Possible C4 questions from past papers P1—P3

Source of the original question is given in brackets, e.g. [P2 January 2001 Question 1]; a question which has been edited is indicated with an asterisk, e.g. [P3 January 2003 Question 8*].

1. A measure of the effective voltage, M volts, in an electrical circuit is given by

$$M^2 = \int_0^1 V^2 \quad dt$$

where V volts is the voltage at time t seconds. Pairs of values of V and t are given in the following table.

t	0	0.25	0.5	0.75	1
V	−48	207	37	−161	−29
V^2					

Use the trapezium rule with five values of V^2 to estimate the value of M.

(6)

[P2 June 2001 Question 3]

2. **Figure 1**

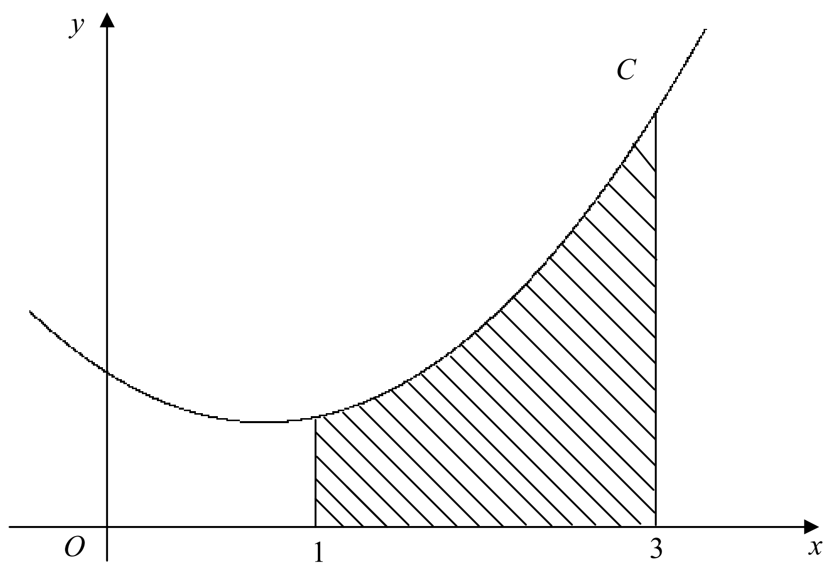

Figure 1 shows part of a curve C with equation $y = x^2 + 3$. The shaded region is bounded by C, the x-axis and the lines $x = 1$ and $x = 3$. The shaded region is rotated 360° about the x-axis.

Using calculus, calculate the volume of the solid generated. Give your answer as an exact multiple of π.

(7)

[P2 June 2001 Question 5]

3. (*a*) Given that $a^x = e^{kx}$, where a and k are constants, $a > 0$ and $x \in \mathbb{R}$, prove that $k = \ln a$. **(2)**

(*b*) Hence, using the derivative of e^{kx}, prove that when $y = 2^x$,

$$\frac{dy}{dx} = 2^x \ln 2.$$

(2)

(*c*) Hence deduce that the gradient of the curve with equation $y = 2^x$ at the point (2, 4) is ln 16.

(2)

[P3 June 2001 Question 3]

4.

$$f(x) = (1 + 3x)^{-1}, \quad |x| < \tfrac{1}{3}.$$

(*a*) Expand f(x) in ascending powers of x up to and including the term in x^3.

(3)

(*b*) Hence show that, for small x,

$$\frac{1+x}{1+3x} \approx 1 - 2x + 6x^2 - 18x^3.$$

(2)

(*c*) Taking a suitable value for x, which should be stated, use the series expansion in part (*b*) to find an approximate value for $\frac{101}{103}$, giving your answer to 5 decimal places.

(3)

[P3 June 2001 Question 4]

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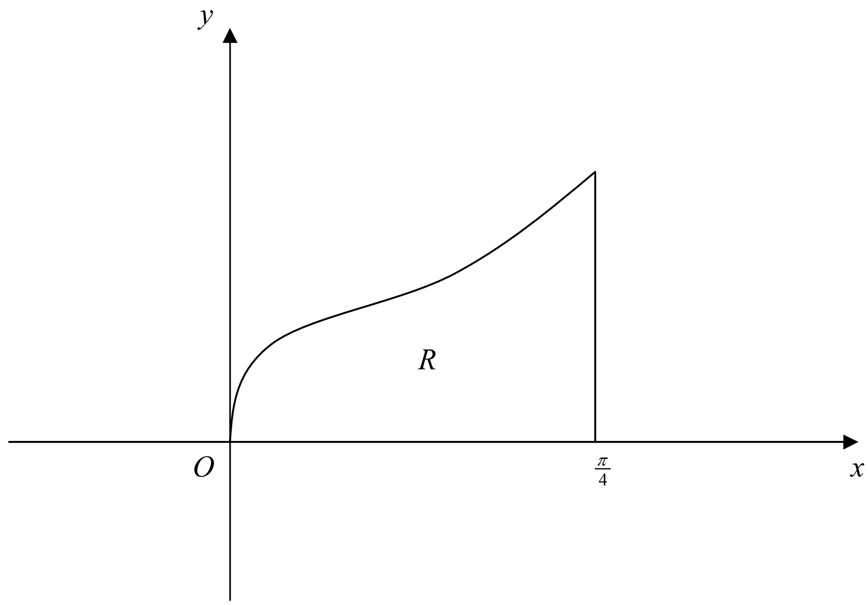
5. (*a*) Use integration by parts to show that

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx = \tfrac{1}{4}\pi - \tfrac{1}{2}\ln 2.$$

(6)

Figure 1



The finite region R, bounded by the equation $y = x^{\frac{1}{2}} \sec x$, the line $x = \dfrac{\pi}{4}$ and the x-axis is shown in Fig. 1. The region R is rotated through 2π radians about the x-axis.

(*b*) Find the volume of the solid of revolution generated.

(2)

(*c*) Find the gradient of the curve with equation $y = x^{\frac{1}{2}} \sec x$ at the point where $x = \dfrac{\pi}{4}$.

(3)

[P3 June 2001 Question 5]

6. Two submarines are travelling in straight lines through the ocean. Relative to a fixed origin, the vector equations of the two lines, l_1 and l_2, along which they travel are

$$\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$
$$\text{and } \mathbf{r} = 9\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \mu(4\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

where λ and μ are scalars.

(*a*) Show that the submarines are moving in perpendicular directions.

(2)

(*b*) Given that l_1 and l_2 intersect at the point A, find the position vector of A.

(5)

The point b has position vector $10\mathbf{j} - 11\mathbf{k}$.

(*c*) Show that only one of the submarines passes through the point B.

(3)

(*d*) Given that 1 unit on each coordinate axis represents 100 m, find, in km, the distance AB.

(2)

[P3 June 2001 Question 6]

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7. In a chemical reaction two substances combine to form a third substance. At time t, $t \geq 0$, the concentration of this third substance is x and the reaction is modelled by the differential equation

$$\frac{dx}{dt} = k(1 - 2x)(1 - 4x), \text{ where } k \text{ is a positive constant.}$$

(*a*) Solve this differential equation and hence show that

$$\ln\left|\frac{1-2x}{1-4x}\right| = 2kt + c, \text{ where } c \text{ is an arbitrary constant.}$$

(7)

(*b*) Given that $x = 0$ when $t = 0$, find an expression for x in terms of k and t.

(4)

(*c*) Find the limiting value of the concentration x as t becomes very large.

(2)

[P3 June 2001 Question 7]

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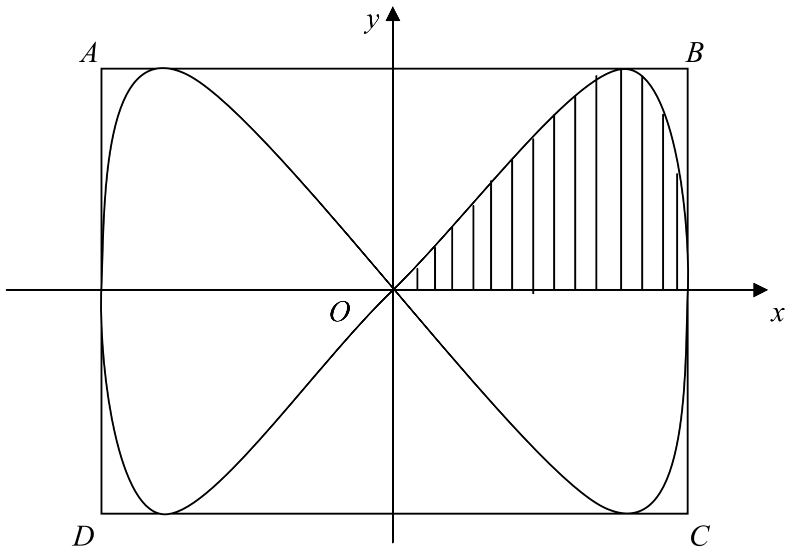
8. **Figure 2**



Part of the design of a stained glass window is shown in Fig. 2. The two loops enclose an area of blue glass. The remaining area within the rectangle *ABCD* is red glass.

The loops are described by the curve with parametric equations

$$x = 3\cos t, \quad y = 9\sin 2t, \quad 0 \le t < 2\pi.$$

(*a*) Find the cartesian equation of the curve in the form $y^2 = \mathrm{f}(x)$.

(4)

(*b*) Show that the shaded area in Fig. 2, enclosed by the curve and the *x*-axis, is given by

$\int_0^{\frac{\pi}{2}} A\sin 2t \sin t \, \mathrm{d}t$, stating the value of the constant A.

(3)

(*c*) Find the value of this integral.

(4)

The sides of the rectangle *ABCD*, in Fig. 2, are the tangents to the curve that are parallel to the coordinate axes. Given that 1 unit on each axis represents 1 cm,

(*d*) find the total area of the red glass.

(4)

[P3 June 2001 Question 8]

9. The following is a table of values for $y = \sqrt{(1 + \sin x)}$, where x is in radians.

x	0	0.5	1	1.5	2
y	1	1.216	p	1.413	q

(*a*) Find the value of p and the value of q.

(2)

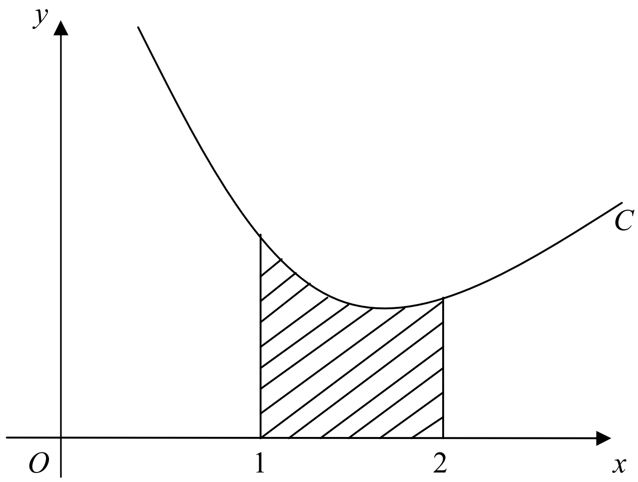
(*b*) Use the trapezium rule and all the values of y in the completed table to obtain an estimate of I, where

$$I = \int_0^2 \sqrt{(1 + \sin x)}\, dx.$$

(4)

[P2 January 2002 Question 1]

10. **Figure 1**



In Fig. 1, the curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x + \frac{2}{x^2}, \quad x > 0.$$

The shaded region is bounded by C, the x-axis and the lines with equations $x = 1$ and $x = 2$. The shaded region is rotated through 2π radians about the x-axis.

Using calculus, calculate the volume of the solid generated. Give your answer in the form $\pi(a + \ln b)$, where a and b are constants.

(8)

[P2 January 2002 Question 4]

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11. Given that

$$\frac{10(2-3x)}{(1-2x)(2+x)} \equiv \frac{A}{1-2x} + \frac{B}{2+x},$$

(a) find the values of the constants A and B.

(3)

(b) Hence, or otherwise, find the series expansion in ascending powers of x, up to and including the term in x^3, of $\dfrac{10(2-3x)}{(1-2x)(2+x)}$, for $|x| < \frac{1}{2}$.

(5)

[P3 January 2002 Question 3]

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12. A radioactive isotope decays in such a way that the rate of change of the number N of radioactive atoms present after t days, is proportional to N.

(*a*) Write down a differential equation relating N and t.

(2)

(*b*) Show that the general solution may be written as $N = Ae^{-kt}$, where A and k are positive constants.

(5)

Initially the number of radioactive atoms present is 7×10^{18} and 8 days later the number present is 3×10^{17}.

(*c*) Find the value of k.

(3)

(*d*) Find the number of radioactive atoms present after a further 8 days.

(2)

[P3 January 2002 Question 5]

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13. Relative to a fixed origin O, the point A has position vector $4\mathbf{i} + 8\mathbf{j} - \mathbf{k}$, and the point B has position vector $7\mathbf{i} + 14\mathbf{j} + 5\mathbf{k}$.

(*a*) Find the vector $\overrightarrow{AB}$.

(1)

(*b*) Calculate the cosine of $\angle OAB$.

(3)

(*c*) Show that, for all values of λ, the point P with position vector

$$\lambda\mathbf{i} + 2\lambda\mathbf{j} + (2\lambda - 9)\mathbf{k}$$

lies on the line through A and B.

(3)

(*d*) Find the value of λ for which OP is perpendicular to AB.

(3)

(*e*) Hence find the coordinates of the foot of the perpendicular from O to AB.

(2)

[P3 January 2002 Question 6]

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14. (i) Use integration by parts to find the exact value of $\int_1^3 x^2 \ln x \;\; \mathrm{d}x$.

(6)

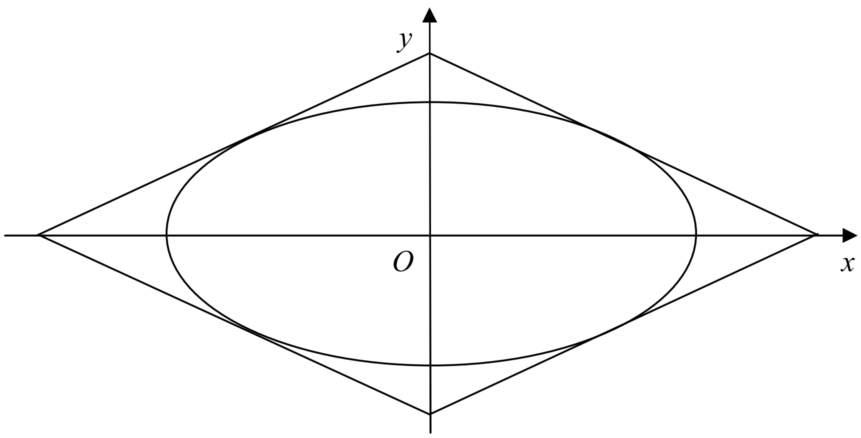
(ii) Use the substitution $x = \sin\theta$ to show that, for $|x| \leq 1$,

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,\mathrm{d}x = \frac{x}{(1-x^2)^{\frac{1}{2}}} + c, \text{ where } c \text{ is an arbitrary constant.}$$

(6)

[P3 January 2002 Question 7]

15. **Figure 1**



A table top, in the shape of a parallelogram, is made from two types of wood. The design is shown in Fig. 1. The area inside the ellipse is made from one type of wood, and the surrounding area is made from a second type of wood.

The ellipse has parametric equations,

$$x = 5\cos\theta, \quad y = 4\sin\theta, \quad 0 \le \theta < 2\pi.$$

The parallelogram consists of four line segments, which are tangents to the ellipse at the points where $\theta = \alpha$, $\theta = -\alpha$, $\theta = \pi - \alpha$, $\theta = -\pi + \alpha$.

(*a*) Find an equation of the tangent to the ellipse at $(5\cos\alpha, 4\sin\alpha)$, and show that it can be written in the form

$$5y\sin\alpha + 4x\cos\alpha = 20.$$

(4)

(*b*) Find by integration the area enclosed by the ellipse.

(4)

(*c*) Hence show that the area enclosed between the ellipse and the parallelogram is

$$\frac{80}{\sin 2\alpha} - 20\pi .$$

(4)

(*d*) Given that $0 < \alpha < \frac{\pi}{4}$, find the value of α for which the areas of two types of wood are equal.

(3)

[P3 January 2002 Question 8]

16. The speed, v m s^{-1}, of a lorry at time t seconds is modelled by

$$v = 5(e^{0.1t} - 1)\sin(0.1t), \quad 0 \le t \le 30.$$

(*a*) Copy and complete the following table, showing the speed of the lorry at 5 second intervals. Use radian measure for $0.1t$ and give your values of v to 2 decimal places where appropriate.

t	0	5	10	15	20	25
v		1.56	7.23	17.36		

(3)

(*b*) Verify that, according to this model, the lorry is moving more slowly at $t = 25$ than at $t = 24.5$.

(1)

The distance, s metres, travelled by the lorry during the first 25 seconds is given by $s = \int_0^{25} v \, dt$.

(*c*) Estimate s by using the trapezium rule with all the values from your table.

(4)

[P2 June 2002 Question 3]

17. **Figure 2**

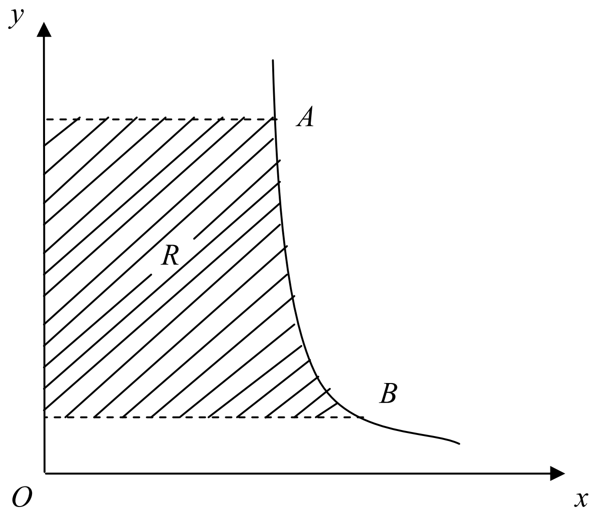


Figure 2 shows a sketch of the curve C with equation $y = \dfrac{4}{x-3}$, $x \neq 3$.

The points A and B on the curve have x-coordinates 3.25 and 5 respectively.

(a) Write down the y-coordinates of A and B.

(1)

(b) Show that an equation of C is $\dfrac{3y+4}{y}$, $y \neq 0$.

(1)

The shaded region R is bounded by C, the y-axis and the lines through A and B parallel to the x-axis. The region R is rotated through 360° about the y-axis to form a solid shape S.

(c) Find the volume of S, giving your answer in the form $\pi(a + b \ln c)$, where a, b and c are integers.

(7)

The solid shape S is used to model a cooling tower. Given that 1 unit on each axis represents 3 metres,

(d) show that the volume of the tower is approximately 15 500 m^3.

(2)

[P2 June 2002 Question 7]

18. (*a*) Use integration by parts to find

$$\int x \cos 2x \quad \mathrm{d}x .$$

(4)

(*b*) Prove that the answer to part (*a*) may be expressed as

$$\tfrac{1}{2} \sin x\, (2x \cos x - \sin x) + C,$$

where C is an arbitrary constant.

(3)

[P3 June 2002 Question 2]

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19. The circle C has equation $x^2 + y^2 - 8x - 16y - 209 = 0$.

(*a*) Find the coordinates of the centre of C and the radius of C.

(3)

The point $P(x, y)$ lies on C.

(*b*) Find, in terms of x and y, the gradient of the tangent to C at P.

(3)

(*c*) Hence or otherwise, find an equation of the tangent to C at the point (21, 8).

(2)

[P3 June 2002 Question 3]

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20. **Figure 1**

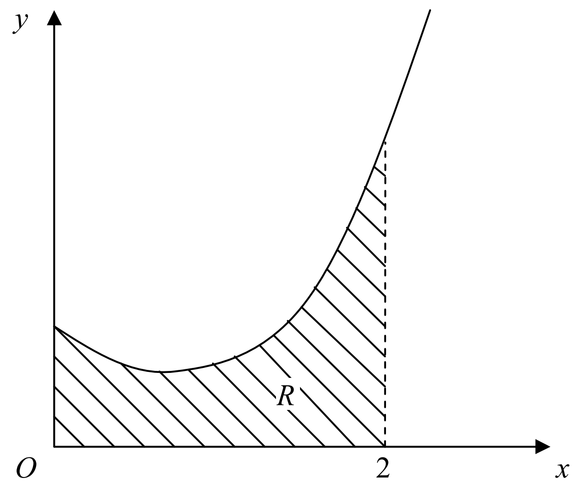

Figure 1 shows part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{x^2 + 1}{(1+x)(3-x)}, \quad 0 \le x < 3.$$

(a) Given that $\mathrm{f}(x) = A + \dfrac{B}{1+x} + \dfrac{C}{3-x}$, find the values of the constants A, B and C.

(4)

The finite region R, shown in Fig. 1, is bounded by the curve with equation $y = \mathrm{f}(x)$, the x-axis, the y-axis and the line $x = 2$.

(b) Find the area of R, giving your answer in the form $p + q \ln r$, where p, q and r are rational constants to be found.

(5)

[P3 June 2002 Question 4]

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21. (a) Prove that, when $x = \frac{1}{15}$, the value of $(1+5x)^{-\frac{1}{2}}$ is exactly equal to sin 60°.

(3)

(b) Expand $(1+5x)^{-\frac{1}{2}}$, $|x| < 0.2$, in ascending powers of x up to and including the term in x^3, simplifying each term.

(4)

(c) Use your answer to part (b) to find an approximation for sin 60°.

(2)

(d) Find the difference between the exact value of sin 60° and the approximation in part (c).

(1)

[P3 June 2002 Question 5

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22. A curve is given parametrically by the equations

$$x = 5 \cos t, \quad y = -2 + 4 \sin t, \quad 0 \leq t < 2\pi.$$

(a) Find the coordinates of all the points at which C intersects the coordinate axes, giving your answers in surd form where appropriate.

(4)

(b) Sketch the graph at C.

(2)

P is the point on C where $t = \frac{1}{6}\pi$.

(c) Show that the normal to C at P has equation

$$8\sqrt{3}y = 10x - 25\sqrt{3}.$$

(4)

[P3 June 2002 Question 6]

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23. A Pancho car has value £V at time t years. A model for V assumes that the rate of decrease of V at time t is proportional to V.

(a) By forming and solving an appropriate differential equation, show that $V = Ae^{-kt}$, where A and k are positive constants.

(3)

The value of a new Pancho car is £20 000, and when it is 3 years old its value is £11 000.

(b) Find, to the nearest £100, an estimate for the value of the Pancho when it is 10 years old.

(5)

A Pancho car is regarded as 'scrap' when its value falls below £500.

(c) Find the approximate age of the Pancho when it becomes 'scrap'.

(3)

[P3 June 2002 Question 7]

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24. Referred to an origin O, the points A, B and C have position vectors $(9\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, $(6\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$ and $(3\mathbf{i} + p\mathbf{j} + q\mathbf{k})$ respectively, where p and q are constants.

(a) Find, in vector form, an equation of the line l which passes through A and B.

(2)

Given that C lies on l,

(b) find the value of p and the value of q,

(2)

(c) calculate, in degrees, the acute angle between OC and AB.

(3)

The point D lies on AB and is such that OD is perpendicular to AB.

(d) Find the position vector of D.

(6)

[P3 June 2002 Question 8]

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25. **Figure 2**

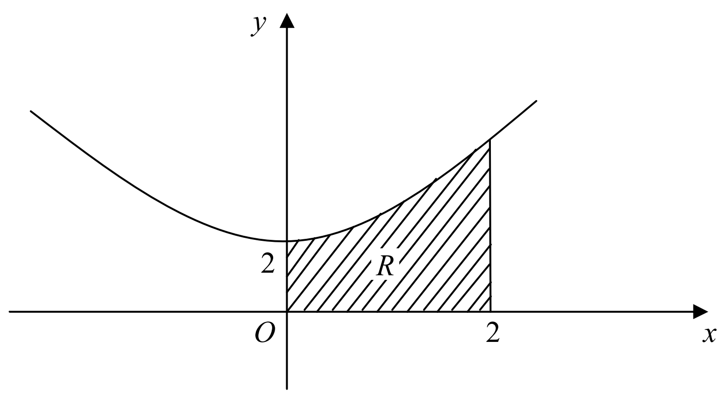

Figure 2 shows part of the curve with equation $y = x^2 + 2$.

The finite region R is bounded by the curve, the x-axis and the lines $x = 0$ and $x = 2$.

(a) Use the trapezium rule with 4 strips of equal width to estimate the area of R.

(5)

(b) State, with a reason, whether your answer in part (a) is an under-estimate or over-estimate of the area of R.

(1)

(c) Using integration, find the volume of the solid generated when R is rotated through 360° about the x-axis, giving your answer in terms of π.

(6)

[P2 November 2002 Question 7]

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26. **Figure 1**

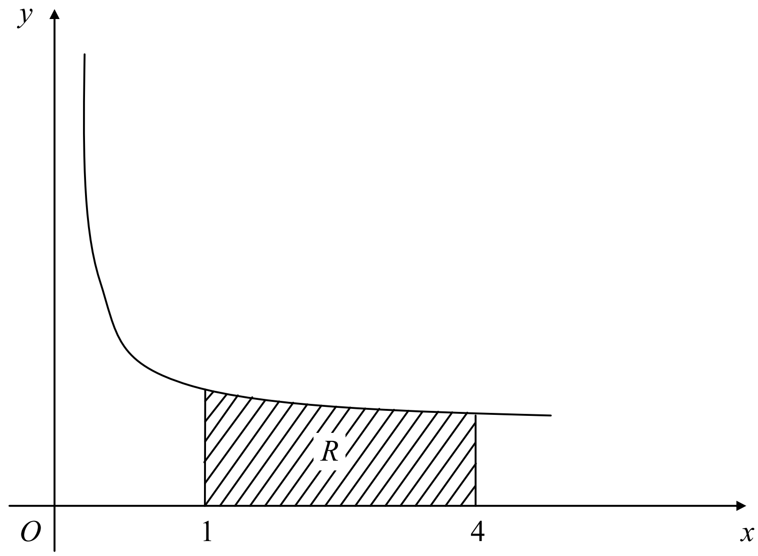

Figure 1 shows part of the curve with equation $y = 1 + \frac{1}{2\sqrt{x}}$. The shaded region R, bounded by the curve, that x-axis and the lines $x = 1$ and $x = 4$, is rotated through 360° about the x-axis. Using integration, show that the volume of the solid generated is $\pi(5 + \frac{1}{2}\ln 2)$.

(8)

[P2 January 2003 Question 4]

27.

Figure 2

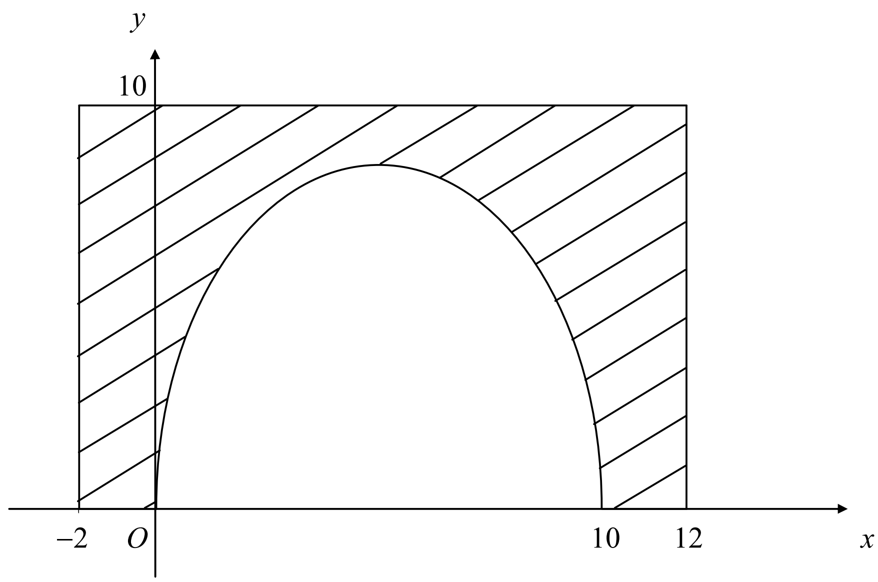

Figure 2 shows the cross-section of a road tunnel and its concrete surround. The curved section of the tunnel is modelled by the curve with equation $y = 8\sqrt{\left(\sin\frac{\pi x}{10}\right)}$, in the interval $0 \leq x \leq 10$. The concrete surround is represented by the shaded area bounded by the curve, the x-axis and the lines $x = -2$, $x = 12$ and $y = 10$. The units on both axes are metres.

(*a*) Using this model, copy and complete the table below, giving the values of y to 2 decimal places.

x	0	2	4	6	8	10
y	0	6.13				0

(2)

The area of the cross-section of the tunnel is given by $\int_0^{10} y \ \mathrm{d}x$.

(*b*) Estimate this area, using the trapezium rule with all the values from your table. **(4)**

(*c*) Deduce an estimate of the cross-sectional area of the concrete surround. **(1)**

(*d*) State, with a reason, whether your answer in part (*c*) over-estimates or under-estimates the true value. **(2)**

[P2 January 2003 Question 5]

28. The function f is given by

$$f(x) = \frac{3(x+1)}{(x+2)(x-1)}, \; x \in \mathbb{R}, x \neq -2, x \neq 1.$$

(*a*) Express f(x) in partial fractions.

(3)

(*b*) Hence, or otherwise, prove that $f'(x) < 0$ for all values of x in the domain.

(3)

[P3 January 2003 Question 1]

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29. (*a*) Expand $(1 + 3x)^{-2}$, $|x| < \frac{1}{3}$, in ascending powers of x up to and including the term in x^3, simplifying each term.

(4)

(*b*) Hence, or otherwise, find the first three terms in the expansion of $\dfrac{x+4}{(1+3x)^2}$ as a series in ascending powers of x.

(4)

[P3 January 2003 Question 4]

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30. Liquid is poured into a container at a constant rate of 30 cm^3 s^{-1}. At time t seconds liquid is leaking from the container at a rate of $\frac{2}{15}V$ cm^3 s^{-1}, where V cm^3 is the volume of liquid in the container at that time.

(*a*) Show that

$$-15\frac{\mathrm{d}V}{\mathrm{d}t} = 2V - 450.$$

(3)

Given that $V = 1000$ when $t = 0$,

(*b*) find the solution of the differential equation, in the form $V = f(t)$.

(7)

(*c*) Find the limiting value of V as $t \to \infty$.

(1)

[P3 January 2003 Question 5]

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31. Referred to a fixed origin O, the points A and B have position vectors $(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ and $(5\mathbf{i} - 3\mathbf{j})$ respectively.

(*a*) Find, in vector form, an equation of the line l_1 which passes through A and B.

(2)

The line l_2 has equation $\mathbf{r} = (4\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$, where μ is a scalar parameter.

(*b*) Show that A lies on l_2.

(1)

(*c*) Find, in degrees, the acute angle between the lines l_1 and l_2.

(4)

The point C with position vector $(2\mathbf{i} - \mathbf{k})$ lies on l_2.

(*d*) Find the shortest distance from C to the line l_1.

(4)

[P3 January 2003 Question 6]

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32. **Figure 2**

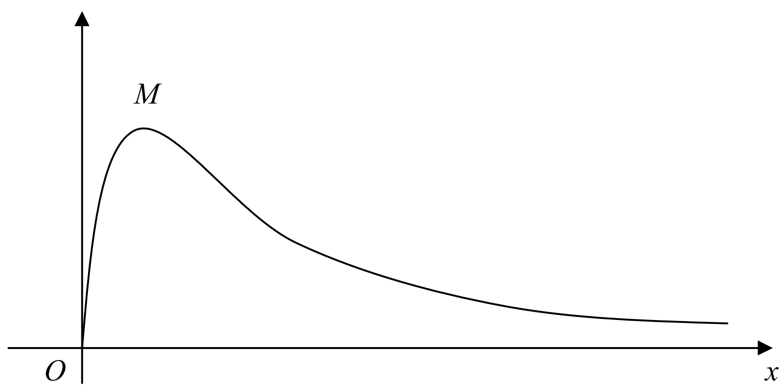

Figure 2 shows the curve with equation $y = x^{\frac{1}{2}} e^{-2x}$.

(*a*) Find the x-coordinate of M, the maximum point of the curve.

(5)

The finite region enclosed by the curve, the x-axis and the line $x = 1$ is rotated through 2π about the x-axis.

(*b*) Find, in terms of π and e, the volume of the solid generated.

(7)

[P3 January 2003 Question 7]

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33. (*a*) Use the identity for cos $(A + B)$ to prove that $\cos 2A = 2\cos^2 A - 1$.

(2)

(*b*) Use the substitution $x = 2\sqrt{2}\sin\theta$ to prove that

$$\int_2^{\sqrt{6}} \sqrt{(8 - x^2)}\,\mathrm{d}x = \tfrac{1}{3}(\pi + 3\sqrt{3} - 6).$$

(7)

A curve is given by the parametric equations

$$x = \sec\theta,\quad y = \ln(1 + \cos 2\theta),\quad 0 \le \theta < \frac{\pi}{2}.$$

(*c*) Find an equation of the tangent to the curve at the point where $\theta = \dfrac{\pi}{3}$.

(5)

[P3 January 2003 Question 8]

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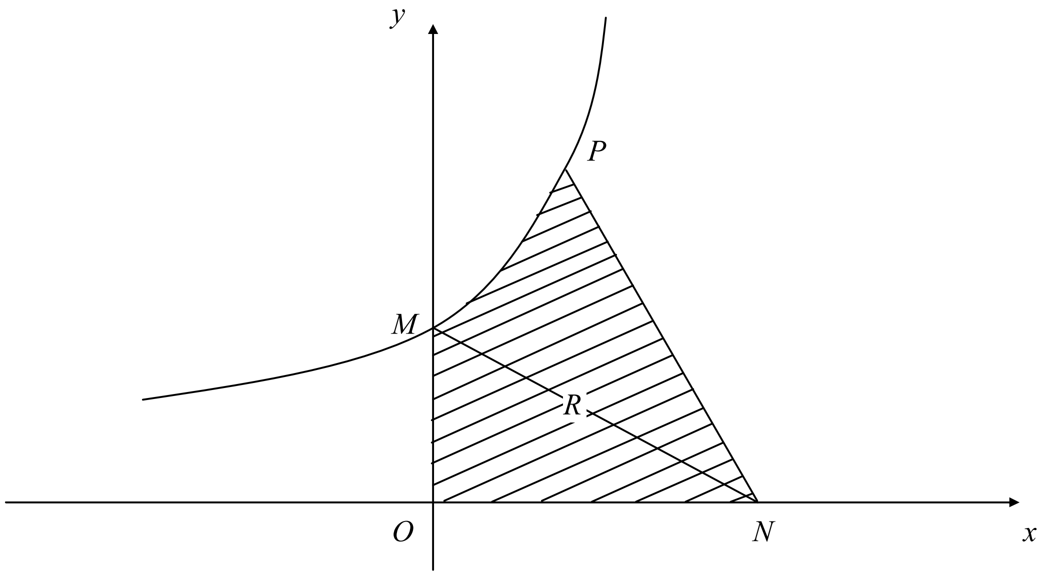
34. **Figure 3**



The curve C with equation $y = 2e^x + 5$ meets the y-axis at the point M, as shown in Fig. 3.

(a) Find the equation of the normal to C at M in the form $ax + by = c$, where a, b and c are integers.

(4)

This normal to C at M crosses the x-axis at the point $N(n, 0)$.

(b) Show that $n = 14$.

(1)

The point $P(\ln 4, 13)$ lies on C. The finite region R is bounded by C, the axes and the line PN, as shown in Fig. 3.

(c) Find the area of R, giving your answers in the form $p + q \ln 2$, where p and q are integers to be found.

(7)

[P2 June 2003 Question 7]

36. The curve C is described by the parametric equations

$$x = 3\cos t, \quad y = \cos 2t, \quad 0 \leq t \leq \pi.$$

(*a*) Find a cartesian equation of the curve C.

(2)

(*b*) Draw a sketch of the curve C.

(2)

[P3 June 2003 Question 1]

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37. A curve has equation $7x^2 + 48xy - 7y^2 + 75 = 0$.

A and B are two distinct points on the curve. At each of these points the gradient of the curve is equal to $\frac{2}{11}$.

(*a*) Use implicit differentiation to show that $x + 2y = 0$ at the points A and B. **(5)**

(*b*) Find the coordinates of the points A and B. **(4)**

[P3 June 2003 Question 4]

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38. **Figure 1**

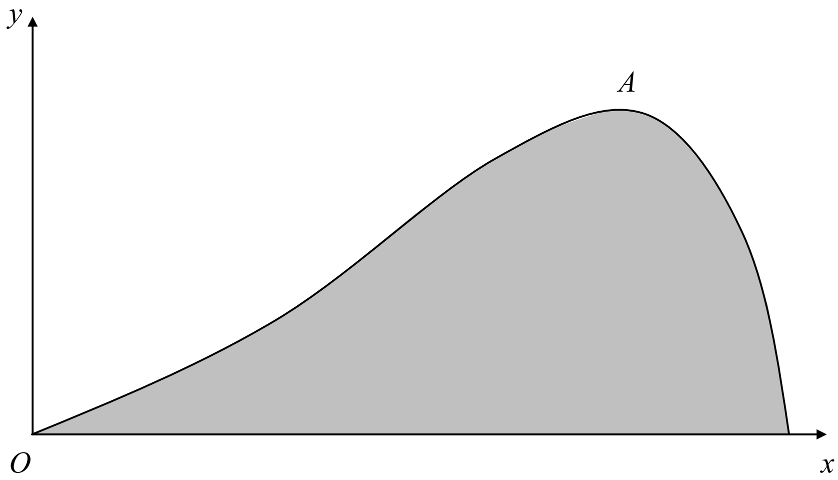

Figure 1 shows a graph of $y = x\sqrt{\sin x}$, $0 < x < \pi$. The maximum point on the curve is A.

(a) Show that the x-coordinate of the point A satisfies the equation $2 \tan x + x = 0$.

(4)

The finite region enclosed by the curve and the x-axis is shaded as shown in Fig. 1.

A solid body S is generated by rotating this region through 2π radians about the x-axis.

(b) Find the exact value of the volume of S.

(7)

[P3 June 2003 Question 5]

39. Relative to a fixed origin O, the point A has position vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, the point B has position vector $5\mathbf{i} + \mathbf{j} + \mathbf{k}$, and the point C has position vector $7\mathbf{i} - \mathbf{j}$.

(*a*) Find the cosine of angle ABC.

(4)

(*b*) Find the exact value of the area of triangle ABC.

(3)

The point D has position vector $7\mathbf{i} + 3\mathbf{k}$.

(*c*) Show that AC is perpendicular to CD.

(2)

(*d*) Find the ratio AD:DB.

(2)

[P3 June 2003 Question 6]

40.

Figure 2

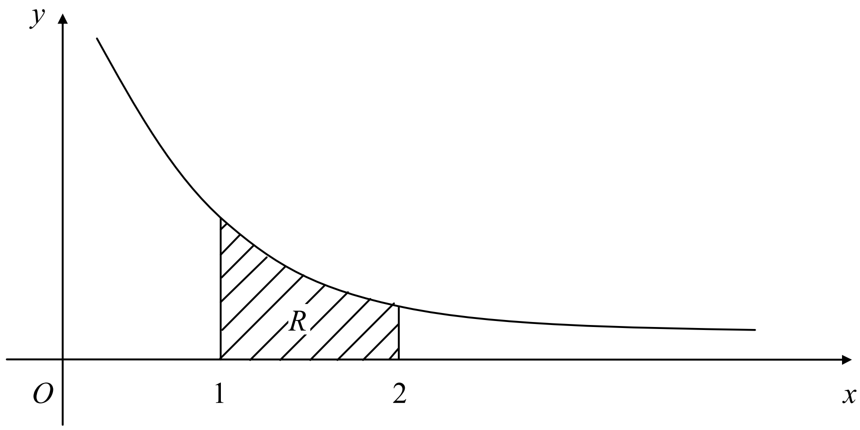

Figure 2 shows part of the curve with equation $y = 1 + \frac{c}{x}$, where c is a positive constant.

The point P with x-coordinate p lies on the curve. Given that the gradient of the curve at P is -4,

(*a*) show that $c = 4p^2$. **(2)**

Given also that the y-coordinate of P is 5,

(*b*) prove that $c = 4$. **(2)**

The region R is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 2$, as shown in Fig. 2. The region R is rotated through 360° about the x-axis.

(*c*) Show that the volume of the solid generated can be written in the form $\pi(k + q \ln 2)$, where k and q are constants to be found. **(7)**

[P2 June 2003 Question 6]

41. Fluid flows out of a cylindrical tank with constant cross section. At time t minutes, $t \geq 0$, the volume of fluid remaining in the tank is V m^3. The rate at which the fluid flows, in m^3 min^{-1}, is proportional to the square root of V.

(a) Show that the depth h metres of fluid in the tank satisfies the differential equation

$$\frac{dh}{dt} = -k\sqrt{h}, \qquad \text{where } k \text{ is a positive constant.}$$

(3)

(b) Show that the general solution of the differential equation may be written as

$$h = (A - Bt)^2, \qquad \text{where } A \text{ and } B \text{ are constants.}$$

(4)

Given that at time $t = 0$ the depth of fluid in the tank is 1 m, and that 5 minutes later the depth of fluid has reduced to 0.5 m,

(c) find the time, T minutes, which it takes for the tank to empty.

(3)

(d) Find the depth of water in the tank at time $0.5T$ minutes.

(2)

[P3 June 2003 Question 7]

42.

$$f(x) = \frac{25}{(3 + 2x)^2(1 - x)}, \quad |x| < 1.$$

(a) Express $f(x)$ as a sum of partial fractions.

(4)

(b) Hence find $\int f(x)\, dx$.

(5)

(c) Find the series expansion of $f(x)$ in ascending powers of x up to and including the term in x^2. Give each coefficient as a simplified fraction.

(7)

[P3 June 2003 Question 8]

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43. **Figure 1**

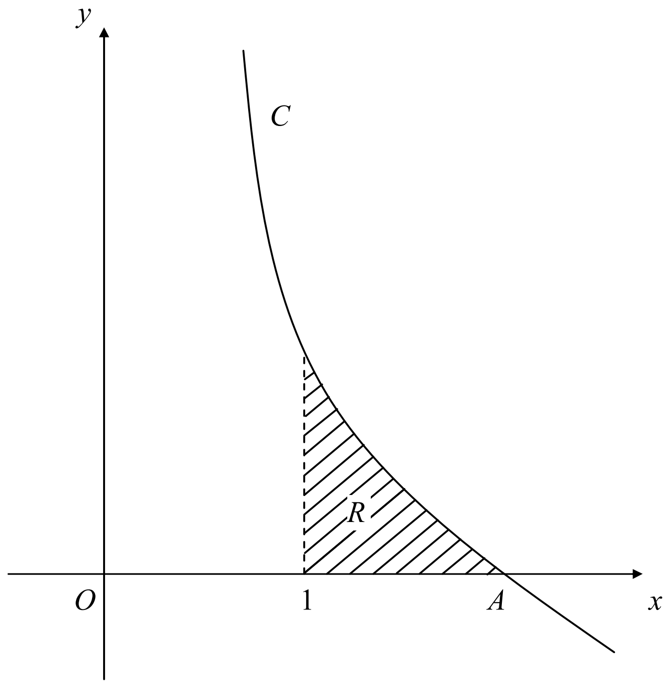


Figure 1 shows the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{8}{x} - x^2, \; x > 0.$$

Given that C crosses the x-axis at the point A,

(*a*) find the coordinates of A.

(3)

The finite region R, bounded by C, the x-axis and the line $x = 1$, is rotated through 2π radians about the x-axis.

(*b*) Use integration to find, in terms of π, the volume of the solid generated.

(7)

[P2 November 2003 Question 4]

44. **Figure 2**

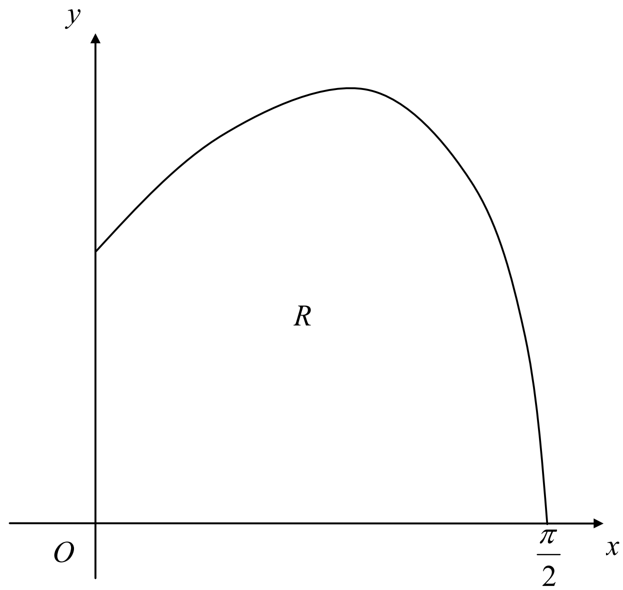


Figure 2 shows part of the curve with equation

$$y = e^x \cos x, \ \ 0 \le x \le \frac{\pi}{2}.$$

The finite region R is bounded by the curve and the coordinate axes.

(*a*) Calculate, to 2 decimal places, the y-coordinates of the points on the curve where $x = 0, \frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{\pi}{2}$.

(3)

(*b*) Using the trapezium rule and all the values calculated in part (*a*), find an approximation for the area of R.

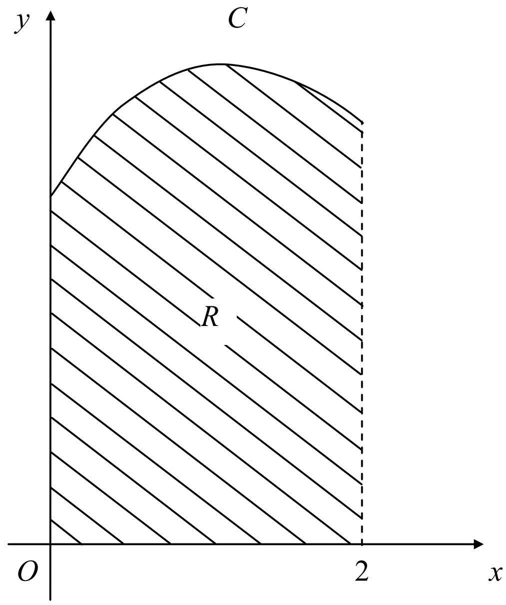
(4)

(*c*) State, with a reason, whether your approximation underestimates or overestimates the area of R.

(2)

[P2 November 2003 Question 6]

45. **Figure 1**



The curve C has equation $y = \mathrm{f}(x)$, $x \in \mathbb{R}$. Figure 1 shows the part of C for which $0 \leq x \leq 2$.

Given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - 2x^2,$$

and that C has a single maximum, at $x = k$,

(a) show that $1.48 < k < 1.49$.

(3)

Given also that the point (0, 5) lies on C,

(b) find $\mathrm{f}(x)$.

(4)

The finite region R is bounded by C, the coordinate axes and the line $x = 2$.

(c) Use integration to find the exact area of R.

(4)

[P2 January 2004 Question 6]

46. A student tests the accuracy of the trapezium rule by evaluating I, where

$$I = \int_{0.5}^{1.5} \left(\frac{3}{x} + x^4 \right) \mathrm{d}x .$$

(*a*) Complete the student's table, giving values to 2 decimal places where appropriate.

x	0.5	0.75	1	1.25	1.5
$\frac{3}{x} + x^4$	6.06	4.32			

(2)

(*b*) Use the trapezium rule, with all the values from your table, to calculate an estimate for the value of I.

(4)

(*c*) Use integration to calculate the exact value of I.

(4)

(*d*) Verify that the answer obtained by the trapezium rule is within 3% of the exact value.

(2)

[P2 January 2004 Question 7]

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47. When $(1 + ax)^n$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 27 respectively.

(*a*) Find the value of a and the value of n.

(5)

(*b*) Find the coefficient of x^3.

(2)

(*c*) State the set of values of x for which the expansion is valid.

(1)

[P3 January 2004 Question 2]

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48. The curve C has equation $5x^2 + 2xy - 3y^2 + 3 = 0$. The point P on the curve C has coordinates (1, 2).

(*a*) Find the gradient of the curve at P.

(5)

(*b*) Find the equation of the normal to the curve C at P, in the form $y = ax + b$, where a and b are constants.

(3)

[P3 January 2004 Question 3]

49.

Figure 1

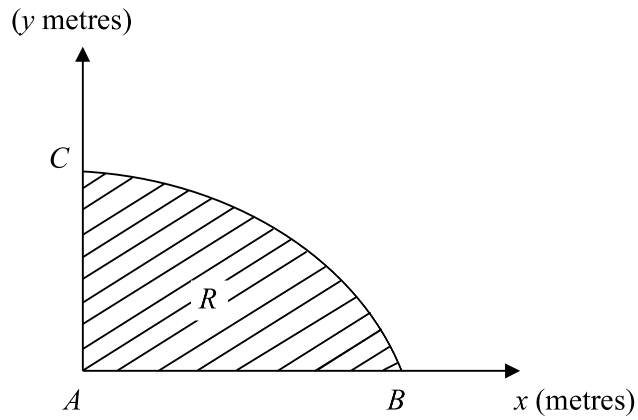

Figure 1 shows a cross-section R of a dam. The line AC is the vertical face of the dam, AB is the horizontal base and the curve BC is the profile. Taking x and y to be the horizontal and vertical axes, then A, B and C have coordinates (0, 0), $(3\pi^2, 0)$ and (0, 30) respectively. The area of the cross-section is to be calculated.

Initially the profile BC is approximated by a straight line.

(*a*) Find an estimate for the area of the cross-section R using this approximation.

(1)

The profile BC is actually described by the parametric equations.

$$x = 16t^2 - \pi^2, \quad y = 30 \sin 2t, \quad \frac{\pi}{4} \leq t \leq \frac{\pi}{2}.$$

(*b*) Find the exact area of the cross-section R.

(7)

(*c*) Calculate the percentage error in the estimate of the area of the cross-section R that you found in part (*a*).

(2)

[P3 January 2004 Question 5]

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50. (*a*) Express $\dfrac{13 - 2x}{(2x - 3)(x + 1)}$ in partial fractions.

(4)

(*b*) Given that $y = 4$ at $x = 2$, use your answer to part (*a*) to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(13 - 2x)}{(2x - 3)(x + 1)}, \quad x > 1.5$$

Express your answer in the form $y = \mathrm{f}(x)$.

(7)

[P3 January 2004 Question 6]

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51. The curve C has equation $y = \dfrac{x}{4 + x^2}$.

(*a*) Use calculus to find the coordinates of the turning points of C.

(5)

Using the result $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{2x(x^2 - 12)}{(4 + x^2)^3}$, or otherwise,

(*b*) determine the nature of each of the turning points.

(3)

(*c*) Sketch the curve C.

(3)

[P3 January 2004 Question 7]

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52. The equations of the lines l_1 and l_2 are given by

$$l_1: \quad \mathbf{r} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k}),$$

$$l_2: \quad \mathbf{r} = -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}),$$

where λ and μ are parameters.

(*a*) Show that l_1 and l_2 intersect and find the coordinates of Q, their point of intersection.

(6)

(*b*) Show that l_1 is perpendicular to l_2.

(2)

The point P with x-coordinate 3 lies on the line l_1 and the point R with x-coordinate 4 lies on the line l_2.

(*c*) Find, in its simplest form, the exact area of the triangle PQR.

(6)

[P3 January 2004 Question 8]

53. **Figure 1**

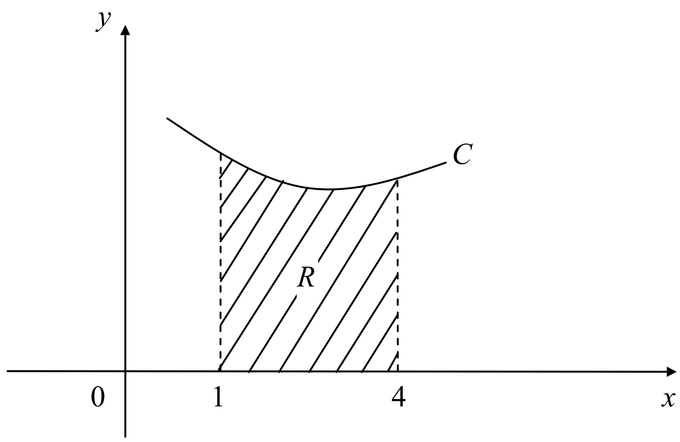

Figure 1 shows parts of the curve C with equation

$$y = \frac{x+2}{\sqrt{x}}.$$

The shaded region R is bounded by C, the x-axis and the lines $x = 1$ and $x = 4$.

This region is rotated through 360° about the x-axis to form a solid S.

(*a*) Find, by integration, the exact volume of S.

(7)

The solid S is used to model a wooden support with a circular base and a circular top.

(*b*) Show that the base and the top have the same radius.

(1)

Given that the actual radius of the base is 6 cm,

(*c*) show that the volume of the wooden support is approximately 630 cm^3.

(2)

[P2 June 2004 Question 4]

54. $$\mathrm{f}(x) = x + \frac{\mathrm{e}^x}{5}, \qquad x \in \mathbb{R}.$$

(*a*) Find $\mathrm{f}'(x)$.

(2)

The curve C, with equation $y = \mathrm{f}(x)$, crosses the y-axis at the point A.

(*b*) Find an equation for the tangent to C at A.

(3)

(*c*) Complete the table, giving the values of $\sqrt{\left(x + \frac{\mathrm{e}^x}{5}\right)}$ to 2 decimal places.

x	0	0.5	1	1.5	2
$\sqrt{\left(x + \frac{\mathrm{e}^x}{5}\right)}$	0.45	0.91			

(2)

(*d*) Use the trapezium rule, with all the values from your table, to find an approximation for the value of

$$\int_0^2 \sqrt{\left(x + \frac{\mathrm{e}^x}{5}\right)}\,\mathrm{d}x.$$

(4)

[P2 June 2004 Question 7]

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55. The circle C has centre (5, 13) and touches the x-axis.

(*a*) Find an equation of C in terms of x and y.

(2)

(*b*) Find an equation of the tangent to C at the point (10, 1), giving your answer in the form $ay + bx + c = 0$, where a, b and c are integers.

(5)

[P3 June 2004 Question 3]

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56. Use the substitution $u = 1 + \sin x$ and integration to show that

$$\int \sin x \cos x\,(1 + \sin x)^5\,\mathrm{d}x = \frac{1}{42}(1 + \sin x)^6\,[6 \sin x - 1] + \text{constant}.$$

(8)

[P3 June 2004 Question 4]

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57. Given that

$$\frac{3+5x}{(1+3x)(1-x)} \equiv \frac{A}{1+3x} + \frac{B}{1-x},$$

(*a*) find the values of the constants A and B.

(3)

(*b*) Hence, or otherwise, find the series expansion in ascending powers of x, up to and including the term in x^2, of

$$\frac{3+5x}{(1+3x)(1-x)}.$$

(5)

(*c*) State, with a reason, whether your series expansion in part (*b*) is valid for $x = \frac{1}{2}$.

(2)

[P3 June 2004 Question 5]

58. **Figure 1**

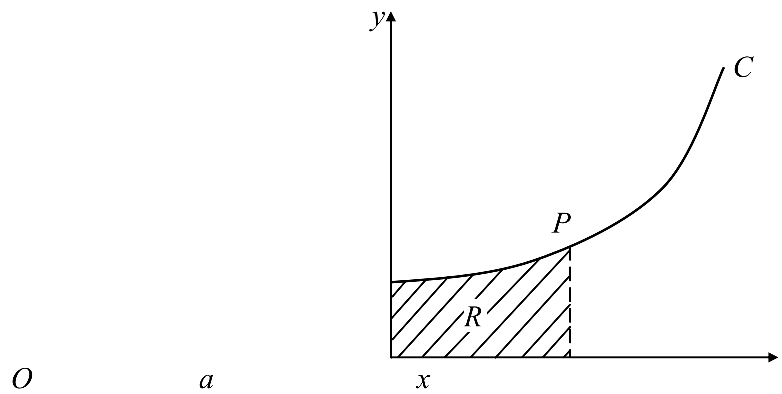

Figure 1 shows a sketch of the curve C with parametric equations

$$x = 3t \sin t,\ y = 2 \sec t, \qquad 0 \le t < \frac{\pi}{2}.$$

The point $P(a, 4)$ lies on C.

(a) Find the exact value of a.

(3)

The region R is enclosed by C, the axes and the line $x = a$ as shown in Fig. 1.

(b) Show that the area of R is given by

$$6\int_0^{\frac{\pi}{3}} (\tan t + t)\,\mathrm{d}t.$$

(4)

(c) Find the exact value of the area of R.

(4)

[P3 June 2004 Question 6]

59. A drop of oil is modelled as a circle of radius r. At time t

$$r = 4(1 - e^{-\lambda t}), \qquad t > 0,$$

where λ is a positive constant.

(a) Show that the area A of the circle satisfies

$$\frac{dA}{dt} = 32\pi\lambda(e^{-\lambda t} - e^{-2\lambda t}).$$

(5)

In an alternative model of the drop of oil its area A at time t satisfies

$$\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{t^2}, \qquad t > 0.$$

Given that the area of the drop is 1 at $t = 1$,

(b) find an expression for A in terms of t for this alternative model.

(7)

(c) Show that, in the alternative model, the value of A cannot exceed 4.

(1)

[P3 June 2004 Question 7]

60. Relative to a fixed origin O, the vector equations of the two lines l_1 and l_2 are

$$l_1: \mathbf{r} = 9\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + t(-8\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}),$$

and

$$l_2: \mathbf{r} = -16\mathbf{i} + \alpha\mathbf{j} + 10\mathbf{k} + s(\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}),$$

where α is a constant.

The two lines intersect at the point A.

(*a*) Find the value of α.

(6)

(*b*) Find the position vector of the point A.

(1)

(*c*) Prove that the acute angle between l_1 and l_2 is 60°.

(5)

Point B lies on l_1 and point C lies on l_2. The triangle ABC is equilateral with sides of length $14\sqrt{2}$.

(*d*) Find one of the possible position vectors for the point B and the corresponding position vector for the point C.

(4)

[P3 June 2004 Question 8]