = a$ near to $x = \alpha$ (decimal search).

In general, the point where the tangent at P, $x = a$, meets the x-axis, $x = b$, will give a better approximation.

At P, $x = a$, the gradient of the tangent is $f'(a)$,

and the gradient of the tangent is also $\frac{PM}{NM}$.

$$PM = y = f(a) \text{ and } NM = a - b$$

$$\Rightarrow f'(a) = \frac{PM}{NM} = \frac{f(a)}{a-b} \Rightarrow b = a - \frac{f(a)}{f'(a)}.$$

Further approximations can be found by repeating the process, which would follow the dotted line converging to the point $(\alpha, 0)$.

This formula can be written as the iteration $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example: (i) Show that there is a root, α, of the equation $f(x) = x^3 - 2x - 5 = 0$ in the interval $[2, 3]$.

(ii) Starting with $x_0 = 2$, use the Newton-Raphson formula to find x_1, x_2 and x_3, giving your answers to 3 D.P. where appropriate.

Solution: (i) $f(2) = 8 - 4 - 5 = -1$, and $f(3) = 27 - 6 - 5 = 16$

There is a **change of sign** and f is **continuous** $\Rightarrow$ there is a root in $[2, 3]$.

(ii) $f(x) = x^3 - 2x - 5 \Rightarrow f'(x) = 3x^2 - 2$

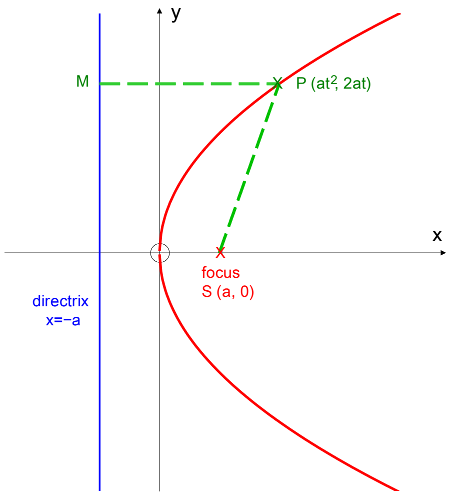
$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8-4-5}{12-2} = 2{\cdot}1$$

$$\Rightarrow x_2 = 2{\cdot}094568121 = 2{\cdot}095$$

$$\Rightarrow x_3 = 2{\cdot}094551482 = 2{\cdot}095$$

3 Coordinate systems

Parabolas



$y^2 = 4ax$ is the equation of a parabola which passes through the origin and has the x-axis as an axis of symmetry.

Parametric form

$x = at^2$, $y = 2at$ satisfy the equation for all values of t. t is a parameter, and these equations are the parametric equations of the parabola $y^2 = 4ax$.

Focus and directrix

The point S $(a, 0)$ is the *focus*, and

the line $x = -a$ is the *directrix*.

Any point P of the curve is equidistant from the focus and the directrix, $PM = PS$.

Proof: $PM = at^2 - (-a) = at^2 + a$

$$PS^2 = (at^2 - a)^2 + (2at)^2 = a^2t^4 - 2a^2t^2 + a^2 + 4a^2t^2$$

$$= a^2t^4 + 2a^2t^2 + a^2 = (at^2 + a)^2 = PM^2$$

$\Rightarrow$ $PM = \text{PS}$.

Gradient

For the parabola $y^2 = 4ax$, with general point P, $(at^2, 2at)$, we can find the gradient in two ways:

1. $y^2 = 4ax$

   $\Rightarrow$ $2y\dfrac{dy}{dx} = 4a$ $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{2a}{y}$, which we can write as $\dfrac{dy}{dx} = \dfrac{2a}{2at} = \dfrac{1}{t}$

2. At P, $x = at^2$, $y = 2at$

   $\Rightarrow$ $\dfrac{dy}{dt} = 2a$, $\dfrac{dx}{dt} = 2at$

   $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2a}{2at} = \dfrac{1}{t}$

Tangents and normals

Example: Find the equations of the tangents to $y^2 = 8x$ at the points where $x = 18$, and show that the tangents meet on the x-axis.

Solution: $x = 18 \quad \Rightarrow \quad y^2 = 8 \times 18 \quad \Rightarrow \quad y = \pm 12$

$2y\frac{dy}{dx} = 8 \quad \Rightarrow \quad \frac{dy}{dx} = \pm\frac{1}{3}$ since $y = \pm 12$

$\Rightarrow$ tangents are $y - 12 = \frac{1}{3}(x - 18) \quad \Rightarrow \quad x - 3y + 18 = 0$ at (18, 12)

and $y + 12 = -\frac{1}{3}(x - 18) \quad \Rightarrow \quad x + 3y + 18 = 0.$ at (18, –12)

To find the intersection, add the equations to give

$2x + 36 = 0 \quad \Rightarrow \quad x = -18 \quad \Rightarrow \quad y = 0$

$\Rightarrow$ tangents meet at (–18, 0) on the x-axis.

Example: Find the equation of the normal to the parabola given by $x = 3t^2,\ y = 6t$.

Solution: $x = 3t^2,\ y = 6t \Rightarrow \quad \frac{dx}{dt} = 6t, \quad \frac{dy}{dt} = 6,$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6}{6t} = \frac{1}{t}$

$\Rightarrow$ gradient of the normal is $\frac{-1}{\frac{1}{t}} = -t$

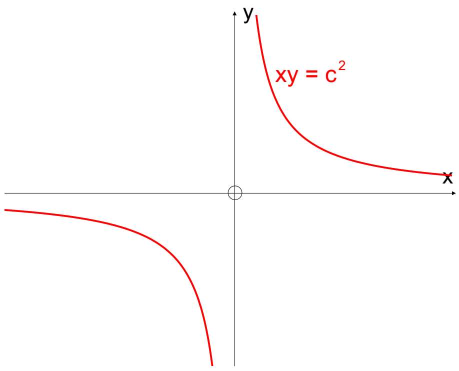
$\Rightarrow$ equation of the normal is $y - 6t = -t(x - 3t^2)$.

Notice that this ‘general equation’ gives the equation of the normal for any particular value of t:– when $t = -3$ the normal is $y + 18 = 3(x - 27) \Leftrightarrow y = 3x - 99$.

Rectangular hyperbolas

A *rectangular* hyperbola is a hyperbola in which the asymptotes meet at 90°.

$xy = c^2$ is the equation of a rectangular hyperbola in which the x-axis and y-axis are perpendicular asymptotes.



Parametric form

$x = ct$, $y = \frac{c}{t}$ are parametric equations of the hyperbola $xy = c^2$.

Tangents and normals

Example: Find the equation of the tangent to the hyperbola $xy = 36$ at the point where $x = 3$.

Solution: $x = 3 \Rightarrow 3y = 36 \Rightarrow y = 12$

$y = \frac{36}{x} \Rightarrow \frac{dy}{dx} = -\frac{36}{x^2} = -4$ when $x = 3$

$\Rightarrow$ tangent is $y - 12 = -4(x - 3) \Rightarrow 4x + y - 24 = 0$.

Example: Find the equation of the normal to the hyperbola given by $x = 3t$, $y = \frac{3}{t}$.

Solution: $x = 3t$, $y = \frac{3}{t} \Rightarrow \frac{dx}{dt} = 3$, $\frac{dy}{dt} = \frac{-3}{t^2}$

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-3}{t^2}}{3} = \frac{-1}{t^2}$

$\Rightarrow$ gradient of the normal is $\frac{-1}{\frac{-1}{t^2}} = t^2$

$\Rightarrow$ equation of the normal is $y - \frac{3}{t} = t^2(x - 3t)$

$\Rightarrow t^3x - ty = 3t^4 - 3$.

4 Matrices

You must be able to add, subtract and multiply matrices.

Order of a matrix

An $r \times c$ matrix has r rows and c columns;

the fi**R**st number is the number of **R**ows

the se**C**ond number is the number of **C**olumns.

Identity matrix

The identity matrix is $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note that $\boldsymbol{MI} = \boldsymbol{IM} = \boldsymbol{M}$ for *any* matrix $\boldsymbol{M}$.

Determinant and inverse

Let $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the *determinant* of $\boldsymbol{M}$ is

Det $\boldsymbol{M} = |\boldsymbol{M}| = ad - bc$.

To find the *inverse* of $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Note that $\boldsymbol{M}^{-1}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{I}$

(i) Find the determinant, $ad - bc$.
If $ad - bc = 0$, there is no inverse.

(ii) Interchange a and d (the leading diagonal)
Change sign of b and c, (the other diagonal)
Divide all elements by the determinant, $ad - bc$.

$$\Rightarrow \quad \boldsymbol{M}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Check:

$$\boldsymbol{M}^{-1}\boldsymbol{M} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} da - bc & 0 \\ 0 & -cb + ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

Similarly we could show that $\boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{I}$.

Example: $\boldsymbol{M} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$ and $\boldsymbol{MN} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$. Find $\boldsymbol{N}$.

Solution: Notice that $\boldsymbol{M}^{-1}(\boldsymbol{MN}) = (\boldsymbol{M}^{-1}\boldsymbol{M})\boldsymbol{N} = \boldsymbol{IN} = \boldsymbol{N}$ multiplying on the **left** by $\boldsymbol{M}^{-1}$

But $\boldsymbol{MNM}^{-1} \neq \boldsymbol{IN}$ we can**not** multiply on the **right** by $\boldsymbol{M}^{-1}$

First find $\boldsymbol{M}^{-1}$

Det $\boldsymbol{M} = 4 \times 3 - 2 \times 5 = 2 \quad \Rightarrow \quad \boldsymbol{M}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$

Using $\boldsymbol{M}^{-1}(\boldsymbol{MN}) = \boldsymbol{IN} = \boldsymbol{N}$

$$\Rightarrow \quad \boldsymbol{N} = \frac{1}{2}\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -7 & 4 \\ 13 & -6 \end{pmatrix} = \begin{pmatrix} -3\cdot 5 & 2 \\ 6\cdot 5 & -3 \end{pmatrix}.$$

Singular and non-singular matrices

If det $\boldsymbol{A} = 0$, then $\boldsymbol{A}$ is a *singular matrix*, and $\boldsymbol{A}^{-1}$ does not exist.

If det $\boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is a *non-singular matrix*, and $\boldsymbol{A}^{-1}$ exists

Linear Transformations

A matrix can represent a transformation, but the point must be written as a column vector before multiplying by the matrix.

Example: The image of (2, 3) under $\boldsymbol{T} = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$ is given by $\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 23 \\ 8 \end{pmatrix}$

$\Rightarrow$ the image of (2, 3) is (23, 8).

Note that the image of (0, 0) is always (0, 0)

$\Leftrightarrow$ the **origin never moves** under a matrix (linear) transformation

Basis vectors

The vectors $\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are called *basis* vectors, and are particularly important in describing the geometrical effect of a matrix, and in finding the matrix for a particular geometric transformation.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} a \\ c \end{pmatrix}$, the *first* column, and $\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} b \\ d \end{pmatrix}$, the *second* column

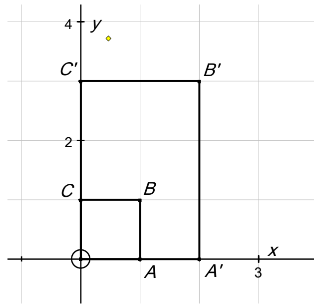
This is a more important result than it seems!

Finding the geometric effect of a matrix transformation

We can easily write down the images of $\underline{\boldsymbol{i}}$ and $\underline{\boldsymbol{j}}$, sketch them and find the geometrical transformation.

Example: Find the transformation represented by the matrix $\boldsymbol{T} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Solution: Find images of $\underline{\boldsymbol{i}}, \underline{\boldsymbol{j}}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and show on a sketch. Make sure that you letter the points



$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & 3 \end{pmatrix}$$

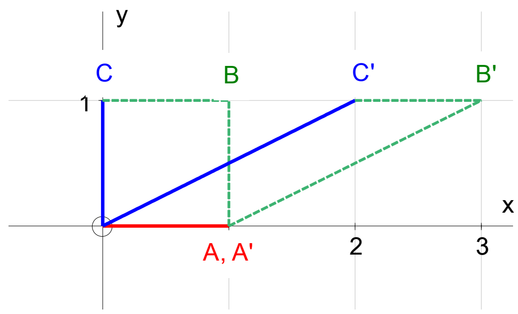
From sketch we can see that the transformation is a two-way stretch, of factor 2 parallel to the x-axis and of factor 3 parallel to the y-axis.

Finding the matrix of a given transformation.

Example: Find the matrix for a shear with factor 2 and invariant line the x-axis.

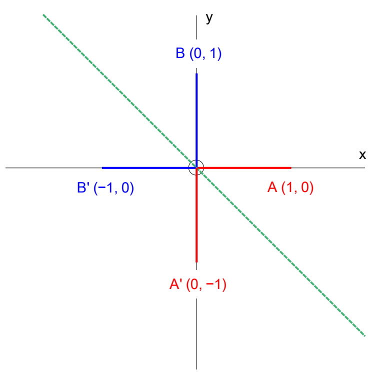
Solution: Each point is moved in the x-direction by a distance of (2 × its y-coordinate).

$\underline{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (does not move as it is on the invariant line).
This will be the first column of the matrix $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$



$\underline{\boldsymbol{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. This will be the second column of the matrix $\begin{pmatrix} * & 2 \\ * & 1 \end{pmatrix}$

$\Rightarrow$ Matrix of the shear is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.



Example: Find the matrix for a reflection in $y = -x$.

Solution: First find the images of $\underline{\boldsymbol{i}}$ and $\underline{\boldsymbol{j}}$. These will be the two columns of the matrix.

$$A \to A' \quad \Rightarrow \quad \underline{\boldsymbol{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

This will be the first column of the matrix $\begin{pmatrix} 0 & * \\ -1 & * \end{pmatrix}$

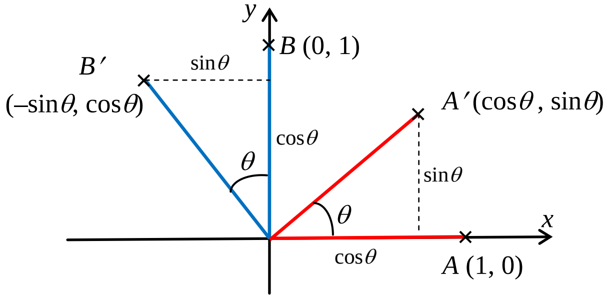
$B \to B' \quad \Rightarrow \quad \underline{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$

This will be the second column of the matrix $\begin{pmatrix} * & -1 \\ * & 0 \end{pmatrix}$

$\Rightarrow$ Matrix of the reflection is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$

Rotation matrix

From the diagram we can see that



$\underline{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$,

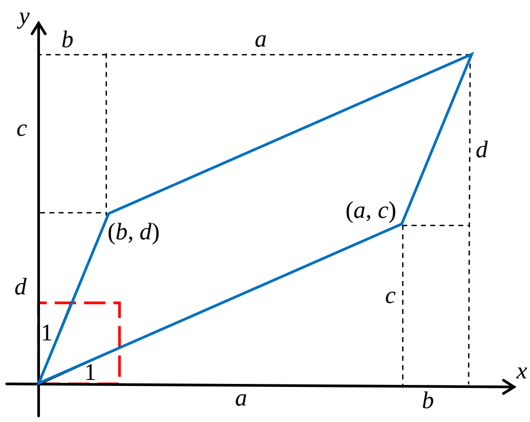
$\underline{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

These will be the first and second columns of the matrix

$\Rightarrow$ matrix is $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$

Determinant and area factor

For the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$



$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$

$\Rightarrow$ the unit square is mapped on to the parallelogram as shown in the diagram.

The area of the unit square = 1.

The area of the parallelogram = $(a + b)(c + d) - 2 \times (bc + \frac{1}{2}ac + \frac{1}{2}bd)$

$= \quad ac + ad + bc + bd - 2bc - ac - bd$

$= \quad ad - bc \quad = \det A.$

All squares of the grid are mapped onto congruent parallelograms

$\Rightarrow$ area factor of the transformation is $\det A = ad - bc$.

5 Series

You need to know the following sums

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

$$= \left(\tfrac{1}{2}n(n+1)\right)^2 = \left(\sum_{r=1}^{n} r\right)^2$$ a fluke, but it helps to remember it

Example: Find $\sum_{r=1}^{n} r(r^2-3)$.

Solution: $\sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$

$= \quad \frac{1}{4}n^2(n+1)^2 - 3 \times \frac{1}{2}n(n+1)$

$= \quad \frac{1}{4}n(n+1)\{n(n+1) - 6\}$

$= \quad \frac{1}{4}n(n+1)(n+3)(n-2)$

Example: Find $S_n = 2^2 + 4^2 + 6^2 + \ldots + (2n)^2$.

Solution: $S_n = 2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = 2^2(1^2 + 2^2 + 3^2 + \ldots + n^2)$

$= \quad 4 \times \frac{1}{6}n(n+1)(2n+1) = \frac{2}{3}n(n+1)(2n+1)$.

Example: Find $\sum_{r=5}^{n+2} r^2$

Solution: $\sum_{r=5}^{n+2} r^2 = \sum_{r=1}^{n+2} r^2 - \sum_{r=1}^{4} r^2$ notice that the top limit is 4 **not** 5

$= \quad \frac{1}{6}(n+2)(n+2+1)(2(n+2)+1) - \frac{1}{6} \times 4 \times 5 \times 9$

$= \quad \frac{1}{6}(n+2)(n+3)(2n+5) - 30.$

6 Proof by induction

1. Show that the result/formula is true for $n = 1$ (and sometimes $n = 2$, 3 ..).
   Conclude
   "therefore the result/formula ………. is true for $n = 1$".

2. Make induction assumption
   "Assume that the result/formula ………. is true for $n = k$".
   Show that the result/formula must then be true for $n = k + 1$
   Conclude
   "therefore the result/formula ………. is true for $n = k + 1$".

3. Final conclusion
   "therefore the result/formula ………… is true for all positive integers, n, by mathematical induction".

Summation

Example: Use mathematical induction to prove that

$$S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

Solution: When $n = 1$, $S_1 = 1^2 = 1$ and $S_1 = \tfrac{1}{6} \times 1(1+1)(2 \times 1 + 1) = \tfrac{1}{6} \times 1 \times 2 \times 3 = 1$

$\Rightarrow$ $S_n = \tfrac{1}{6}n(n+1)(2n+1)$ is true for $n = 1$.

Assume that the formula is true for $n = k$

$$\Rightarrow \quad S_k = 1^2 + 2^2 + 3^2 + \ldots + k^2 = \tfrac{1}{6}k(k+1)(2k+1)$$

$$\Rightarrow \quad S_{k+1} = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

$$= \tfrac{1}{6}(k+1)\{k(2k+1) + 6(k+1)\}$$

$$= \tfrac{1}{6}(k+1)\{2k^2 + 7k + 6\} = \tfrac{1}{6}(k+1)(k+2)(2k+3)$$

$$= \tfrac{1}{6}(k+1)\{(k+1)+1\}\{2(k+1)+1\}$$

$\Rightarrow$ The formula is true for $n = k + 1$

$\Rightarrow$ $S_n = \tfrac{1}{6}n(n+1)(2n+1)$ is true for all positive integers, n, by mathematical induction.

Recurrence relations

Example: A sequence, 4, 9, 19, 39, … is defined by the recurrence relation

$u_1 = 4$, $u_{n+1} = 2u_n + 1$. Prove that $u_n = 5 \times 2^{n-1} - 1$.

Solution: When $n = 1$, $u_1 = 4$, and $u_1 = 5 \times 2^{1-1} - 1 = 5 - 1 = 4$, $\Rightarrow$ formula true for $n = 1$.

Assume that the formula is true for $n = k$, $\Rightarrow$ $u_k = 5 \times 2^{k-1} - 1$.

From the recurrence relation,

$$u_{k+1} = 2u_k + 1 = 2(5 \times 2^{k-1} - 1) + 1$$

$$\Rightarrow \quad u_{k+1} = 5 \times 2^k - 2 + 1 = 5 \times 2^{(k+1)-1} - 1$$

$\Rightarrow$ the formula is true for $n = k + 1$

$\Rightarrow$ the formula is true for all positive integers, n, by mathematical induction.

Divisibility problems

Considering $f(k + 1) - f(k)$, will lead to a proof which sometimes has hidden difficulties,

and a more reliable way is to consider $f(k + 1) - m \times f(k)$, where m is chosen to eliminate the exponential term.

Example: Prove that $f(n) = 5^n - 4n - 1$ is divisible by 16 for all positive integers, n.

Solution: When $n = 1$, $f(1) = 5^1 - 4 - 1 = 0$, which is divisible by 16, and so $f(n)$ is divisible by 16 when $n = 1$.

Assume that the result is true for $n = k$, $\Rightarrow$ $f(k) = 5^k - 4k - 1$ is divisible by 16.

Considering $f(k + 1) - 5 \times f(k)$ we will eliminate the 5^k term.

$$f(k+1) - 5 \times f(k) = (5^{k+1} - 4(k+1) - 1) - 5 \times (5^k - 4k - 1)$$

$$= 5^{k+1} - 4k - 4 - 1 - 5^{k+1} + 20k + 5 = 16k$$

$$\Rightarrow \quad f(k+1) = 5 \times \boldsymbol{f(k)} + \mathbf{16}k$$

Since $\boldsymbol{f(k)}$ is divisible by 16 (induction assumption), and $\mathbf{16}k$ is divisible by 16, then $f(k + 1)$ must be divisible by 16,

$\Rightarrow$ $f(n) = 5^n - 4n - 1$ is divisible by 16 for $n = k + 1$

$\Rightarrow$ $f(n) = 5^n - 4n - 1$ is divisible by 16 for all positive integers, n, by mathematical induction.

Example: Prove that $f(n) = 2^{2n+3} + 3^{2n-1}$ is divisible by 5 for all positive integers *n*.

Solution: When $n = 1$, $f(1) = 2^{2+3} + 3^{2-1} = 32 + 3 = 35 = \mathbf{5} \times 7$, and so the result is true for $n = 1$.

Assume that the result is true for $n = k$

$\Rightarrow \quad f(k) = 2^{2k+3} + 3^{2k-1}$ is divisible by 5

We could consider either (it does not matter which)

$f(k+1) - 2^2 \times f(k)$, which would eliminate the 2^{2k+3} term **I**

or $f(k+1) - 3^2 \times f(k)$, which would eliminate the 3^{2k-1} term **II**

$$\textbf{I} \Rightarrow \quad f(k+1) - 2^2 \times f(k) = 2^{2(k+1)+3} + 3^{2(k+1)-1} - 2^2 \times (2^{2k+3} + 3^{2k-1})$$

$$= 2^{2k+5} + 3^{2k+1} - 2^{2k+5} - 2^2 \times 3^{2k-1}$$

$$\Rightarrow \quad f(k+1) - 4 \times f(k) = 9 \times 3^{2k-1} - 4 \times 3^{2k-1} = 5 \times 3^{2k-1}$$

$$\Rightarrow \quad f(k+1) = 4 \times \boldsymbol{f(k)} - \mathbf{5} \times 3^{2k-1}$$

Since $\boldsymbol{f(k)}$ is divisible by **5** (induction assumption), and $\mathbf{5} \times 3^{2k-1}$ is divisible by **5**, then $f(k+1)$ must be divisible by **5**.

$\Rightarrow \quad f(n) = 2^{2n+3} + 3^{2n-1}$ is divisible by 5 for all positive integers, *n*, by mathematical induction.

Powers of matrices

Example: If $M = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$, prove that $M^n = \begin{pmatrix} 2^n & 1-2^n \\ 0 & 1 \end{pmatrix}$ for all positive integers *n*.

Solution: When $n = 1$, $M^1 = \begin{pmatrix} 2^1 & 1-2^1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = M$

$\Rightarrow$ the formula is true for $n = 1$.

Assume the formula is true for $n = k \Rightarrow M^k = \begin{pmatrix} 2^k & 1-2^k \\ 0 & 1 \end{pmatrix}$.

$$M^{k+1} = MM^k = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^k & 1-2^k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \times 2^k & 2 - 2 \times 2^k - 1 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow \quad M^{k+1} = \begin{pmatrix} 2^{k+1} & 1-2^{k+1} \\ 0 & 1 \end{pmatrix} \Rightarrow$ The formula is true for $n = k + 1$

$\Rightarrow \quad M^n = \begin{pmatrix} 2^n & 1-2^n \\ 0 & 1 \end{pmatrix}$ is true for all positive integers, *n*, by mathematical induction.

7 Appendix

Complex roots of a real polynomial equation

Preliminary results:

I $(z_1 + z_2 + z_3 + z_4 + \ldots + z_n)^* = z_1^* + z_2^* + z_3^* + z_4^* + \ldots + z_n^*$,

by repeated application of $(z + w)^* = z^* + w^*$

II $(z^n)^* = (z^*)^n$

$(zw)^* = z^*w^*$

$\Rightarrow (z^n)^* = (z^{n-1}z)^* = (z^{n-1})^*(z)^* = (z^{n-2}z)^*(z)^* = (z^{n-2})^*(z)^*(z)^* \ \ldots = (z^*)^n$

Theorem: **If** $z = a + bi$ is a root of $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \alpha_{n-2} z^{n-2} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$,
and if all the α_i are **real**,
then the conjugate, $z^* = a - bi$ is also a root.

Proof: If $z = a + bi$ is a root of the equation $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_1 z + \alpha_0 = 0$

then $\alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$

$\Rightarrow (\alpha_n z^n + \alpha_{n-1} z^{n-1} + \ldots + \alpha_2 z^2 + \alpha_1 z + \alpha_0)^* = 0$ since $0^* = 0$

$\Rightarrow (\alpha_n z^n)^* + (\alpha_{n-1} z^{n-1})^* + \ldots + (\alpha_2 z^2)^* + (\alpha_1 z)^* + (\alpha_0)^* = 0$ using **I**

$\Rightarrow \alpha_n^*(z^n)^* + \alpha_{n-1}^*(z^{n-1})^* + \ldots + \alpha_2^*(z^2)^* + \alpha_1^*(z)^* + \alpha_0^* = 0$ since $(zw)^* = z^*w^*$

$\Rightarrow \alpha_n(z^n)^* + \alpha_{n-1}(z^{n-1})^* + \ldots + \alpha_2(z^2)^* + \alpha_1(z)^* + \alpha_0 = 0$ α_i **real** $\Rightarrow \alpha_i^* = \alpha_i$

$\Rightarrow \alpha_n(z^*)^n + \alpha_{n-1}(z^*)^{n-1} + \ldots + \alpha_2(z^*)^2 + \alpha_1(z^*) + \alpha_0 = 0$ using **II**

$\Rightarrow$ $z^* = a - bi$ is also a root of the equation.

Formal definition of a linear transformation

A linear transformation $\boldsymbol{T}$ has the following properties:

(i) $$\boldsymbol{T}\begin{pmatrix} kx \\ ky \end{pmatrix} = k\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix}$$

(ii) $$\boldsymbol{T}\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \boldsymbol{T}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \boldsymbol{T}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

It can be shown that **any** matrix transformation is a linear transformation, and that **any** linear transformation can be represented by a matrix.

Derivative of x^n, for any integer

We can use proof by induction to show that $\frac{d}{dx}(x^n) = nx^{n-1}$, for any integer n.

1) We know that the derivative of x^0 is 0 which equals $0x^{-1}$,

since $x^0 = 1$, and the derivative of 1 is 0

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = 0$.

2) We know that the derivative of x^1 is 1 which equals $1 \times x^{1-1}$

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = 1$

Assume that the result is true for $n = k$

$\Rightarrow \quad \frac{d}{dx}(x^k) = kx^{k-1}$

$\Rightarrow \quad \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \times x^k) = x \times \frac{d}{dx}(x^k) + 1 \times x^k$ product rule

$\Rightarrow \quad \frac{d}{dx}(x^{k+1}) = x \times kx^{k-1} + x^k = kx^k + x^k = (k+1)x^k$

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = k + 1$

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for all positive integers, n, by mathematical induction.

3) We know that the derivative of x^{-1} is $-x^{-2}$ which equals $-1 \times x^{-1-1}$

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = -1$

Assume that the result is true for $n = k$

$\Rightarrow \quad \frac{d}{dx}(x^k) = kx^{k-1}$

$\Rightarrow \quad \frac{d}{dx}(x^{k-1}) = \frac{d}{dx}\left(\frac{x^k}{x}\right) = \frac{x \times \frac{d}{dx}(x^k) - x^k \times 1}{x^2}$ quotient rule

$\Rightarrow \quad \frac{d}{dx}(x^{k+1}) = \frac{x \times kx^{k-1} - x^k}{x^2} = \frac{(k-1)x^k}{x^2} = (k-1)x^{k-2} = (k-1)x^{(k-1)-1}$

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for $n = k - 1$

We are going backwards (**from** $n = k$ **to** $n = k - 1$), and, since we started from $n = -1$,

$\Rightarrow \quad \frac{d}{dx}(x^n) = nx^{n-1}$ is true for all negative integers, n, by mathematical induction.

Putting 1), 2) and 3), we have proved that

$\frac{d}{dx}(x^n) = nx^{n-1}$, for **any** integer n.

8 Index