REVISION SHEET – FP1 (AQA)

ALGEBRA

The main ideas are:

- The relationships between roots and coefficients in polynomial (quadratic) equations
- Finding polynomial equations with roots related to that of a given one

Before the exam you should know:

- Use the relations between the symmetric functions of the roots of polynomial equations and the coefficients.
- The method of substitution which is available for finding a polynomial equation with roots related to a given one.

  E.g., if $2x^2 - 7x + 4 = 0$ has roots α, β then $2(y-3)^2 - 7(y-3) + 4 = 0$ will have roots $\alpha + 3, \beta + 3$.

Roots and coefficients in polynomial equations

Quadratic

If $ax^2 + bx + c = 0$ has roots α and β then: i) $\alpha + \beta = -\dfrac{b}{a}$ ii) $\alpha\beta = \dfrac{c}{a}$

Example

The quadratic equation $4x^2 + 3x + 1 = 0$ has roots α and β.

i) Write down the values of $\alpha + \beta$ and $\alpha\beta$.

ii) Find a quadratic equation with integer coefficients with roots $2\alpha - 1, 2\beta - 1$.

Solution

i) $\alpha + \beta = -\dfrac{3}{4}$ and $\alpha\beta = \dfrac{1}{4}$.

ii) $(2\alpha - 1) + (2\beta - 1) = 2(\alpha + \beta) - 2 = 2\left(-\dfrac{3}{4}\right) - 2 = -\dfrac{5}{2}$.

$$(2\alpha - 1)(2\beta - 1) = 4\alpha\beta - 2(\alpha + \beta) + 1 = 4\left(\frac{1}{4}\right) - 2\left(-\frac{3}{4}\right) + 1 = \frac{7}{2}$$

Therefore a quadratic with integer coefficients with roots $2\alpha - 1, 2\beta - 1$ is $2x^2 + 5x + 7 = 0$

Example (Substitution Method)

The quadratic equation $4x^2 + 3x + 1 = 0$ has roots α and β. Find a quadratic equation with integer coefficients with roots $2\alpha + 1, 2\beta + 1$.

Solution

Let $w = 2z + 1$ so that $z = \dfrac{w-1}{2}$. Since α and β are the roots of $4x^2 + 3x + 1 = 0$, $2\alpha + 1, 2\beta + 1$ are the roots of $4\left(\dfrac{w-1}{2}\right)^2 + 3\left(\dfrac{w-1}{2}\right) + 1 = 0$.

Multiplying this out gives:

$$2(w-1)^2 + 3(w-1) + 2 = 0$$
$$2w^2 - 4w + 2 + 3w - 3 + 2 = 0$$
$$2w^2 - w + 1 = 0$$

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CALCULUS and TRIGONOMETRY

The main ideas are:

- Finding the Gradient of the Tangent to a Curve at a Point
- Evaluating Simple Improper Integrals
- Solving Trigonometric Equations using Exact Values

Before the exam you should know:

- How to find the gradient of the tangent to a curve at a point and in particular that: $$f'(x) = \frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
- How to evaluating simple improper integrals.
- How to solve trigonometric equations using exact values.

Gradient of a Tangent

Considering chords

For an equation of the form $y = f(x)$, where $f(x)$ is a simple polynomial, i.e. $x^3 - 3x$, then a point A could have coordinates $(x, f(x))$ and a point B could have coordinates $(x+h, f(x+h))$.

The straight line AB has gradient: $\dfrac{f(x+h)-f(x)}{h}$

By letting $h \to 0$ the gradient of $y = f(x)$ at a given point A can be found, provided the appropriate limit on h can be evaluated, and the resulting limit is the derivative of f with respect to x.

Hence, $f'(x) = \dfrac{df}{dx} = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$. This is also referred to as differentiating from first principles.

Example

A graph has the equation $y = x^2 - x - 4$, find the gradient of the chord from the point where $x = 3$ to the point where $x = 3 + h$ and hence find the gradient of the tangent when $x = 3$.

Solution

When $x = 3$, $\quad y = 2$

When $x = 3 + h$, $\quad y = (3+h)^2 - (3+h) - 4$

$= 9 + 6h + h^2 - 3 - h - 4 \quad = 2 + 5h + h^2$

Gradient of chord $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{(2+5h+h^2)-2}{h} = 5 + h$

As $h \to 0$, the gradient of the chord $\to 5$. Hence, the gradient of the tangent at $x = 3$ is 5.

Improper Integrals

An improper integral is a definite integral for which the integrand (the expression to be integrated) is undefined either within or at one or both of the limits of integration, or at some point between the limits of integration. You can decide whether or not an improper integral has a finite value, and if so, what it is, by considering its limits.

Examples

Find, if possible, the values of i) $\int_1^\infty \frac{2}{x^2}dx$ ii) $\int_0^\infty 3\sqrt{x}dx$

Solutions

i) $\int_1^a \frac{2}{x^2}dx \quad = \int_1^\infty 2x^{-2}dx \quad = \left[-2x^{-1}\right]_1^a = -\frac{2}{a}+2$, as $a \to \infty, -\frac{2}{a} \to 0$, so the value of the integral is 2.

ii) $\int_0^a 3\sqrt{x}dx \quad = \int_0^a 3x^{1/2}dx \quad = \left[x^{3/2}\right]_0^a \quad = a^{3/2}$, as $a \to \infty, a^{3/2} \to \infty$, so the value of the integral cannot be found.

Trigonometry

In order to solve trigonometric function using exact values you will need to recall the specific values (which can be seen in the table below). You may also prefer to think about an equilateral triangle (of side length 2) to find the values for 30^o and 60^o and a right angles isosceles triangle of lengths 1 unit, 1 unit and $\sqrt{2}$ units (on the hypotenuse) to calculate the values for 45^o.

θ	0^0	$30^0, \frac{\pi}{6}$	$45^0, \frac{\pi}{4}$	$60^0, \frac{\pi}{3}$	$90^0, \pi$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}\left(or\ \frac{\sqrt{2}}{2}\right)$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}\left(or\ \frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{3}}{2}$	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}\left(or\ \frac{\sqrt{3}}{3}\right)$	1	$\sqrt{3}$	Undefined

It is also important that you recall in which 'quadrant', i.e. $0 < x < 90^o$, $90^o < x < 180^o$ etc. trigonometric functions are equivalent, i.e. $\sin(180^o-\theta)=\sin\theta, \cos(180^o-\theta)=-\cos\theta$ etc.

Example Find the exact value of cos 225^o

Solution 225^o is in the third quadrant, so cos 225^o is negative.
cos 225^o = cos (180^o + 45^o) = – cos 45^o = $-\frac{1}{\sqrt{2}}\left(\text{or } -\frac{\sqrt{2}}{2}\right)$

Example Find the general solution of the equation $\sin\theta = 0.5$

Solution

The general solution of the equation $\sin\theta = \sin\alpha$ is given by: $\theta = (360n+\alpha)^o$ or $\theta = (360n+180-\alpha)^o$

You know that $\sin 30^o = 0.5$, so $\theta = (360n+30)^o$ or $\theta = (360n+150)^o$.

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GRAPHS AND INEQUALITIES

The main ideas are:

- Sketching Graphs of Rational Functions and Parabolas, Ellipses and Hyperbolas
- Solving Inequalities

Before the exam you should know:

- There are three main cases of horizontal asymptotes.
  - One is a curve which is a linear polynomial divided by a linear polynomial, for example $y = \frac{4x+1}{3x-2}$. This has a horizontal asymptote at $y = \frac{\text{coefficient of } x \text{ on the top}}{\text{coefficient of } x \text{ on the bottom}}$. Here this would be $y = \frac{4}{3}$.
  - The second is a curve given by a quadratic polynomial divided by a quadratic polynomial, for example, $y = \frac{5x^2+x+5}{2x^2-2x+1}$ as $x \to \pm\infty$, This has a horizontal asymptote at $y = \frac{\text{coefficient of } x^2 \text{ on the top}}{\text{coefficient of } x^2 \text{ on the bottom}}$. Here this would be $y = \frac{5}{2}$.
  - Thirdly, when the curve is given by a linear polynomial divided by a quadratic polynomial, it will generally have the x-axis (y = 0) as a horizontal asymptote.
- When you solve an inequality, try substituting a few of the values for which you are claiming it is true back into the original inequality as a check.
- About graphs of parabolas, ellipses and hyperbolas.

Graph Sketching.

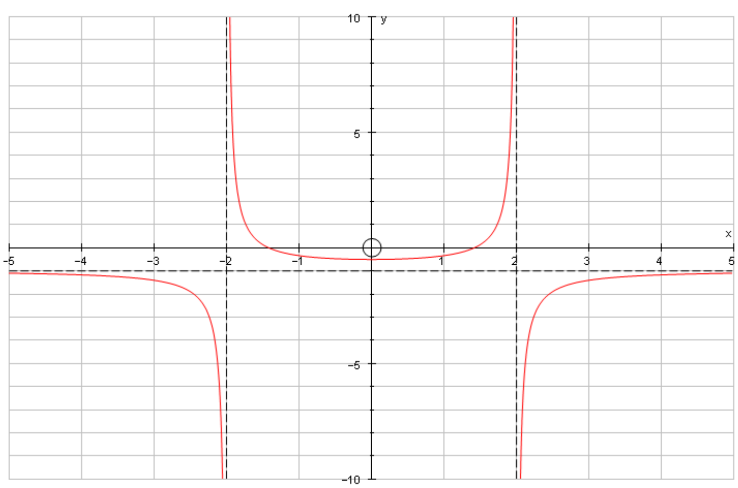
Rational functions.

To sketch the graph of $y = \frac{N(x)}{D(x)}$:

- Find the intercepts – that is where the graph cuts the axes.
- Find any asymptotes – the vertical asymptotes occur at values of x which make the denominator zero.
- Examine the behaviour of the graph near to the vertical asymptotes; a good way to do this is to find out what the value of y is for values of x very close to the vertical asymptote
- Examine the behaviour around any non-vertical asymptotes, i.e. as x tends to $\pm\infty$.

Example Sketch the curve $y = \frac{x^2-2}{4-x^2}$

Solution (Sketch)



The curve can be written as $y = \frac{x^2-2}{(x+2)(x-2)}$.

If $x = 0$ then y = -0.5. So the y intercept is (0, -0.5)
Setting $y = 0$ gives, x = -2 and x = 2. So the x intercepts are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$.
The denominator is zero when $x = -2$ and when $x = 2$ so these are the vertical asymptotes

Also $y = \frac{x^2-2}{4-x^2} = \frac{-1(-x^2+4)+2}{4-x^2} = -1 + \frac{2}{4-x^2}$

so y = -1 is a horizontal asymptote.

Rational functions.

You are also required to know the graphs of parabolas, ellipses and hyperbolas with equations:

i) $y^2 = 4ax$ ii) $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ iii) $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ iv) $xy = c^2$

Solving Inequalities

Broadly speaking inequalities can be solved in one of three ways, or sometimes in a combination of more than one of these ways.

Method 1

Draw a "sketch" of the inequality. For example, if you are asked to solve an inequality of the form $g(x) \le f(x)$ then sketch both f and g, and identify points where the graph of f is lower than the graph of g. These points will lie between points x_1 for which $g(x_1) = f(x_1)$ and so these usually need to be calculated.

Method 2

Use algebra to find an equivalent inequality which is easier to solve. When dealing with inequalities remember there are certain rules which need to be obeyed when performing algebraic manipulations. The main one is "DON'T MULTIPLY BY A NUMBER UNLESS YOU KNOW IT'S SIGN, IF IT'S NEGATIVE YOU MUST REVERSE THE INEQUALITY SIGN, IF IT'S POSITIVE THEN LEAVE THE INEQUALITY SIGN AS IT IS." For example, don't multiply by $(x – 2)$ because that's positive when $x > 2$ and negative when $x < 2$. On the other hand $(x – 2)^2$ is always positive so you can safely multiply by this (with no need to reverse the inequality sign).

Method 3

Sometimes is easier to rearrange an inequality of the from $g(x) \le f(x)$ to $g(x) - f(x) \le 0$ (you don't have to worry about reversing the inequality for such a rearrangement). Identify points where $g(x) – f(x) = 0$ or where $g(x) – f(x)$ has a vertical asymptote. Finally test whether the inequality is true in the various regions between these points.

Example Solve the inequality $3x - 2 \le \dfrac{x+2}{x-1}$

Solution (using Method 3)

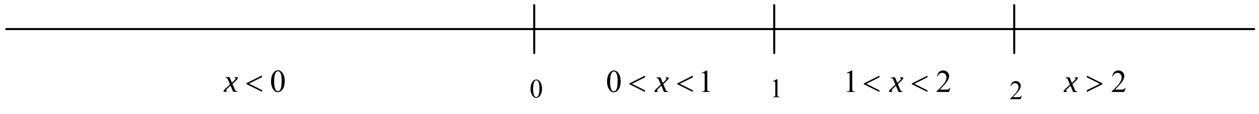
$$3x - 2 \le \frac{x+2}{x-1} \iff 3x - 2 - \frac{x+2}{x-1} \le 0$$

$$\iff \frac{(3x-2)(x-1)-(x+2)}{x-1} \le 0$$

$$\iff \frac{3x^2-6x}{x-1} \le 0$$

$$\iff \frac{3x(x-2)}{x-1} \le 0$$

Looking at the expression, $3x = 0$ if $x = 0$, $x – 2 = 0$ if $x = 2$ and $x – 1 = 0$ if $x = 1$.
This means that the truth of the inequality should be tested in each of the following regions



It can be seen that the inequality is TRUE if $x < 0$, false if $0 < x < 1$, TRUE if $1 < x < 2$ and FALSE if $x > 2$. The solution is therefore $x \le 0$, $1 < x \le 2$. Can you see why $x = 0$ and $x = 2$ are included as values for which the inequality is true, but $x = 1$ is not?

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COMPLEX NUMBERS

The main ideas are:

- Manipulating complex numbers
- Complex conjugates and roots of equations

Before the exam you should know:

- How to add and subtract complex numbers.
- How to multiply two complex numbers quickly and in one step as this will save a lot of time in the exam.
- That non-real roots of quadratic equations with real coefficients occur in conjugate pairs.

Manipulating Complex Numbers

Adding, subtracting and multiplying

- Adding and subtracting both involve fairly straightforward calculations. It is important to remember to equate real and imaginary parts.
- Multiplying is slightly more difficult and care should be taken to make sure that the correct sign is written when two 'imaginary' parts are multiplied.

Example Multiply the two complex number $3+i$ and $2-3i$.

Solution
$$\begin{aligned}(3+i)(2-3i) &= 6-9i+2i-3i^2\\ &= 6-7i-3(\sqrt{-1})^2\\ &= 6-7i+3\\ &= 9-7i\end{aligned}$$

Complex Conjugates and Roots of Equations

The complex conjugate of $z = a+bi$ **is** $z* = a-bi$.

- Remember $zz*$ is a real number and it equals the square of the modulus of z.
- Complex roots of polynomial equations with real coefficients occur in conjugate pairs. This means that if you are told one complex root of a polynomial equation with real coefficients you are in fact being told two roots. This is key to answering some typical exam questions.

An example of an algebraic trick that it is very useful to know is:

$$\begin{aligned}(z-(3+2i))(z-(3-2i)) &= ((z-3)-2i)((z-3)+2i)\\ &= (z-3)^2-4i^2\\ &= z^2-6z+13\end{aligned}$$

You should be comfortable with manipulating equations primarily involving comparing real and imaginary parts. E.g. 2z+z* = 1+*i*, where z* is the conjugate of z.

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MATRICES

The main ideas are:

- Manipulating matrices
- Using matrices to represent transformations

Before the exam you should know:

- How to add, subtract and multiply matrices.
- That matrix multiplication is associative, so $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ but not commutative, so $\mathbf{AB} \neq \mathbf{BA}$.
- The standard matrices for rotation, reflection and enlargement and understand how matrices can be combined to represent composite transformations.

Manipulating matrices.

Adding and subtracting matrices are straightforward.

Multiplying matrices is slightly more difficult. Matrices may only be multiplied if they are conformable, that is if the number of columns in the multiplying matrix (the left-hand one) is the same as the number of rows in the matrix being multiplied (the right-hand one). Matrix multiplication is **not** commutative. This means that for two matrices, **A** and **B**, it is **not** generally true that **AB** = **BA** (it is vital that you remember this).

If $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 \\ 24 \end{pmatrix}$ then $\mathbf{AB} = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} 7 \\ 24 \end{pmatrix} = \begin{pmatrix} 1\times 7 + 3\times 24 \\ 2\times 7 + 7\times 24 \end{pmatrix} = \begin{pmatrix} 79 \\ 182 \end{pmatrix}$

Notice that **BA** is not possible as, this way round, the matrices are not conformable.

I is the identity matrix. The 2×2 identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. All identity matrices have 1's on the left to right downwards diagonal and 0's everywhere else.

Using matrices to represent transformations

These are some of the standard transformation matrices, and are worth remembering. Remember that the first column of a matrix is where (1, 0) moves to and the second column is where (0, 1) moves to. This can be useful in exams because using this idea it is possible to derive the matrix of a transformation described in words.

- Rotation through angle θ, anticlockwise about the origin: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
- Enlargement, scale factor k, centre the origin: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.
- Stretch, factor a horizontally, factor b vertically: $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

A composite transformation is made up of two or more standard transformations, for example a rotation through angle π, followed by a reflection in $y = x$. The matrix representing a composite transformation is obtained by multiplying the component transformation matrices together. The order is important. The matrix for the composite of the transformation with matrix **M**, followed by the transformation with matrix **N** is **NM**. The order is right to left.

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NUMERICAL METHODS

There main ideas are:

- Using the method of interval bisection, linear interpolation and the Newton Raphson Method to approximate the solution of an equation.
- The step by step method for solving differential equations
- Reducing a Relation to a Linear Law

Before the exam you should know

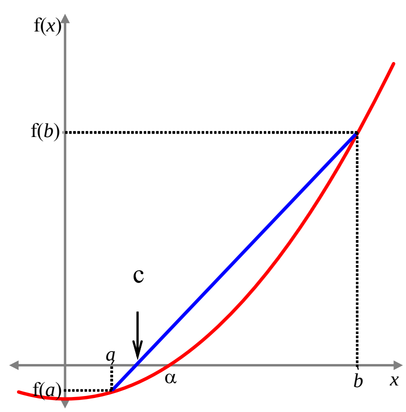
- The formula associated with linear interpolation
$$c = \frac{af(b) - bf(a)}{\mathrm{f}(b) - \mathrm{f}(a)}$$
- The formula associated with the Newton Raphson Method: $x_{r+1} = x_r - \frac{\mathrm{f}(x_r)}{\mathrm{f}'(x_r)}$
- That in the step by step method for solving $\frac{dy}{dx} = \mathrm{f}(x)$:
$$x_{n+1} = x_n + h, y_{n+1} = y_n + hf(x_n)$$

The Bisection Method

If a root of an equation f(x) = 0 lies between a and b, then $c = \frac{a+b}{2}$ gives an approximation to the root. You can then determine whether the root lies between a and c or between b and c and repeat this idea, obtaining better and better interval estimates to the root. Eventually you can give an approximation of the root to the desired degree of accuracy.

Linear Interpolation

If a root of an equation f(x) = 0 lies between a and b, then $c = \frac{af(b) - bf(a)}{\mathrm{f}(b) - \mathrm{f}(a)}$ gives an approximation of the root. If needed you can determine whether the root lies between a and c or between b and c and repeat this process to obtain a sequence of approximations to the root.



Example

Show that the equation $\mathrm{f}(x) = x^2 - 7 = 0$ has a root, α, between 2 and 3. By using the rule of false position, starting with these two values, find an estimate to this root.

Solution

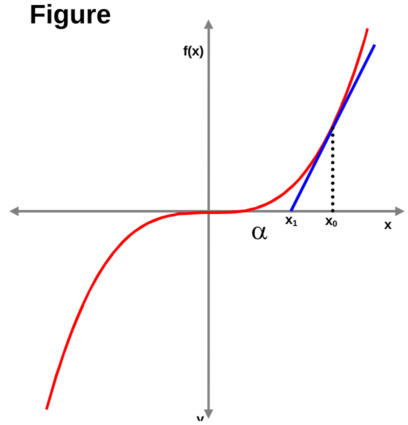
Since $\mathrm{f}(2) = -5 < 0$ and $\mathrm{f}(3) = 2 > 0$ the function must cross the x-axis between $x = 2$ and $x = 3$ and so there will be a root of f(x) = 0 there. With $a = 2$ and $b = 3$, the linear interpolation gives a first approximation to the solution of $c = \frac{af(b) - bf(a)}{\mathrm{f}(b) - \mathrm{f}(a)}$. In this case this is

$$\frac{(2\times \mathrm{f}(3)) - (3\times \mathrm{f}(2))}{\mathrm{f}(3) - \mathrm{f}(2)} = \frac{(2\times 2) - (3\times -3)}{2-(-3)} = \frac{13}{5} = 2.6.$$

Newton Raphson Method

The sequence of values generated by $x_{r+1} = x_r - \dfrac{f(x_r)}{f'(x_r)}$ with x_0 an appropriate estimate, usually converges to a root of $f(x) = 0$ near to x_0.

Figure



Example

Use the Newton Raphson method to find the root of the equation $x^4 + x - 3 = 0$ near $x = 1.5$. With $x_0 = 1.5$, use the method three times (in other words calculate as far as x_4). Hence, give an approximation to the root and state its accuracy.

Solution

The iterative formula in the case of $f(x) = x^4 + x - 3$ is as follows: $x_{r+1} = x_r - \dfrac{f(x_r)}{f'(x_r)} = x_r - \left(\dfrac{x_r^{\ 4} + x_r - 3}{4x_r^{\ 3} + 1}\right)$.

Thus $x_1 = x_0 - \left(\dfrac{x_0^{\ 4} + x_0 - 3}{4x_0^{\ 3} + 1}\right) = 1.5 - \left(\dfrac{1.5^4 + 1.5 - 3}{\left(4 \times 1.5^3\right) + 1}\right) = 1.254310$

Similarly $x_2 = x_1 - \left(\dfrac{x_1^{\ 4} + x_1 - 3}{4x_1^{\ 3} + 1}\right) = 1.172278$, $x_3 = 1.164110$, and $x_4 = 1.164035$.

From this evidence it looks as though 1.164 may be an estimate to the root which is correct to three decimal places. You can check this by verifying that the function changes sign between 1.1635 and 1.1645. In fact

$$f(1.1635) = -0.00391 < 0 \text{ and } f(1.1645) = 0.003399 > 0.$$

Step by Step Method for solving Differential Equations

$$\boxed{x_{n+1} = x_n + h, y_{n+1} = y_n + hf(x_n)}$$

Example

A solution of the differential equation $\dfrac{dy}{dx} = x\sqrt{x+1}$ satisfies y = 5 when x = 1. Approximate the value of y when x = 1.4 using step size $h = 0.1$.

Solution

n	x_n	y_n	dy/dx=f(x)
1	1	5	1.4142136
2	1.1	5.1414214	1.5940514
3	1.2	5.3008265	1.7798876
4	1.3	5.4788153	1.9715476

In each case, this value is simply *dy/dx* at the current value of *x*

In each case, this is the previous *y* value plus (*h* × the value of dy/dx at the previous *x* value), *h* × the value of dy/dx at the previous *x* value is a good approximation to how much *y* changes by when *x* changes by *h*.

Reducing a Relation to a Linear Law

Suppose you have some pairs of data (*x*, *y*) that are supposed to be in the relationship $y = ax^n$. You wish to determine values of *a* and *n* which give a best fit to your data.

Taking logarithms gives $\log(y) = n\log(x) + \log(a)$. Therefore by plotting log *y* against log *x* you should see a straight line whose gradient is *n* and whose intercept with the *y*-axis is at log(*a*). Think "*y* = m*x* + c".

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SERIES

The main ideas are:

- Summing Series using standard formulae
- Finding Polynomial Expressions

Before the exam you should know:

- The standard formula:
$$\sum_{r=1}^{n} r, \sum_{r=1}^{n} r^2, \sum_{r=1}^{n} r^3$$
- And be able to see that a series like
$$(1\times 3)+(2\times 4)+...+n(n+2)$$
can be written in sigma notation as:
$$\sum_{r=1}^{n} r(r+2)$$

Summing Series

Using standard formulae

It is important to know the standard results for summations, namely:

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1),$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Also required is fluency in manipulating and simplify the standard formulae sums.

Example

$$\begin{aligned}\sum_{r=1}^{n} r\left(r^2+1\right) = \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r &= \frac{1}{4}n^2\left(n+1\right)^2 + \frac{1}{2}n\left(n+1\right) \\ &= \frac{1}{4}n\left(n+1\right)\left[n\left(n+1\right)+2\right] \\ &= \frac{1}{4}n\left(n+1\right)\left(n^2+n+2\right).\end{aligned}$$

It is also useful to understand that a series such as $(1\times 2)+(2\times 3)+...+n(n+1)$ can be written in sigma notation.

For example, the sequence just described can be written as $\sum_{r=1}^{n} r(r+1)$ and then solved as in the previous example.

It is usual that the nth term of the sequence gives what the sigma notation should be, so if the nth term is not given in a sequence it is important to be able to establish what it is.