MEI

Mathematics in Education and Industry

MEI Structured Mathematics

Module Summary Sheets

C2, Concepts for Advanced Mathematics

(Version B—reference to new book)

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Summary C2 Topic 1: **Algebra—Logarithms**

References:
Chapter 11
Pages 319-323

If $x = a^m$ then $m = \log_a x$

The following laws are derived from the laws of indices.

1. If $y = a^n$ then $n = \log_a y$, so $xy = a^{m+n}$

$\Rightarrow m + n = \log_a xy \Rightarrow \log_a x + \log_a y = \log_a xy$

Similarly:

2. $\log x - \log y = \log \frac{x}{y}$
3. $\log x^n = n \log x$
4. $\log 1 = 0, \log_a a = 1$
5. $\log \frac{1}{x} = -\log x$

In numerical work logx usually refers to base 10. In algebraic work the laws of logs are true for any base, and the base is therefore usually unspecified.

Exercise 11A
Q. 1(i),
6, 7(ii)

E.g. $\log 3 + \log 4 = \log 12$

$\log 6 - \log 2 = \log 3$

$\log 4 = \log 2^2 = 2\log 2$

$\log \frac{1}{2} = -\log 2$

E.g. Evaluate $\log_2 8$

$\log_2 8 = \log_2 2^3 = 3\log_2 2 = 3$

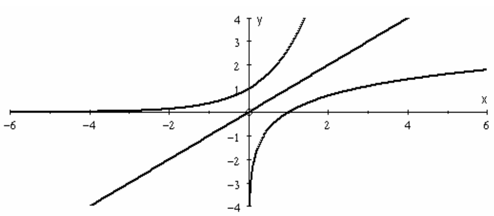
E.g. Solve $2^x = 3$

Take logs: $\log 2^x = x \log 2 = \log 3$

$\Rightarrow x = \frac{\log 3}{\log 2} = 1.585$

References:
Chapter 11
Page 324

Exponential functions
If $y = \log_a x$ then $x = a^y$.
The exponential function is the inverse of the log function and so their graphs are reflections in the line $y = x$.



Exercise 11A
Q. 2(i), (vi)

E.g. Find the smallest integer value of n for which $1.12^n = 2$

$1.12^n = 2 \Rightarrow n\log 1.12 = \log 2 \Rightarrow n = \frac{\log 2}{\log 1.12} = 6.12 \Rightarrow n = 7$

(N.B. this was done using logs to base 10 - any base would do if you could find the value of the logs.)

E.g. Find values of a and b so that the data below fit the graph $y = ax^b$.

x	1	2	3	4	5
y	3.00	4.24	5.20	6.00	6.71

Take logs so that the formula becomes $\log y = \log a + b\log x$

$\log x$	0	0.30	0.48	0.60	0.70
$\log y$	0.48	0.63	0.72	0.78	0.83

These points lie on a straight line with intercept 0.48.
i.e. $\log a = 0.48$ giving $a = 10^{0.48}$ ~3

gradient $= \frac{0.83\text{-}0.48}{0.70 - 0} \sim 0.5 = b \Rightarrow y = 3x^{\frac{1}{2}} = 3\sqrt{x}$

References:
Chapter 11
Pages 326-329

Exercise 11B
Q. 3

Reduction to linear form
The straight line $y = mx + c$ has gradient m and intercept c. A straight line graph can be identified and these two values found. A curve cannot be identified so easily.
Some curves can be transformed to straight lines by the use of loges. A set of data thought to fit one of two specific curves can therefore be tested.

If a set of data is thought to obey the rule $y = ab^x$, then taking logs gives
$\log y = \log a + x\log b$ ($Y = Mx + c$)
i.e. plot $\log y$ against x, giving line with gradient $\log b$ and intercept $\log a$.

If a set of data is thought to obey the rule $y = ax^n$ then taking logs gives
$\log y = \log a + n \log x$ ($Y = MX + C$)
i.e. plot $\log y$ against $\log x$, giving gradient n and intercept $\log a$.

E.g. Find values of a and b so that the data below fit the graph $y = ab^x$.

x	0	1	2	3	4
y	5.00	1.50	0.45	0.14	0.04

Take logs so that the formula becomes $\log y = \log a + x\log b$

x	0	1	2	3	4
$\log y$	0.70	0.18	-0.35	-0.85	-1.40

These points lie on a straight line with intercept 0.70.
i.e. $\log a = 0.7$ giving $a = 10^{0.7}$ ~5

gradient $= \frac{-1.40 - 0.70}{4} \sim -0.525 = \log b$

$\Rightarrow b = 10^{-0.525} \sim 0.3 \Rightarrow y = 5(0.3)^x$

C2; Concepts for Advanced Mathematics 2
Version B: page 2
Competence statements a1, a2, a3, a4, a5, a6, a7

References: Chapter 7 Pages 160-162

Sequences and Series
A sequence is a set of numbers in a given order. Each number is called a term. The k^{th} term is denoted a_k.
The terms in the sequence may be defined by a law of connection.
A series is a sequence of numbers added together.

E.g. A sequence of numbers: 1, 3, 5, 7, ….
$a_1 = 1$, $a_2 = 3$, etc

A series of numbers S = 1 + 3 + 5 + 7…..
If there are a finite number of terms then the series may be summed to give a value
This is written:

$$S_n = \sum_{k=1}^{k=n} a_k = a_1 + a_2 + \ldots.. a_n$$

A series may have an infinite number of terms.

Exercise 7A Q. 1(i),(iv),2(i),(iv)

E.g. find the rule for the following sequences:
(i) 1, 3, 5, 7,…...
(ii) 1, 2, 4, 8, ….
(iii) 2, 5, 10, 17,……

(i) $a_k = a_{k-1} + 2$ with $a_1 = 1$
or $a_k = 1 + 2(k-1) = 2k - 1$
(ii) $a_k = 2a_{k-1}$ with $a_1 = 1$
or $a_k = 2^{(k-1)}$
(iii) $a_k = k^2 + 1$

E.g. If $a_k = k(k+2)$ find $\sum_{k=1}^{5} a_k$

$$\sum_{k=1}^{5} a_k = 1\times3 + 2\times4 + 3\times5 + 4\times6 + 5\times7 = 85$$

References: Chapter 7 Pages 162-163 Pages 169-171

Arithmetic Sequences and Series
A sequence of terms where the difference between consecutive terms is constant.

i.e. $a_{k+1} - a_k = d$, called the common difference.
If $a_1 = a$ and the common difference $= d$
then $a_k = a + (k-1)d$

The sum to n terms, $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

If l is the last term then $S_n = \frac{n}{2}\left(a + l\right)$

Exercise 7B Q. 8

E.g. given the A.P. 2 + 5 + 8 +… find (i) the 17th term and (ii) the sum to 20 terms.

$a_{17} = a + 16d = 2 + 16\times3 = 50$
The sum to 20 terms is S_{20}
where $S_{20} = \frac{20}{2}\left(2\times2 + 19\times3\right) = 610$

References: Chapter 7 Pages 163-164 Pages 176-180

Geometric Sequences and Series
A sequence of terms where the ratio of consecutive terms is constant.

i.e. $a_{k+1} \div a_k = r$, called the common ratio.
If $a_1 = a$ and the common ratio $= r$
then $a_k = ar^{n-1}$

The sum to n terms, $S_n = \frac{a(r^n - 1)}{(r-1)} = \frac{a(1-r^n)}{(1-r)}$

Exercise 7C Q. 3, 4

E.g. given the G.P. 2 + 6 + 18 + …. Find (i) the 5th term and (ii) the sum to 7 terms.

$a = 3,\ r = 3 \Rightarrow a_5 = ar^{n-1} = 2\times3^{4-1} = 54$

The sum to 7 terms, $S_7 = \frac{2(3^7 - 1)}{(3-1)} = 2186$

References: Chapter 7 Page 181

Infinite G.P.

If $|r| < 1$ then the terms converge as $r^n \to 0$ as $n \to \infty$

The sum to infinity, $S = \frac{a}{(1-r)}$

Exercise 7C Q. 8, 9

E.g. given the G.P. 16 + 8 + 4 + … find the sum to infinity.

$a = 16,\ r = \frac{1}{2} \Rightarrow$ The sum to infinity, $S = \frac{a}{1-r}$

$$= \frac{16}{(1-\frac{1}{2})} = 32$$

References: Chapter 7 Page 164

Periodicity and Oscillations
A sequence which repeats itself at regular intervals is called ***Periodic***.
A sequence in which terms lie above and below a middle term is said to ***oscillate***.

E.g. a sequence in which $a_{k+5} = a_k$ for all integers k has period 5.

An oscillating sequence may be periodic, but it does not have to be.

E.g. a G.P. with $a = 81$ and $r = -\frac{1}{3}$

$\Rightarrow 81, -27,\ 9, -3,\ 1, -\frac{1}{3}, \ldots$

Summary C2 Topic 2: **Trigonometry - 1**

References:
Chapter 10
Pages 271-276

Ratios for any angles
The value of the ratios for any angles should be known

$$\sin\theta = \frac{y}{1},\ \cos\theta = \frac{x}{1} : \tan\theta = \frac{y}{x}$$

Exercise 10A
Q. 6

Special angles
There are special values for the ratios of the angles 0^0, 30^0, 45^0, 60^0, 90^0 etc. These need to be learnt in surd form as well as in decimals.

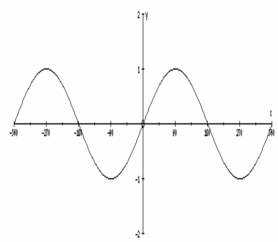
E.g. $\sin 45 = \cos 45 = \frac{1}{\sqrt{2}}, \tan 45 = 1$

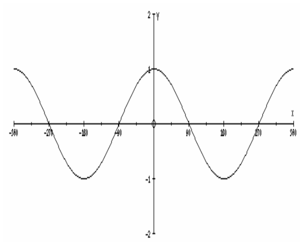
$\sin 30 = \cos 60 = \frac{1}{2},\ \sin 60 = \cos 30 = \frac{\sqrt{3}}{2}$

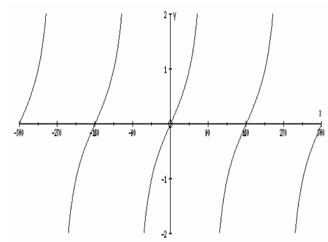
$\tan 30 = \frac{1}{\sqrt{3}}, \tan 60 = \sqrt{3}$

References:
Chapter 10
Pages 277-279

Graphs
The graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ should be known.


$y = \sin x$


$y = \cos x$


$y = \tan x$

E.g. from the graphs it can be seen that
sin150 = sin30
sin210 = –sin30
cos200 = –cos20
cos(-30) = cos30
tan120 = –tan60
tan200 = tan20

References:
Chapter 10
Pages 280-283

Exercise 10A
Q. 2

Solution of equations

The use of calculators to solve trigonometrical equations will only give one answer (the "Principal angle").

When a specific range is given there may be more than one root or a root that is not the Principal angle, so the result from the calculator needs to be used with the information above to give the full solution.

E.g. Find values of x in the range $0 \le x \le 360$ for which $\sin x = -0.3$

calculators will give $x = -17.5$ which is not in the range.
Reference to graphs will confirm two roots,
$x = 180 + 17.5 = 197.5$
$x = 360 - 17.5 = 342.5$

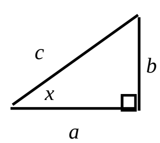
References:
Chapter 10
Page 276

Exercise 10A
Q. 1

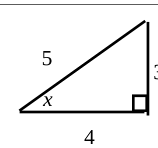
Pythagoras

In the triangle $a^2 + b^2 = c^2 \Rightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$

$\Rightarrow \cos^2 x + \sin^2 x = 1$



E.g. $\sin x = \frac{3}{5}, \cos x = \frac{4}{5}$



$\Rightarrow \sin^2 x + \cos^2 x = \frac{9}{25} + \frac{16}{25}$

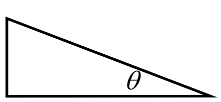
$= \frac{25}{25} = 1$

References:
Chapter 10
Pages 270-271

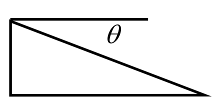
Exercise 10D
Q. 1

Applications - Draw careful diagrams!!

Angles of inclination and depression



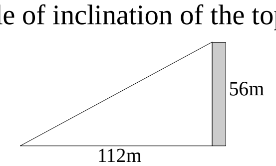
Angle of inclination



Angle of depression

E.g. Find the angle of inclination of the top of the tower:

$\tan\theta = \frac{56}{112}$



$\Rightarrow \theta = 26.6^0$

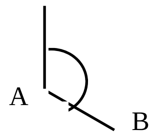
C2; Concepts for Advanced Mathematics 2
Version B: page 4
Competence statements t1, t2, t3, t4, t5, t6, t7

References: Chapter 10 Page 271

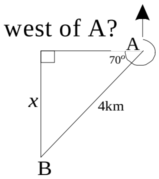
Exercise 10D Q. 3

Bearings

The bearing of one place from another is the compass direction measured as an angle clockwise from North to the line between the two places.



E.g. A boat B leaves a harbour A. It travels for 4km on a bearing 200^0. How far north must it travel before it is due west of A?



$\sin 70^o = \dfrac{x}{4}$

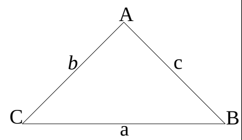
$x = 4 \sin 70^o \Rightarrow x = 3.76\,\text{km}$

References: Chapter 10 Pages 285-297

Exercise 10B Q. 1(i), 2(i)

Exercise 10C Q. 1(i), 2(i)

General Triangle



Cosine rule : $a^2 = b^2 + c^2 - 2bc\cos A$

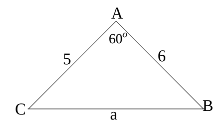
Sine rule : $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

(care should be taken in the 'ambiguous' case:

eg. $\sin x = 0.5 \Rightarrow x = 30^o$ or 150^o)

Area $= \dfrac{1}{2} ab\sin C$

E.g. Solve the triangle



$a^2 = 5^2 + 6^2 - 2.5.6 \cos 60$

$a^2 = 31 \Rightarrow a = 5.57$

$\dfrac{5.57}{\sin 60^o} = \dfrac{6}{\sin C}$

$\sin C = \dfrac{6 \sin 60^o}{5.57} \Rightarrow C = 69^0$

$B = 180 - 60 - 69 = 51^0$

Area $= \dfrac{1}{2} \times 5 \times 6 \times \sin 60^o = 13\,\text{units}^2$

Exercise 10E Q. 1(iv)

Problem solving

- Draw a good clear diagram, especially in 3-D problems
- Pick out and redraw any relevant triangles
- Use appropriate trigonometrical formulae

References: Chapter 10 Pages 299-302

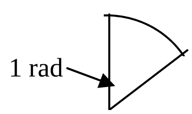
Exercise 10F Q. 1(i),(iv),(ix) 2(i),(iv), (ix)

Circular Measure

Angles are measured in degrees and also in radians.
Radians are particularly useful for circular measurement.

1 radian = the angle subtended by an arc length of 1 unit in a circle of radius 1 unit.

1 revolution = 360° = 2π radians



E.g. $30^0 = 30 \times \dfrac{2\pi}{360} = \dfrac{\pi}{6}$ rad

3 rad $= 3 \times \dfrac{360}{2\pi} \approx 171.9^0$

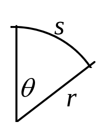
E.g. $\sin \pi = 0,\ \cos \dfrac{\pi}{3} = 0.5$

References: Chapter 10 Pages 303-304

Exercise 10G Q. 5

Length of arc and area of sector

Arc length, $s = r\theta$



Area of sector, $A = \dfrac{1}{2} r^2 \theta$

E.g. Find the arc length and area of the sector of a circle, radius 3 cm and angle 60^0.

$60^0 = \dfrac{\pi}{3} \Rightarrow$ Arc $= 3 \times \dfrac{\pi}{3} = \pi$ cm

$\Rightarrow$ Area $= \dfrac{1}{2} \times 3^2 \times \dfrac{\pi}{3} = \dfrac{3\pi}{2}$ cm^2

References: Chapter 10 Pages 311-314

Exercise 10H Q. 1(ii),(iv),(v)5

Transformations

$y = \sin x + a$ represents a translation a units parallel to the y-axis.

$y = a \sin x$ represents a one way stretch of scale factor a parallel to the y-axis.

$y = \sin ax$ represents a one way stretch of scale factor $\dfrac{1}{a}$ parallel to the x-axis.

References: Chapter 8 Pages 191-196

The **gradient** of a curve at a point is given by the gradient of the tangent at that point.

The gradient function $\frac{dy}{dx}$ is defined by $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

References: Chapter 8 Pages 197-201

Exercise 8B Q. 1(i),(ix)

The Gradient function.
Given a function $y = f(x)$ we can differentiate to find the *gradient function* or the *derivative*.

If $y = f(x) = ax^n$ then $\frac{dy}{dx} = f'(x) = nax^{n-1}$

The function must be a summation (or difference) of terms before it can be differentiated.

If $y = f(x) + g(x)$ then $\frac{dy}{dx} = f'(x) + g'(x)$

References: Chapter 12 Pages 339-340

Exercise 12A Q. 1(i), (viii)

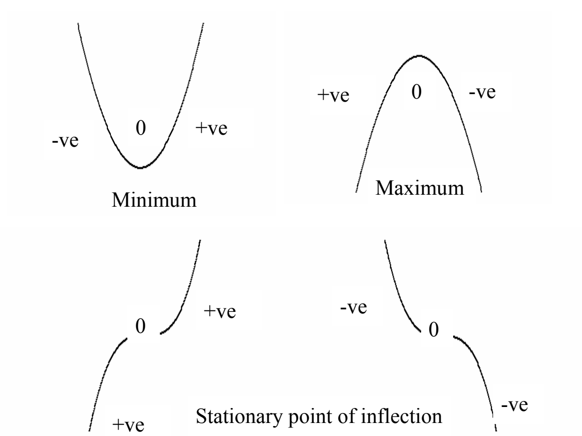
Tangents and Normals
The gradient of the tangent at a point on the curve is the gradient of the curve.
The normal to the curve at a point is perpendicular to the tangent.
Hence the equation of the tangent at the point (x_1, y_1)
Is $y - y_1 = m(x - x_1)$ where m is the gradient of the curve at the point found above.
The equation of the normal is $y - y_1 = m'(x - x_1)$ where $mm' = -1$.

References: Chapter 8 Pages 206-207

Exercise 8D Q. 5

References: Chapter 8 Pages 210-215 Pages 217-220

Stationary points
At stationary points, (maxima, minima or points of inflexion) the gradient = 0.
The nature of a stationary point can be determined by looking at the sign of the gradient just either side of it.



Exercise 8E Q. 2, 11

Exercise 8F Q. 2

References: Chapter 8 Pages 221-226

Exercise 8G Q. 1

2nd derivatives
The gradient of a curve is itself a function and so has a gradient.

This is denoted $f''(x) = \frac{d^2y}{dx^2}$

E.g. Find the gradient function of the following:

(i) $y = x^3$

(ii) $y = 3x^2 + 4x - 2 + \frac{2}{x}$

(i) $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$

(ii) $y = 3x^2 + 4x - 2 + \frac{2}{x} \Rightarrow \frac{dy}{dx} = 6x + 4 - \frac{2}{x^2}$

(Note that the gradient function of kx is k and of a constant is 0.)

E.g. Find the gradient of the tangent to the curve $y = x^2 + 1$ at the point (2, 5).

$y = x^2 + 1 \Rightarrow \frac{dy}{dx} = 2x$

When $x = 2$, $\frac{dy}{dx} = 2 \times 2 = 4$

E.g. Find the stationary points on the curve $y = x^3 - 6x^2 + 9x - 4$ and determine which is a maximum and which is a minimum.

$y = x^3 - 6x^2 + 9x - 4 \Rightarrow \frac{dy}{dx} = 3x^2 - 12x + 9$

$= 0$ when $3(x^2 - 4x + 3) = 0 \Rightarrow 3(x-3)(x-1) = 0$

$\Rightarrow x = 1, 3.$ When $x = 1, y = 0$; When $x = 3, y = -4$

$\frac{dy}{dx} = 3(x-3)(x-1)$;

For the point $(1, 0)$; When $x < 1$, $\frac{dy}{dx}$ = -ve. - ve = +ve

and when $x > 1$, $\frac{dy}{dx}$ = -ve. + ve = -ve, giving maximum.

For the point $(3, -4)$; When $x < 3$, $\frac{dy}{dx}$ = -ve. + ve = -ve

and when $x > 3$, $\frac{dy}{dx}$ = +ve. + ve = +ve, giving minimum.

E.g. Find the equation of the tangent and normal to the curve $y = x^2 - 5x + 3$ at the point (1, –1).

$\frac{dy}{dx} = 2x - 5$. When $x = 1$, $\frac{dy}{dx} = -3$

$\Rightarrow$ Gradient of tangent $= -3$, Gradient of normal $= \frac{1}{3}$

$\Rightarrow$ Tangent : $y + 1 = -3(x - 1) \Rightarrow y + 3x = 2$

Normal : $y + 1 = \frac{1}{3}(x - 1) \Rightarrow 3y - x = -4$

For the use of 2nd derivatives to identify the nature of stationary points, see Page 8.

C2; Concepts for Advanced Mathematics 2
Version B: page 6
Competence statements c1, c2, c3, c4, c5, c6, c7, c8

Summary C2 Topic 3: **Calculus; Integration**

MEI

References: Chapter 9 Pages 234-236

Exercise 9A Q. 2

Exercise 9B Q. 1(i)

Integration is the reverse process of differentiation.

$$\int kx^n \,dx = \frac{kx^{n+1}}{n+1} + c \quad \text{providing } n \neq -1$$

c is the constant of integration.

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int af(x)dx = a\int f(x)dx$$

E.g. $\int \left(2x^3 + 3x^2 - x + 1 + \frac{1}{x^3}\right)dx$

$= \int 2x^3\,dx + \int 3x^2 dx - \int x\,dx + \int 1\,dx + \int \frac{1}{x^3}dx$

$= 2\int x^3\,dx + 3\int x^2 dx - \int x\,dx + \int 1\,dx + \int x^{-3}dx$

$= \frac{x^4}{2} + x^3 - \frac{x^2}{2} + x - \frac{1}{2x^2} + c$

References: Chapter 12 Pages 347-349

Exercise 12B Q. 1(i), (viii)

Evaluating c.
We must be given corresponding values of x and y. These may be a point on the curve $y = f(x)$ or they may be from initial values of a problem.
We substitute these values in to the equation, and then find c.

E.g. Find the equation of the curve through $(3, 4)$ for which $\frac{dy}{dx} = x^2 - 3$.

$y = \int (x^2 - 3)dx \Rightarrow y = \frac{x^3}{3} - 3x + c$

When $x = 3, y = 4 \Rightarrow y = 9 - 9 + c \Rightarrow c = 4$

$\Rightarrow y = \frac{x^3}{3} - 3x + 4$

References: Chapter 9 Pages 239-246 Pages 250-251

Definite integration

If $\frac{d}{dx}(F(x)) = f(x)$ then the definite integral from a to b of $f(x)$ is given by

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

E.g. Evaluate $\int_1^2 (x^2 - x + 1)\,dx$

$\int_1^2 (x^2 - x + 1)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_1^2$

$= \left(\frac{8}{3} - 2 + 2\right) - \left(\frac{1}{3} - \frac{1}{2} + 1\right) = \frac{8}{3} - \frac{5}{6} = \frac{11}{6}$

Exercise 9B Q. 2(i), (ix), 5

Exercise 9C Q. 1(ii),(viii)

The area under a graph
The area under a graph can be found as the limit of a sum of areas of rectangles.

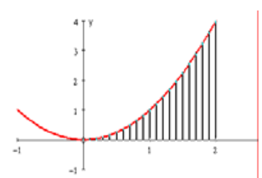
$$\text{Area} = \lim_{\delta x \to 0} \sum_a^b y\delta x = \int_a^b y\,dx$$

Note that the area under the x-axis will be negative.

E.g. Find the area bounded by the curve $y = x^2$, the lines $x = 0, x = 2$ and the x-axis.

Area $= \int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2$

$= \left(\frac{8}{3}\right) - (0) = \frac{8}{3}$

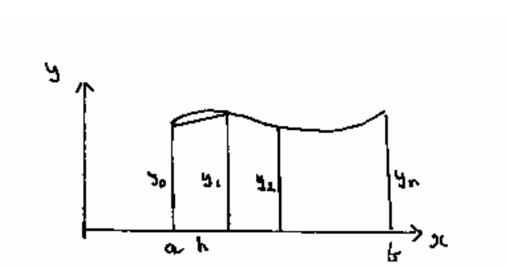


References: Chapter 9 Pages 260-264

Exercise 9F Q. 6

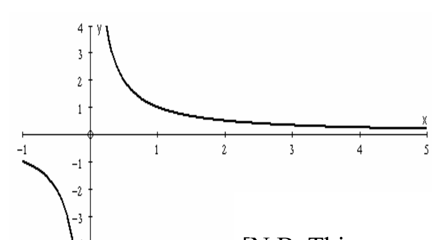
Numerical integration—Trapezium Rule

The area is estimated by trapezia:



$$\int_a^b y\,dx \sim \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n\right] \text{ where } h = \frac{b-a}{n}$$

E.g. Estimate the area under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 4$.



x	1	2	3	4
y	1	0.5	0.33	0.25

[N.B. This area can be found directly by integration - the topic appears in C3.]

$h = \frac{4-1}{3} = 1$

$\int_1^4 \frac{1}{x}dx \approx \frac{1}{2}[1 + 2(0.5 + 0.33) + 0.25] = \frac{1}{2} \times 2.91 = 1.46$

See C1: Topic 5 "Curve Sketching"

Plotting a curve gives an indication of some precision. Points (x, y) are calculated (and perhaps a table of values is constructed).
Sketching a curve gives an indication of shape.
In C1 this included the points where it crosses the axes and the behaviour for large x.

In addition it is helpful to know the position and nature of the stationary points and any asymptotes.

When you sketch a curve:

- Find where it crosses the axes
  i.e. find f(0) and also the value(s) of x for which $f(x) = 0$
- Check for any discontinuities.
  If $f(x)$ contains a fraction involving x and if there is a value of x that makes the denominator zero, then there is a discontinuity at that value of x.

  E.g. $f(x) = \dfrac{1}{x^2 - x}$ has two discontinuities, at $x = 0$ and $x = 1$

- Examine the behaviour as $x \Rightarrow \pm\infty$
- Look for any stationary points.

References:
Chapter 8
Pages 222-226

Exercise 8G
Q. 10

Exercise 8H
Q. 6

2nd derivatives

$\dfrac{dy}{dx} = 0$ gives a stationary point.

$\dfrac{d^2y}{dx^2} > 0$ gives a minimum.

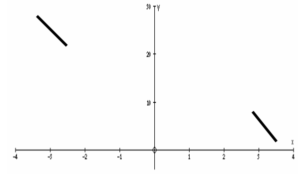
$\dfrac{d^2y}{dx^2} < 0$ gives a maximum.

You can visualise this by seeing that the gradient function is a changing function. If the gradient is negative just before the stationary value, zero at the value and positive after the stationary value then it is an increasing function giving a minimum (See the diagrams on page 6)

C2; Concepts for Advanced Mathematics 2
Version B; page 8
Competence statements C1, C2

E.g. Sketch the curve $y = 6 + 12x + 3x^2 - 2x^3$.

$x = 0$ gives $y = 6$
There are no discontinuities
As $x \to \infty$, $y \to -\infty$
As $x \to -\infty$ $y \to \infty$



$y = 6 + 12x + 3x^2 - 2x^3$

$\dfrac{dy}{dx} = 12 + 6x - 6x^2 = 0$ when $x^2 - x - 2 = 0$

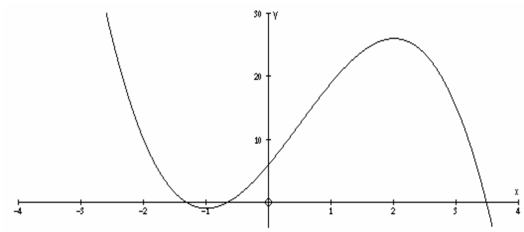
$\Rightarrow (x-2)(x+1) = 0$ i.e. $x = -1, 2$

Stationary points: $(-1, -1), (2, 26)$

$\dfrac{d^2y}{dx^2} = 6 - 12x;$

When $x = -1, \dfrac{d^2y}{dx^2} > 0 \Rightarrow$ minimum

When $x = 2, \dfrac{d^2y}{dx^2} < 0 \Rightarrow$ maximum



E.g. The equation of a curve is given by
$y = x^4 + x^2 - 20$.

(i) Show that there is only one turning point and find the coordinates.
(ii) By treating the expression $x^4 + x^2 - 20$ as a quadratic in x^2, find where the curve cuts the x-axis.
(iii) Sketch the curve.

(i) $\dfrac{dy}{dx} = 4x^3 + 2x = 0$ when $x(4x^2 + 2) = 0$

This is so only at $x = 0 \Rightarrow y = -20$

$\dfrac{d^2y}{dx^2} = 12x^2 + 2 > 0$ when $x = 0$ so minimum.

(ii) $x^4 + x^2 - 20 = 0 \Rightarrow (x^2 - 4)(x^2 + 5) = 0$

$\Rightarrow x^2 = 4 \Rightarrow (2, 0)$ and $(-2, 0)$

(iii)