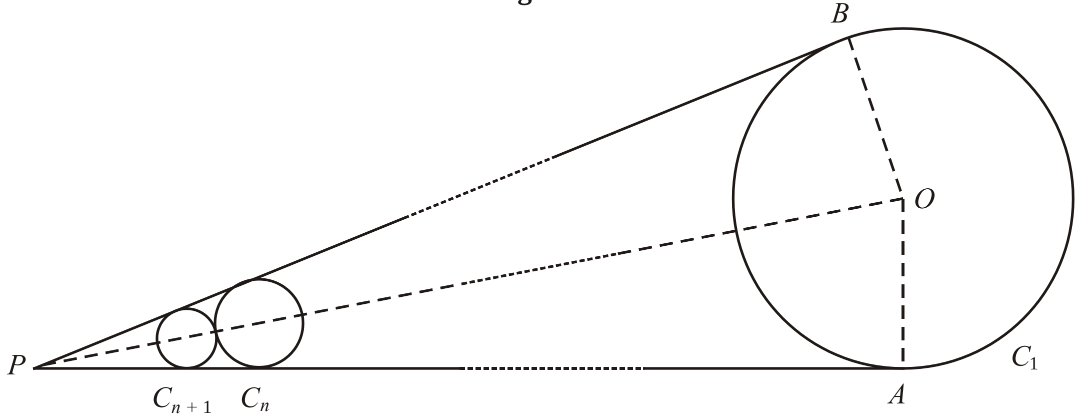
1.

Figure 2



The circle C_1 has centre O and radius R. The tangents AP and BP to C_1 meet at the point P and angle $APB = 2\alpha$, $0 < \alpha < \frac{\pi}{2}$. A sequence of circles $C_1, C_2, ..., C_n, ...$ is drawn so that each new circle C_{n+1} touches each of C_n, AP and BP for $n = 1, 2, 3, ...$ as shown in the figure above. The centre of each circle lies on the line OP.

(a) Show that the radii of the circles form a geometric sequence with common ratio

$$\frac{1 - \sin\alpha}{1 + \sin\alpha}.$$

(5)

(b) Find, in terms of R and α, the total area enclosed by all the circles, simplifying your answer.

(3)

The area inside the quadrilateral $PAOB$, not enclosed by part of C_1 or any of the other circles, is S.

(c) Show that

$$S = R^2 \left(\alpha + \cot\alpha - \frac{\pi}{4}\operatorname{cosec}\alpha - \frac{\pi}{4}\sin\alpha\right).$$

(5)

(d) Show that, as α varies,

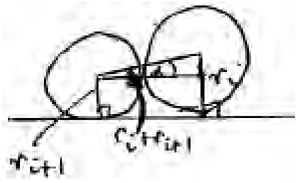
$$\frac{\mathrm{d}S}{\mathrm{d}\alpha} = R^2 \cot^2 \alpha \left(\frac{\pi}{4} \cos \alpha - 1 \right).$$

(4)

(e) Find, in terms of R, the least value of S for $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{4}$.

(3)
(Total 20 marks)

1. (a)



Appropriate figure M1

$\Rightarrow \sin\alpha = \dfrac{r_i - r_{i+1}}{r_i + r_{i+1}}$ (exp for $\sin\alpha$) M1, A1

$\therefore r_i + r_{i+1})\sin\alpha = r_i - r_{i+1}$ $\left(\dfrac{r_{i+1}}{r_i}\right)$ M1

$\therefore$ ration of radii $= \dfrac{1-\sin\alpha}{1+\sin\alpha}$ * (=r) A1 c.s.o. 5

(b) Total area $= \pi R^2 + \pi r_2^{\,2} + \pi r_3^{\,2} + \ldots$

$= \pi R^2(1 + r^2 + r^4 + \ldots)$ (correct "r") B1

$= \dfrac{\pi R^2}{1-r^2} = \pi R^2 \dfrac{1}{1-\left(\dfrac{1-\sin\alpha}{1+\sin\alpha}\right)^2}$ M1

$= \dfrac{\pi R^2(1+\sin\alpha)^2}{(1+\sin\alpha)^2 - (1-\sin\alpha)^2}$ $= \dfrac{\pi R^2(1+\sin\alpha)^2}{4\sin\alpha}$ A1 3

(c) Required area = 2 × Area Δ POA + Area major sector AOB

– Area found in (b) M1

Area Δ POA = $\dfrac{1}{2}R(R\cot\alpha)$ B1

$\angle POA = \dfrac{\pi}{2} - \alpha$ $\therefore$ angle of major sector to B $= \pi + 2\alpha$ M1

$\therefore$ Area sector AOB = $\dfrac{1}{2}R^2(\pi + 2\alpha)$ A1

$\therefore$ Required area = $R^2(\cot\alpha + \dfrac{\pi}{2} + d - \dfrac{\pi}{4}\left(\dfrac{1 + 2\sin\alpha + \sin^2\alpha}{\sin\alpha}\right)$ A1 c.s.o. 5

$= R^2(\alpha + \cot\alpha - \dfrac{\pi}{4}\operatorname{cosec}\alpha - \dfrac{\pi}{4}\sin\alpha)$

(d) $\dfrac{ds}{d\alpha} = R^2\left(1 - \operatorname{cosec}^2\alpha + \dfrac{\pi}{4}\operatorname{cosec}\alpha\cot\alpha - \dfrac{\pi}{4}\cos\alpha\right)$ M1, A1

$= R^2(-\cot^2\alpha + \frac{\pi}{4}\frac{\cos\alpha}{\sin\alpha} - \frac{\pi}{4}\cos\alpha)$

$= R^2(-\cot^2\alpha + \frac{\pi}{4}\cos\alpha(\text{cosec}^2\alpha - 1))$ M1

$= R^2\cot^2\alpha(\frac{\pi}{4}\cos\alpha - 1)$ A1 (c.s.o) 4

(use of $\cot^2\alpha = \text{cosec}^{\alpha} - 1$) o.e.

(e) In the given range $R^2\cot^2\alpha > 0$

In the interval

$(0, \frac{\pi}{4})$; $\frac{\pi}{4}\cos\alpha - 1$ is a decreasing function ($\because$ $\cos\alpha$ is decreasing).

At $\alpha = 0$, $\frac{\pi}{4}\cos\alpha - 1 = \frac{\pi}{4} - 1 < 0$

$\therefore \frac{\pi}{4}\cos\alpha - 1 < 0$ in $(\frac{\pi}{6}, \frac{\pi}{4})$

$\therefore \frac{ds}{d\alpha} < 0$ throughout the interval (convincing argument) M1

$\therefore$ Least value of S occurs at $\alpha = \frac{\pi}{4}$ A1

Min S $= R^2\left(\frac{\pi}{4} + 1 - \frac{\pi}{4}.\sqrt{2} - \frac{\pi}{4}.\frac{1}{\sqrt{2}}\right)$

$= R^2\left(1 - \frac{\pi}{4}\left(-1 + \sqrt{2} + \frac{1}{\sqrt{2}}\right)\right)$ o.e. A1 3

[20]

1. There were mixed responses to this question. Many candidates made very little progress and quite a number just carried out the differentiation in part (d). Reasonable diagrams to help with part (a) were rarely seen. Often terms were used without either a diagram or an explanation, leaving it to the examiner to interpret what the candidate was trying to do. The most successful approach was to consider two similar triangles POB and PO_2B_2 and forming sin α for each. Many were then unable to formulate the geometric sequence for the total area of the circles, so there were even fewer correct answers in the required simplified form. It was disappointing that so many attempts were dimensionally incorrect.

Part (c) proved to be difficult. Few dealt with the major arc of circle C_1. Answers to part (e) proved to be even more elusive. Many equated the derivative to zero and seemed happy to state that the least value occurred when $\cos\alpha = 4/\pi$. Some better efforts arrived at this point, realised that this had no solution and then tried to show that S was either a decreasing or an increasing function in the interval $[\pi/6, \pi/4]$. There were very few complete solutions to this part. It seems that even the best candidates for this paper are unaware that maxima and minima are local events firstly and only sometimes global maxima/minima.