Possible C3 questions from past papers P1—P3

Source of the original question is given in brackets, e.g. [P2 January 2001 Question 1]; a question which has been edited is indicated with an asterisk, e.g. [P3 January 2003 Question 8*].

1. The function f, defined for $x \in \mathbb{R}$, $x > 0$, is such that

$$f'(x) = x^2 - 2 + \frac{1}{x^2}.$$

(*a*) Find the value of $f''(x)$ at $x = 4$. **(3)**

(*b*) Given that $f(3) = 0$, find $f(x)$. **(4)**

(*c*) Prove that f is an increasing function. **(3)**

[P1 June 2001 Question 5]

2. The curve C has equation $y = 2e^x + 3x^2 + 2$. The point A with coordinates (0, 4) lies on C. Find the equation of the tangent to C at A. **(5)**

[P2 June 2001 Question 1]

3. The root of the equation $f(x) = 0$, where

$$f(x) = x + \ln 2x - 4$$

is to be estimated using the iterative formula $x_{n+1} = 4 - \ln 2x_n$, with $x_0 = 2.4$.

(*a*) Showing your values of $x_1, x_2, x_3, \ldots$, obtain the value, to 3 decimal places, of the root. **(4)**

(*b*) By considering the change of sign of $f(x)$ in a suitable interval, justify the accuracy of your answer to part (*a*). **(2)**

[P2 June 2001 Question 2]

4. (i) Prove, by counter-example, that the statement

"$\sec(A + B) \equiv \sec A + \sec B$, for all A and B"

is false. **(2)**

(ii) Prove that

$$\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta, \quad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}.$$ **(5)**

[P2 June 2001 Question 4]

5. The function f is given by

$$\mathrm{f} : x \mapsto \frac{x}{x^2-1} - \frac{1}{x+1}, \quad x > 1.$$

(*a*) Show that $\mathrm{f}(x) = \dfrac{1}{(x-1)(x+1)}$. **(3)**

(*b*) Find the range of f. **(2)**

The function g is given by

$$\mathrm{g} : x \mapsto \frac{2}{x}, \quad x > 0.$$

(*c*) Solve $\mathrm{gf}(x) = 70$. **(4)**

[P2 June 2001 Question 7]

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6. (*a*) Express $2 \cos \theta + 5 \sin \theta$ in the form $R \cos (\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$.
Give the values of R and α to 3 significant figures. **(3)**

(*b*) Find the maximum and minimum values of $2 \cos \theta + 5 \sin \theta$ and the smallest possible value of θ for which the maximum occurs. **(2)**

The temperature T °C, of an unheated building is modelled using the equation

$$T = 15 + 2\cos\left(\frac{\pi t}{12}\right) + 5\sin\left(\frac{\pi t}{12}\right), \quad 0 \le t < 24,$$

where t hours is the number of hours after 1200.

(*c*) Calculate the maximum temperature predicted by this model and the value of t when this maximum occurs. **(4)**

(*d*) Calculate, to the nearest half hour, the times when the temperature is predicted to be 12 °C. **(6)**

[P2 June 2001 Question 9]

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7. The function f is defined by

$$\mathrm{f} : x \mapsto |2x - a|, \quad x \in \mathbb{R},$$

where a is a positive constant.

(*a*) Sketch the graph of $y = \mathrm{f}(x)$, showing the coordinates of the points where the graph cuts the axes. **(2)**

(*b*) On a separate diagram, sketch the graph of $y = \mathrm{f}(2x)$, showing the coordinates of the points where the graph cuts the axes. **(2)**

(*c*) Given that a solution of the equation $\mathrm{f}(x) = \frac{1}{2}x$ is $x = 4$, find the two possible values of a. **(4)**

[P2 January 2002 Question 3]

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8. (*a*) Prove that

$$\frac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta, \; \theta \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$ **(3)**

(*b*) Solve, giving exact answers in terms of π,

$$2(1 - \cos 2\theta) = \tan\theta, \qquad 0 < \theta < \pi.$$ **(6)**

[P2 January 2002 Question 6]

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9. **Figure 2**

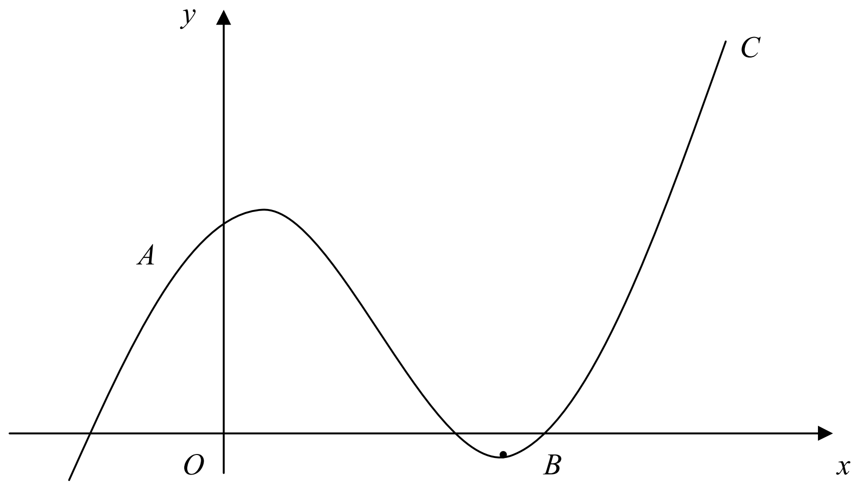

Figure 2 shows part of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 0.5\mathrm{e}^x - x^2.$$

The curve C cuts the y-axis at A and there is a minimum at the point B.

(a) Find an equation of the tangent to C at A. **(4)**

The x-coordinate of B is approximately 2.15. A more exact estimate is to be made of this coordinate using iterations $x_{n+1} = \ln \mathrm{g}(x_n)$.

(b) Show that a possible form for $\mathrm{g}(x)$ is $\mathrm{g}(x) = 4x$. **(3)**

(c) Using $x_{n+1} = \ln 4x_n$, with $x_0 = 2.15$, calculate x_1 , x_2 and x_3. Give the value of x_3 to 4 decimal places. **(2)**

[P2 January 2002 Question 7]

10. $$f(x) = \frac{2}{x-1} - \frac{6}{(x-1)(2x+1)}, \quad x > 1.$$

(*a*) Prove that $f(x) = \dfrac{4}{2x+1}$. **(4)**

(*b*) Find the range of f. **(2)**

(*c*) Find $f^{-1}(x)$. **(3)**

(*d*) Find the range of $f^{-1}(x)$. **(1)**

[P2 January 2002 Question 8]

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11. Use the derivatives of $\sin x$ and $\cos x$ to prove that the derivative of $\tan x$ is $\sec^2 x$. **(4)**

[P3 January 2002 Question 2]

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12. Express $\dfrac{3}{x^2 + 2x} + \dfrac{x-4}{x^2-4}$ as a single fraction in its simplest form. **(7)**

[P2 June 2002 Question 2]

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13. **Figure 1**

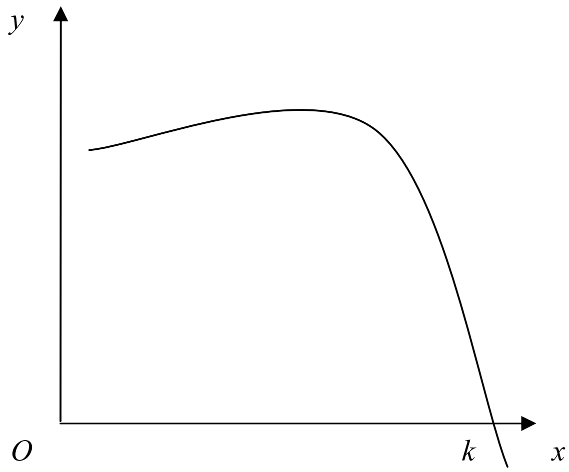

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 10 + \ln(3x) - \tfrac{1}{2}\mathrm{e}^{x}, \quad 0.1 \le x \le 3.3.$$

Given that $\mathrm{f}(k) = 0$,

(*a*) show, by calculation, that $3.1 < k < 3.2$. **(2)**

(*b*) Find $\mathrm{f}'(x)$. **(3)**

The tangent to the graph at $x = 1$ intersects the y-axis at the point P.

(*c*) (i) Find an equation of this tangent.

(ii) Find the exact y-coordinate of P, giving your answer in the form $a + \ln b$. **(5)**

[P2 June 2002 Question 6]

14. $$\mathrm{f}(x) = x^2 - 2x - 3,\ x \in \mathbb{R}, x \geq 1.$$

(*a*) Find the range of f. **(1)**

(*b*) Write down the domain and range of f^{-1}. **(2)**

(*c*) Sketch the graph of f^{-1}, indicating clearly the coordinates of any point at which the graph intersects the coordinate axes. **(4)**

Given that $\mathrm{g}(x) = |x - 4|, x \in \mathbb{R}$,

(*d*) find an expression for $\mathrm{gf}(x)$. **(2)**

(*e*) Solve $\mathrm{gf}(x) = 8$. **(5)**

[P2 June 2002 Question 8]

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15. Express $\dfrac{y+3}{(y+1)(y+2)} - \dfrac{y+1}{(y+2)(y+3)}$ as a single fraction in its simplest form.

(5)

[P2 November 2002 Question 1]

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16. (*a*) Express $1.5 \sin 2x + 2 \cos 2x$ in the form $R \sin (2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving your values of R and α to 3 decimal places where appropriate. **(4)**

(*b*) Express $3 \sin x \cos x + 4 \cos^2 x$ in the form $a \cos 2x + b \sin 2x + c$, where a, b and c are constants to be found. **(2)**

(*c*) Hence, using your answer to part (*a*), deduce the maximum value of $3 \sin x \cos x + 4 \cos^2 x$. **(2)**

[P2 November 2002 Question 3]

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17. The curve C with equation $y = p + q\mathrm{e}^x$, where p and q are constants, passes through the point $(0, 2)$. At the point $P(\ln 2, p + 2q)$ on C, the gradient is 5.

(*a*) Find the value of p and the value of q. **(5)**

The normal to C at P crosses the x-axis at L and the y-axis at M.

(*b*) Show that the area of $\Delta\ OLM$, where O is the origin, is approximately 53.8. **(5)**

[P2 November 2002 Question 5]

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18. **Figure 1**

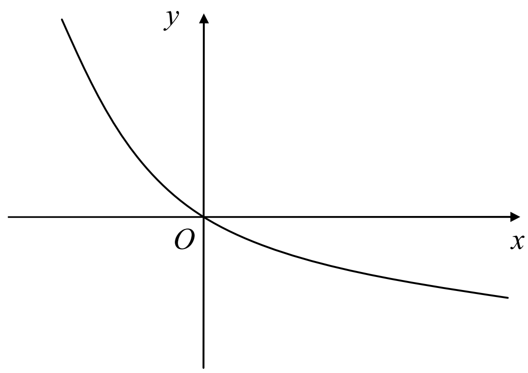

Figure 1 shows a sketch of the curve with equation $y = e^{-x} - 1$.

(a) Copy Fig. 1 and on the same axes sketch the graph of $y = \frac{1}{2}|x - 1|$. Show the coordinates of the points where the graph meets the axes. **(2)**

The x-coordinate of the point of intersection of the graph is α.

(b) Show that $x = \alpha$ is a root of the equation $x + 2e^{-x} - 3 = 0$. **(3)**

(c) Show that $-1 < \alpha < 0$. **(2)**

The iterative formula $x_{n+1} = -\ln[\frac{1}{2}(3 - x_n)]$ is used to solve the equation $x + 2e^{-x} - 3 = 0$.

(d) Starting with $x_0 = -1$, find the values of x_1 and x_2. **(2)**

(e) Show that, to 2 decimal places, $\alpha = -0.58$. **(2)**

[P2 November 2002 Question 6]

19. The function f is defined by f: $x \mapsto \dfrac{3x-1}{x-3}$, $x \in \mathbb{R}$, $x \neq 3$.

(a) Prove that $f^{-1}(x) = f(x)$ for all $x \in \mathbb{R}$, $x \neq 3$. **(3)**

(b) Hence find, in terms of k, ff(k), where $x \neq 3$. **(2)**

Figure 3

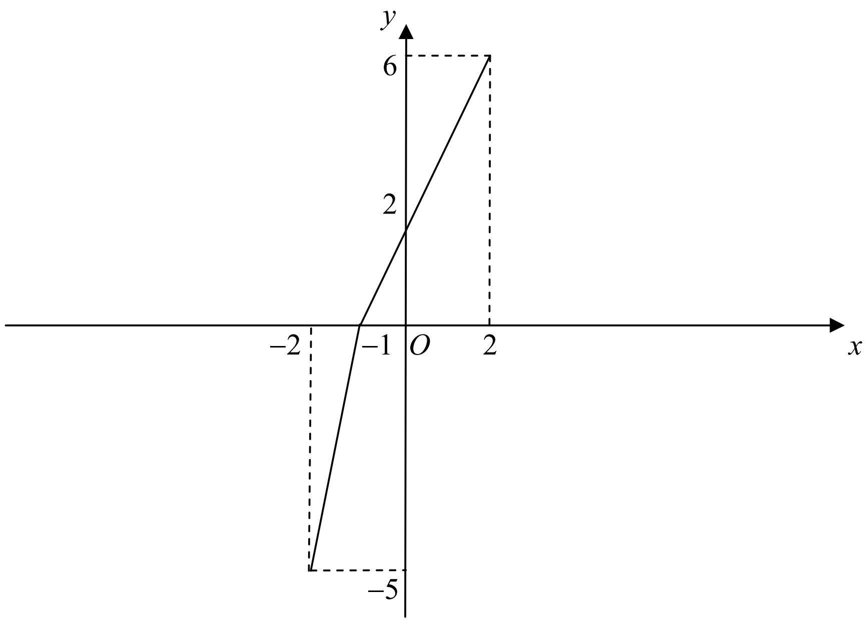

Figure 3 shows a sketch of the one-one function g, defined over the domain $-2 \leq x \leq 2$.

(c) Find the value of fg(–2). **(3)**

(d) Sketch the graph of the inverse function g^{-1} and state its domain. **(3)**

The function h is defined by h: $x \mapsto 2g(x-1)$.

(e) Sketch the graph of the function h and state its range. **(3)**

[P2 November 2002 Question 8]

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20. Express $\dfrac{x}{(x+1)(x+3)} + \dfrac{x+12}{x^2-9}$ as a single fraction in its simplest form. **(6)**

[P2 January 2003 Question 1]

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21. (*a*) Sketch the graph of $y = |2x + a|$, $a > 0$, showing the coordinates of the points where the graph meets the coordinate axes. **(2)**

(*b*) On the same axes, sketch the graph of $y = \dfrac{1}{x}$. **(1)**

(*c*) Explain how your graphs show that there is only one solution of the equation

$$x|2x + a| - 1 = 0.$$ **(1)**

(*d*) Find, using algebra, the value of x for which $x|2x + 1| - 1 = 0$. **(3)**

[P2 January 2003 Question 3]

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22. The curve with equation $y = \ln 3x$ crosses the x-axis at the point P $(p, 0)$.

(*a*) Sketch the graph of $y = \ln 3x$, showing the exact value of p. **(2)**

The normal to the curve at the point Q, with x-coordinate q, passes through the origin.

(*b*) Show that $x = q$ is a solution of the equation $x^2 + \ln 3x = 0$. **4)**

(*c*) Show that the equation in part (*b*) can be rearranged in the form $x = \frac{1}{3}e^{-x^2}$. **(2)**

(*d*) Use the iteration formula $x_{n+1} = \frac{1}{3}e^{-x_n^2}$, with $x_0 = \frac{1}{3}$, to find x_1, x_2, x_3 and x_4. Hence write down, to 3 decimal places, an approximation for q. **(3)**

[P2 January 2003 Question 6]

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23. (*a*) Express $\sin x + \sqrt{3} \cos x$ in the form $R \sin (x + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. **(4)**

(*b*) Show that the equation $\sec x + \sqrt{3} \operatorname{cosec} x = 4$ can be written in the form

$$\sin x + \sqrt{3} \cos x = 2 \sin 2x.$$ **(3)**

(*c*) Deduce from parts (*a*) and (*b*) that $\sec x + \sqrt{3} \operatorname{cosec} x = 4$ can be written in the form

$$\sin 2x - \sin (x + 60°) = 0.$$ **(1)**

[P2 January 2003 Question 7*]

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24. **Figure 3**

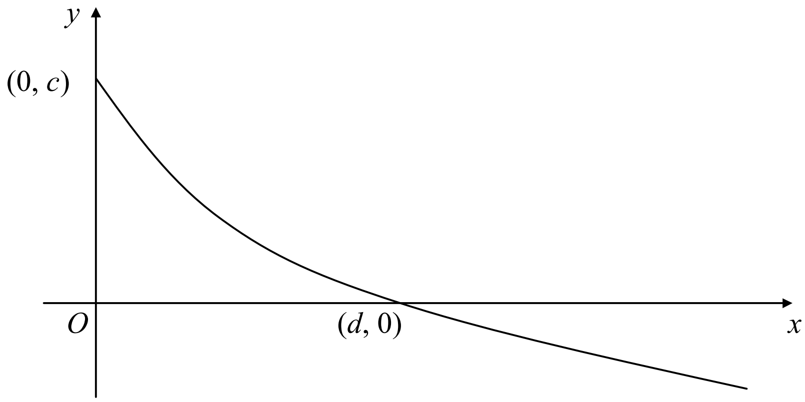

Figure 3 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, $x \geq 0$. The curve meets the coordinate axes at the points $(0, c)$ and $(d, 0)$.

In separate diagrams sketch the curve with equation

(*a*) $y = \mathrm{f}^{-1}(x)$, **(2)**

(*b*) $y = 3\mathrm{f}(2x)$. **(3)**

Indicate clearly on each sketch the coordinates, in terms of c or d, of any point where the curve meets the coordinate axes.

Given that f is defined by

$$\mathrm{f} : x \mapsto 3(2^{-x}) - 1,\ x \in \mathbb{R},\ x \geq 0,$$

(*c*) state

(i) the value of c,

(ii) the range of f. **(3)**

(*d*) Find the value of d, giving your answer to 3 decimal places. **(3)**

The function g is defined by

$$\mathrm{g} : x \mapsto \log_2 x,\ x \in \mathbb{R},\ x \geq 1.$$

(*e*) Find $\mathrm{fg}(x)$, giving your answer in its simplest form. **(3)**

[P2 January 2003 Question 8

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25. (*a*) Simplify $\dfrac{x^2 + 4x + 3}{x^2 + x}$. **(2)**

(*b*) Find the value of x for which $\log_2(x^2 + 4x + 3) - \log_2(x^2 + x) = 4$. **(4)**

[P2 June 2003 Question 1]

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26. The functions f and g are defined by

$$\mathrm{f}: x \mapsto x^2 - 2x + 3,\ x \in \mathbb{R},\ \ 0 \le x \le 4,$$

$$\mathrm{g}: x \mapsto \lambda x^2 + 1, \text{ where } \lambda \text{ is a constant, } x \in \mathbb{R}.$$

(*a*) Find the range of f. **(3)**

(*b*) Given that gf(2) = 16, find the value of λ. **(3)**

[P2 June 2003 Question 2]

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27. **Figure 1**

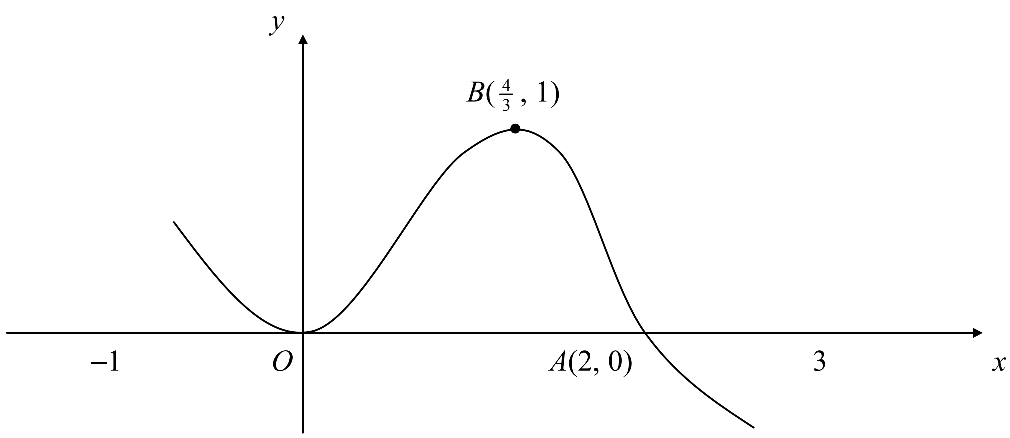

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, $-1 \le x \le 3$. The curve touches the x-axis at the origin O, crosses the x-axis at the point $A(2, 0)$ and has a maximum at the point $B(\frac{4}{3}, 1)$.

In separate diagrams, show a sketch of the curve with equation

(*a*) $y = \mathrm{f}(x + 1)$, **(3)**

(*b*) $y = |\mathrm{f}(x)|$, **(3)**

(*c*) $y = \mathrm{f}(|x|)$, **(4)**

marking on each sketch the coordinates of points at which the curve

(i) has a turning point,

(ii) meets the x-axis.

[P2 June 2003 Question 4]

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28. (*a*) Sketch, on the same set of axes, the graphs of

$$y = 2 - e^{-x} \text{ and } y = \sqrt{x}.$$ **(3)**

[It is not necessary to find the coordinates of any points of intersection with the axes.]

Given that $f(x) = e^{-x} + \sqrt{x} - 2, \; x \geq 0$,

(*b*) explain how your graphs show that the equation $f(x) = 0$ has only one solution,
(1)

(*c*) show that the solution of $f(x) = 0$ lies between $x = 3$ and $x = 4$. **(2)**

The iterative formula $x_{n+1} = (2 - e^{-x_n})^2$ is used to solve the equation $f(x) = 0$.

(*d*) Taking $x_0 = 4$, write down the values of x_1, x_2, x_3 and x_4, and hence find an approximation to the solution of $f(x) = 0$, giving your answer to 3 decimal places. **(4)**

[P2 June 2003 Question 5]

28a. (i) Given that $\cos(x + 30)° = 3\cos(x - 30)°$, prove that $\tan x° = -\dfrac{\sqrt{3}}{2}$.
(5)

(ii) (*a*) Prove that $\dfrac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$.
(3)

(*b*) Verify that $\theta = 180°$ is a solution of the equation $\sin 2\theta = 2 - 2\cos 2\theta$.
(1)

(*c*) Using the result in part (*a*), or otherwise, find the other two solutions, $0 < \theta < 360°$, of the equation using $\sin 2\theta = 2 - 2\cos 2\theta$.
(4)

[P2 June 2003 Question 8]

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29. (*a*) Express as a fraction in its simplest form

$$\frac{2}{x-3}+\frac{13}{x^2+4x-21}. \qquad \textbf{(3)}$$

(*b*) Hence solve

$$\frac{2}{x-3}+\frac{13}{x^2+4x-21}=1. \qquad \textbf{(3)}$$

[P2 November 2003 Question 1]

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30. Prove that

$$\frac{1-\tan^2\theta}{1+\tan^2\theta}\equiv\cos 2\theta. \qquad \textbf{(4)}$$

[P2 November 2003 Question 5*]

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31. The functions f and g are defined by

$$\text{f}: x \mapsto |x-a|+a, \quad x \in \mathbb{R},$$

$$\text{g}: x \mapsto 4x+a, \qquad x \in \mathbb{R}.$$

where a is a positive constant.

(*a*) On the same diagram, sketch the graphs of f and g, showing clearly the coordinates of any points at which your graphs meet the axes. **(5)**

(*b*) Use algebra to find, in terms of a, the coordinates of the point at which the graphs of f and g intersect. **(3)**

(*c*) Find an expression for fg(x). **(2)**

(*d*) Solve, for x in terms of a, the equation

$$\text{fg}(x) = 3a. \qquad \textbf{(3)}$$

[P2 November 2003 Question 7]

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32. The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 3 \ln x + \frac{1}{x}, \quad x > 0.$$

The point P is a stationary point on C.

(*a*) Calculate the x-coordinate of P. **(4)**

(*b*) Show that the y-coordinate of P may be expressed in the form $k - k \ln k$, where k is a constant to be found. **(2)**

The point Q on C has x-coordinate 1.

(*c*) Find an equation for the normal to C at Q. **(4)**

The normal to C at Q meets C again at the point R.

(*d*) Show that the x-coordinate of R

(i) satisfies the equation $6 \ln x + x + \frac{2}{x} - 3 = 0$,

(ii) lies between 0.13 and 0.14. **(4)**

[P2 November 2003 Question 8]

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33. The function f is given by $\mathrm{f} : x \mapsto 2 + \frac{3}{x+2}, \quad x \in \mathbb{R}, \quad x \neq -2.$

(*a*) Express $2 + \frac{3}{x+2}$ as a single fraction. **(1)**

(*b*) Find an expression for $\mathrm{f}^{-1}(x)$. **(3)**

(*c*) Write down the domain of f^{-1}. **(1)**

[P2 January 2004 Question 1]

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34. The function f is even and has domain $\mathbb{R}$. For $x \geq 0$, $f(x) = x^2 - 4ax$, where a is a positive constant.

(*a*) In the space below, sketch the curve with equation $y = f(x)$, showing the coordinates of all the points at which the curve meets the axes. **(3)**

(*b*) Find, in terms of a, the value of $f(2a)$ and the value of $f(-2a)$. **(2)**

Given that $a = 3$,

(*c*) use algebra to find the values of x for which $f(x) = 45$. **(4)**

[P2 January 2004 Question 4]

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35. Given that $y = \log_a x, \quad x > 0$, where a is a positive constant,

(*a*) (i) express x in terms of a and y, **(1)**

(ii) deduce that $\ln x = y \ln a$. **(1)**

(*b*) Show that $\dfrac{dy}{dx} = \dfrac{1}{x \ln a}$. **(2)**

The curve C has equation $y = \log_{10} x, \quad x > 0$. The point A on C has x-coordinate 10. Using the result in part (*b*),

(*c*) find an equation for the tangent to C at A. **(4)**

The tangent to C at A crosses the x-axis at the point B.

(*d*) Find the exact x-coordinate of B. **(2)**

[P2 January 2004 Question 5]

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36. (i) (*a*) Express $(12 \cos \theta - 5 \sin \theta)$ in the form $R \cos (\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$.

(4)

(*b*) Hence solve the equation

$$12 \cos \theta - 5 \sin \theta = 4,$$

for $0 < \theta < 90°$, giving your answer to 1 decimal place. **(3)**

(ii) Solve

$$8 \cot \theta - 3 \tan \theta = 2,$$

for $0 < \theta < 90°$, giving your answer to 1 decimal place. **(5)**

[P2 January 2004 Question 8]

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37. Express as a single fraction in its simplest form

$$\frac{x^2 - 8x + 15}{x^2 - 9} \times \frac{2x^2 + 6x}{(x-5)^2}.$$

(4)

[P2 June 2004 Question 1]

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38. (i) Given that $\sin x = \frac{3}{5}$, use an appropriate double angle formula to find the exact value of $\sec 2x$.

(4)

(ii) Prove that

$$\cot 2x + \operatorname{cosec} 2x \equiv \cot x, \qquad \left(x \neq \frac{n\pi}{2}, n \in \mathbb{Z}\right).$$

(4)

[P2 June 2004 Question 2]

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39. $$f(x) = x^3 + x^2 - 4x - 1.$$

The equation $f(x) = 0$ has only one positive root, α.

(*a*) Show that $f(x) = 0$ can be rearranged as

$$x = \sqrt{\left(\frac{4x+1}{x+1}\right)}, \ x \neq -1.$$

(2)

The iterative formula $x_{n+1} = \sqrt{\left(\frac{4x_n+1}{x_n+1}\right)}$ is used to find an approximation to α.

(*b*) Taking $x_1 = 1$, find, to 2 decimal places, the values of x_2, x_3 and x_4.

(3)

(*c*) By choosing values of x in a suitable interval, prove that $\alpha = 1.70$, *correct* to 2 decimal places.

(3)

(*d*) Write down a value of x_1 for which the iteration formula $x_{n+1} = \sqrt{\left(\frac{4x_n+1}{x_n+1}\right)}$ does *not* produce a valid value for x_2.

Justify your answer.

(2)

[P2 June 2004 Question 5]

40. $$f(x) = x + \frac{e^x}{5}, \qquad x \in \mathbb{R}.$$

(*a*) Find $f'(x)$.

(2)

The curve C, with equation $y = f(x)$, crosses the y-axis at the point A.

(*b*) Find an equation for the tangent to C at A.

(3)

(*c*) Complete the table, giving the values of $\sqrt{\left(x + \frac{e^x}{5}\right)}$ to 2 decimal places.

x	0	0.5	1	1.5	2
$\sqrt{\left(x + \frac{e^x}{5}\right)}$	0.45	0.91			

(2)

[P2 June 2004 Question 7*]

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41. The function f is given by

$$f: x \mapsto \ln(3x - 6), \qquad x \in \mathbb{R}, \quad x > 2.$$

(*a*) Find $f^{-1}(x)$.

(3)

(*b*) Write down the domain of f^{-1} and the range of f^{-1}.

(2)

(*c*) Find, to 3 significant figures, the value of x for which $f(x) = 3$.

(2)

The function g is given by

$$g: x \mapsto \ln|3x - 6|, \qquad x \in \mathbb{R}, \quad x \neq 2.$$

(*d*) Sketch the graph of $y = g(x)$.

(3)

(*e*) Find the exact coordinates of all the points at which the graph of $y = g(x)$ meets the coordinate axes.

(3)

[P2 June 2004 Question 8]
