Pure Core 2

Revision Notes

June 2016

Pure Core 2

1 Algebra

Polynomials

A polynomial is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ . \ . \ . \ a_2 x^2 + a_1 x + a_0$$

where all the powers of x are positive integers or 0.

Addition, subtraction and multiplication of polynomials are easy, division must be done by long division.

Factorising

General examples of factorising:

$$2ab + 6ac^2 = 2a(b + 3c^2)$$

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

$$x^2 - 6x = x(x - 6)$$

$$6x^2 - 11x - 10 = (3x + 2)(2x - 5)$$

$$\begin{aligned} 2ax - 3by - 6ay + bx &= 2ax - 6ay + bx - 3by \\ &= 2a(x - 3y) + b(x - 3y) \\ &= (2a + b)(x - 3y) \end{aligned}$$

Standard results

$x^2 - y^2 = (x - y)(x + y)$, difference of two squares

$(x + y)^2 = x^2 + 2xy + y^2$,

$(x - y)^2 = x^2 - 2xy + y^2$

Long division

Example:

$$\begin{array}{r|rrrrr}
 & & & 3x^2 & -\ 5x & +\ 9 \\ \hline
2x^2 + 3x - 1 & 6x^4 & -\ x^3 & & +\ x & -\ 3 \\
 & 6x^4 & +\ 9x^3 & -\ 3x^2 & & \\ \hline
 & & -10x^3 & +\ 3x^2 & +\ x & -\ 3 \\
 & & -10x^3 & -\ 15x^2 & +\ 5x & \\ \hline
 & & & 18x^2 & -\ 4x & -\ 3 \\
 & & & 18x^2 & +\ 27x & -\ 9 \\ \hline
 & & & & -\ 31x & +\ 6
\end{array}$$

$\Rightarrow$ when $6x^4 - x^3 + x - 3$ is divided by $2x^2 + 3x - 1$,
the quotient is $3x^2 - 5x + 9$, and the remainder is $-31x + 6$.

Remainder theorem

If 627 is divided by 6 the quotient is 104 and the remainder is 3.
This can be written as $627 = 6 \times 104 + 3$.

In the same way, if a polynomial

$P(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$ is divided by $(cx + d)$ to give a quotient, $Q(x)$ with a remainder r, then r will be a constant (since the divisor is of degree one) and we can write

$$P(x) = (cx + d) \times Q(x) + r$$

If we now choose the value of x which makes $(cx + d) = 0 \Rightarrow x = {}^{-d}/_c$

then we have $P\left({}^{-d}/_c\right) = 0 \times Q(x) + r$

$\Rightarrow \quad P\left({}^{-d}/_c\right) = r.$

Theorem: If we put $x = {}^{-d}/_c$ in the polynomial we obtain r, the remainder that we would have after dividing the polynomial by $(cx + d)$.

Example: The remainder when $P(x) = 2x^3 + ax^2 + bx + 9$ is divided by $(2x – 3)$ is $–6$, and when $P(x)$ is divided by $(x + 2)$ the remainder is 1.

Find the values of a and b.

Solution: $(2x – 3) = 0$ when $x = {}^3/_2$,

$\Rightarrow$ dividing $P(x)$ by $(2x – 3)$ gives a remainder

$P\left({}^3/_2\right) = 2 \times \left({}^3/_2\right)^3 + a \times \left({}^3/_2\right)^2 + b \times \left({}^3/_2\right) + 9 = -6$

$\Rightarrow \quad 3a + 2b = -29$ **I**

and $(x + 2) = 0$ when $x = -2$,

$\Rightarrow$ dividing $P(x)$ by $(x + 2)$ gives a remainder

$P(-2) = 2 \times (-2)^3 + a \times (-2)^2 + b \times (-2) + 9 = 1$

$\Rightarrow \quad 4a - 2b = 8$ **II**

I + **II** $\Rightarrow \quad 7a = -21$

$\Rightarrow \quad a = -3$

using **I** we get $b = -10$

Factor theorem

Theorem: If, in the remainder theorem, $r = 0$ then $(cx + d)$ is a factor of $P(x)$

$\Rightarrow$ $P\left(^{-d}/_{c}\right) = 0$ $\Leftrightarrow$ $(cx + d)$ is a factor of $P(x)$.

Example: A quadratic equation has solutions (roots) $x = {}^{-1}/_{2}$ and $x = 3$. Find the quadratic equation in the form $ax^2 + bx + c = 0$

Solution: The equation has roots $x = {}^{-}½$ and $x = 3$

$\Rightarrow$ it must have factors $(2x + 1)$ and $(x - 3)$ by the factor theorem

$\Rightarrow$ an equation is $(2x + 1)(x - 3) = 0$

$\Rightarrow$ $2x^2 - 5x - 3 = 0$. or any multiple

Example: Show that $(x - 2)$ is a factor of $P(x) = 6x^3 - 19x^2 + 11x + 6$ and hence factorise the expression completely.

Solution: Choose the value of x which makes $(x - 2) = 0$, i.e. $x = 2$

$\Rightarrow$ remainder $= P(2) = 6 \times 8 - 19 \times 4 + 11 \times 2 + 6 = 48 - 76 + 22 + 6 = 0$

$\Rightarrow$ $(x - 2)$ is a factor by the factor theorem.

We have started with a cubic and so we the other factor must be a quadratic, which can be found by long division or by 'common sense'.

$\Rightarrow$ $6x^3 - 19x^2 + 11x + 6 = (x - 2)(6x^2 - 7x - 3)$

$=$ $(x - 2)(2x - 3)(3x + 1)$

which is now factorised completely.

Choosing a suitable factor

To choose a suitable factor we look at the coefficient of the highest power of x and the constant (the term without an x).

Example: Factorise $2x^3 + x^2 - 13x + 6$.

Solution: 2 is the coefficient of x^3 and 2 has factors of 2 and 1.

6 is the constant term and 6 has factors of 1, 2, 3 and 6

$\Rightarrow$ the possible linear factors of $2x^3 + x^2 - 13x + 6$ are

$(x \pm 1)$, $(x \pm 2)$, $(x \pm 3)$, $(x \pm 6)$

$(2x \pm 1)$, $(2x \pm 2)$, $(2x \pm 3)$, $(2x \pm 6)$

But $(2x \pm 2) = 2(x \pm 1)$ and $(2x \pm 6) = 2(x \pm 3)$, so they are not *new* factors.

We now test the possible factors using the factor theorem until we find one that works.

Test $(x - 1)$, put $x = 1$ giving $2 \times 1^3 + 1^2 - 13 \times 1 + 6 \neq 0$

Test $(x + 1)$, put $x = -1$ giving $2 \times (-1)^3 + (-1)^2 - 13 \times (-1) + 6 \neq 0$

Test $(x - 2)$, put $x = 2$ giving $2 \times 2^3 + 2^2 - 13 \times 2 + 6 = 16 + 4 - 26 + 6 = 0$

and since the result is zero $(x - 2)$ is a factor.

We now divide to give

$$2x^3 + x^2 - 13x + 6 = (x - 2)(2x^2 + 5x - 3)$$

$$= (x - 2)(2x - 1)(x + 3).$$

Cubic equations

Factorise using the factor theorem then solve.

N.B. The quadratic factor might not factorise in which case you will need to use the formula for this part.

Example: Solve the equation $2x^3 + x^2 - 3x + 1 = 0$.

Solution: Possible factors are $(x \pm 1)$ and $(2x \pm 1)$.

Put $x = 1$ we have $2 \times 1^3 + 1^2 - 3\times1 + 1 = 1 \neq 0$
$\Rightarrow$ $(x - 1)$ is **not** a factor

Put $x = -1$ we have $2 \times (-1)^3 + (-1)^2 - 3\times(-1) + 1 = 3 \neq 0$
$\Rightarrow$ $(x + 1)$ is **not** a factor

Putting $x = \frac{1}{2}$ we have $2 \times \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 3\times\frac{1}{2} + 1 = 0$

$\Rightarrow$ $(2x - 1)$ is a factor

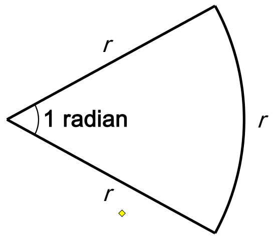
$\Rightarrow$ $2x^3 + x^2 - 3x + 1 = (2x - 1)(x^2 + x - 1) = 0$

$\Rightarrow$ $x = \frac{1}{2}$ or $x^2 + x - 1 = 0$ – this will not factorise so we use the formula

$\Rightarrow$ $x = \frac{1}{2}$ or $x = \dfrac{-1 \pm \sqrt{(-1)^2 - 4\times1\times(-1)}}{2\times1} = 0{\cdot}618$ or $-1{\cdot}618$ to 3 D.P.

2 Trigonometry

Radians



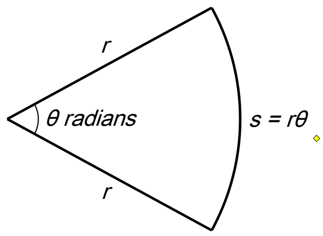
A radian is the angle subtended at the centre of a circle by an arc of length equal to the radius.

Connection between radians and degrees

$180^{\circ} = \pi^{c}$

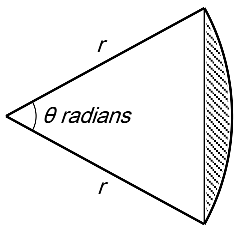
Degrees	30	45	60	90	120	135	150	180	270	360
Radians	$^{\pi}/_{6}$	$^{\pi}/_{4}$	$^{\pi}/_{3}$	$^{\pi}/_{2}$	$^{2\pi}/_{3}$	$^{3\pi}/_{4}$	$^{5\pi}/_{6}$	π	$^{3\pi}/_{2}$	2π

Arc length , area of a sector and area of a segment



Arc length $s = r\theta$

Area of sector $A = \frac{1}{2}r^2\theta$.



Area of segment

= area sector – area of triangle

$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$.

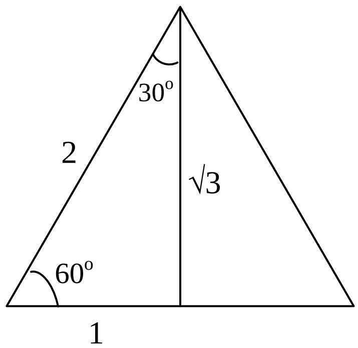
Trigonometric functions

Basic results

$\tan A = \frac{\sin A}{\cos A}$; $\sin(-A) = -\sin A$; $\cos(-A) = \cos A$; $\tan(-A) = -\tan A$.

Exact values for 30°, 45° and 60°

From the equilateral triangle of side 2 we can see that

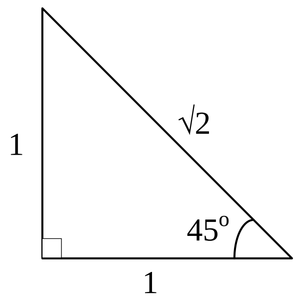


$\sin 60^\circ = \sqrt{3}/2$ $\sin 30^\circ = 1/2$

$\cos 60^\circ = 1/2$ $\cos 30^\circ = \sqrt{3}/2$

$\tan 60^\circ = \sqrt{3}$ $\tan 30^\circ = 1/\sqrt{3}$

From the isosceles right–angled triangle with sides 1, 1, $\sqrt{2}$ we can see that

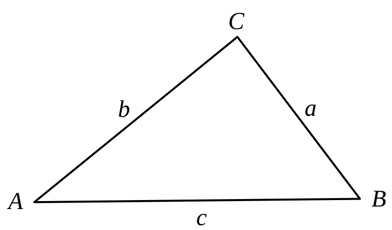


$\sin 45^\circ = 1/\sqrt{2}$

$\cos 45^\circ = 1/\sqrt{2}$

$\tan 45^\circ = 1$

Sine and cosine rules and area of triangle



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

be careful – the sine rule always gives you two answers for each angle,

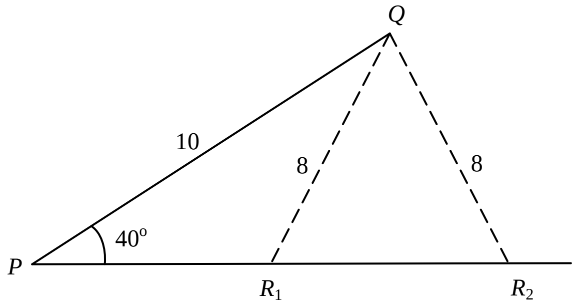
so if possible do **not** use the **sine rule** to find the **largest** angle as it might be obtuse;

you may be able to use the cosine rule.

Ambiguous case

Example: In a triangle PQR, $PQ = 10$, $\angle QPR = 40^{\circ}$ and $QR = 8$.

Find $\angle PRQ$.



Solution: If we draw PQ and an angle of 40°, there are two possible positions for R, giving two values of $\angle PRQ$.

The sine rule gives

$$\frac{\sin 40}{8} = \frac{\sin R}{10}$$

$\Rightarrow \sin R = 0{\cdot}80348\ldots$

$\Rightarrow R = 53{\cdot}5^{\circ}$ or $180 - 53{\cdot}5 = 126{\cdot}5^{\circ}$ **both** answers are correct

Cosine rule

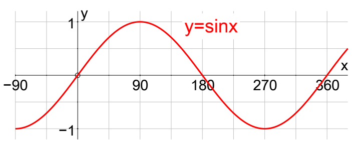
$a^2 = b^2 + c^2 - 2bc \cos A$

You will always have unique answers with the cosine rule.

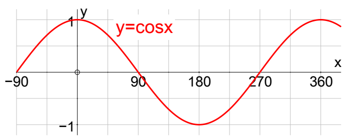
Area of triangle

Area of a triangle $= \tfrac{1}{2}ab\sin C = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ac\sin B$.

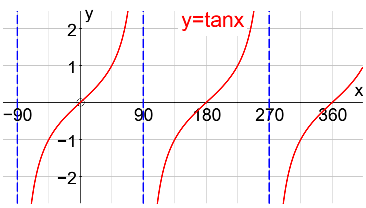
Graphs of trigonometric functions



$y = \sin x$



$y = \cos x$

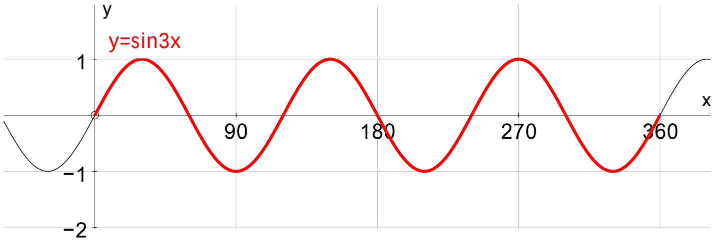


$y = \tan x$

Graphs of $y = \sin nx$, $y = \sin(-x)$, $y = \sin(x + n)$ etc.

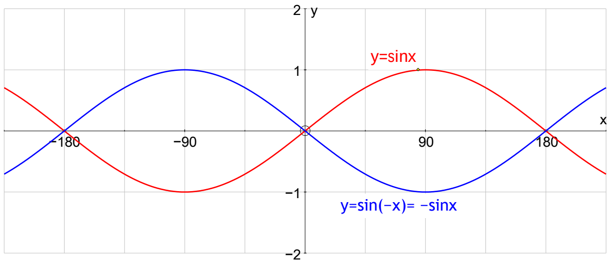
You should know the shapes of these graphs

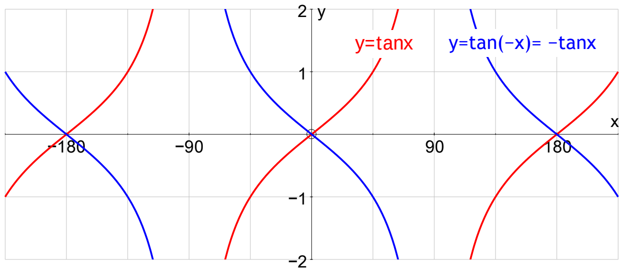
y* = sin 3*x



$y = \sin 3x$ is like $y = \sin x$

but repeats itself **3** times for $0^o \le x \le 360^o$, or $0 \le x \le 2\pi^c$





$y = f(x) = \sin x$

$\Rightarrow$ for a reflection in the *y*-axis,
$f(-x) = \sin(-x) = -\sin x$,
and for a reflection in the *x*-axis,
$-f(x) = -\sin x$

$\Rightarrow$ same graph for both reflections

$y = f(x) = \tan x$

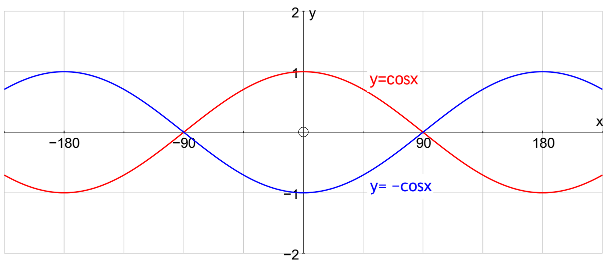
$\Rightarrow$ for a reflection in the *y*-axis,
$f(-x) = \tan(-x) = -\tan x$,
and for a reflection in the *x*-axis,
$-f(x) = -\tan x$

$\Rightarrow$ same graph for both reflections

$y = \cos(-x)$ and $-\cos x$

$y = \cos(-x)$ is the same as the graph of $y = f(x) = \cos x$, since the graph of $y = \cos x$ is symmetrical about the *y*-axis, and $f(-x) = \cos(-x) = \cos x$.

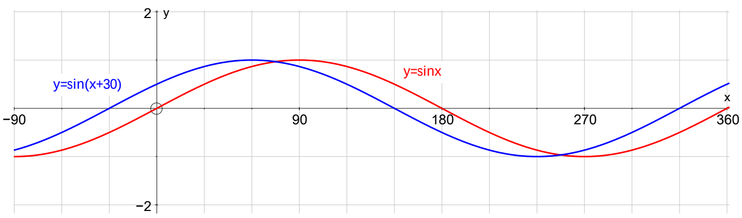
But $y = -\cos x = -f(x)$ is a reflection of $y = f(x) = \cos x$ in the *x*-axis.



$y = \sin(x + 30)$

$y = \sin(x + 30)$ is the graph of

$y = \sin x$ translated through $\begin{pmatrix} -30 \\ 0 \end{pmatrix}$.



Solving trigonometrical equations

Examples: Solve (*a*) $\sin x = 0{\cdot}453$, (*b*) $\cos x = -0{\cdot}769$, (c) $\sin x = -0{\cdot}876$,

(*d*) $\tan x = 1{\cdot}56$, for $0 \le x < 360^o$.

Solutions:

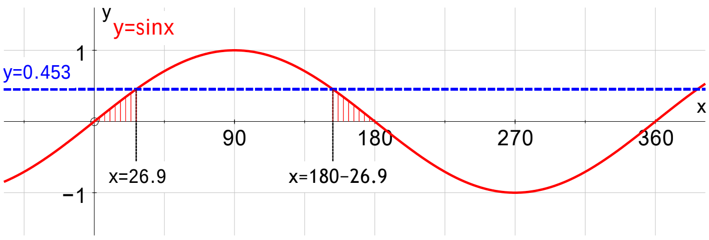
(*a*) $\sin x = 0{\cdot}453$

$\Rightarrow$ $x = 26{\cdot}9$

using the graph we see that

$x = 180 - 26{\cdot}9$

$\Rightarrow$ $x = 26{\cdot}9$ or $153{\cdot}1$



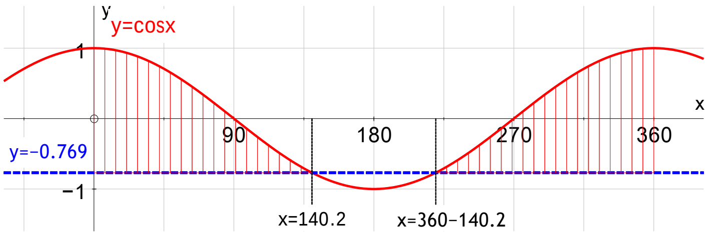
(*b*) $\cos x = -0{\cdot}769$

$\Rightarrow$ $x = 140{\cdot}3$

using the graph we see that

$x = 360 - 140{\cdot}3$

$\Rightarrow$ $x = 140{\cdot}3$ or $219{\cdot}7$



(*c*) $\sin x = -\ 0{\cdot}876$

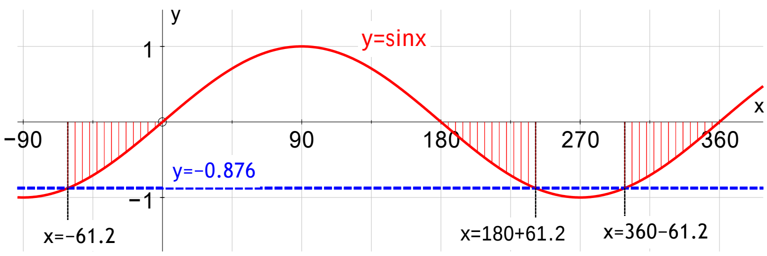
$\Rightarrow$ $x = -61{\cdot}2$

using the graph we see that

$x = 180 + 61{\cdot}2$

or $x = 360 - 61{\cdot}2$

$\Rightarrow$ $x = 241{\cdot}2$ or $298{\cdot}8$



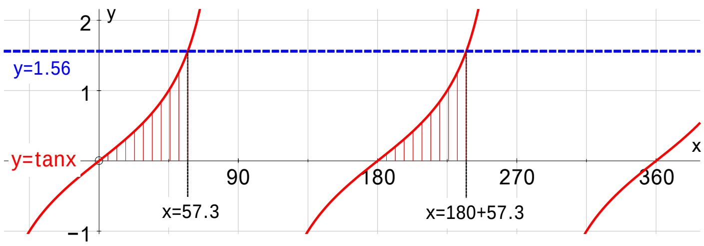
(*d*) $\tan x = 1{\cdot}56$

$\Rightarrow$ $x = 57{\cdot}3$

using the graph we see that

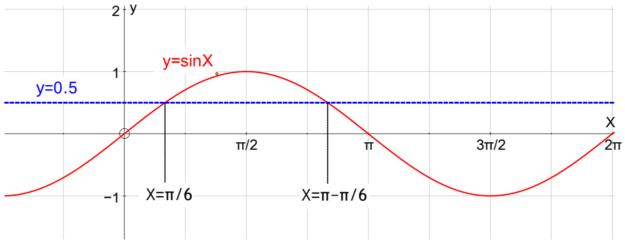
$x = 180 + 57{\cdot}3$

$\Rightarrow$ $x = 57{\cdot}3$ or $237{\cdot}3$



Example: Solve $\sin(x - \frac{\pi}{4}) = 0.5$ for $0^c \le x \le 2\pi^c$, giving your answers in radians in terms of π.

Solution: First put $X = x - \frac{\pi}{4}$



$\sin X = 0{\cdot}5 \quad \Rightarrow \quad X = \frac{\pi}{6} \text{ or } \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\Rightarrow \quad X = x - \frac{\pi}{4} = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$

$\Rightarrow \quad x = \frac{5\pi}{12} \text{ or } \frac{13\pi}{12}.$

Example: Solve $\cos 2x = 0{\cdot}473$ for $0^o \le x \le 360^o$, giving your answers to the nearest degree.

Solution: First put $X = 2x$ and find **all** solutions of $\cos X = 0{\cdot}473$ for $0^o \le X \le 720^o$

$\Rightarrow \quad X = 61{\cdot}77\ldots, \text{ or } 360 - 61{\cdot}77\ldots = 298{\cdot}22\ldots$

or $61{\cdot}77\ldots + 360 = 421.77\ldots$, or $298.22\ldots + 360 = 658{\cdot}22\ldots$

i.e. $X = 61{\cdot}77\ldots,\ 298{\cdot}22\ldots,\ 421.77\ldots,\ 658{\cdot}22\ldots$

$\Rightarrow \quad x = \frac{1}{2}X = 31^o, 149^o, 211^o, 329^o$ to the nearest degree.

Using identities

(i) using $\tan A \equiv \dfrac{\sin A}{\cos A}$

Example: Solve $3 \sin x = 4 \cos x$.

Solution: First divide both sides by $\cos x$

$\Rightarrow \quad 3\frac{\sin x}{\cos x} = 4 \quad \Rightarrow \quad 3 \tan x = 4 \quad \Rightarrow \quad \tan x = {}^4/_3$

$\Rightarrow \quad x = 53{\cdot}1^o, \text{ or } 180 + 53{\cdot}1 = 233{\cdot}1^o.$

(ii) using $\sin^2 A + \cos^2 A = 1$

Example: Given that $\cos A = \frac{5}{13}$ and that $270^\circ < A < 360^\circ$, find $\sin A$ and $\tan A$.

Solution: We know that $\sin^2 A + \cos^2 A \equiv 1$

$\Rightarrow \quad \sin^2 A = 1 - \cos^2 A$

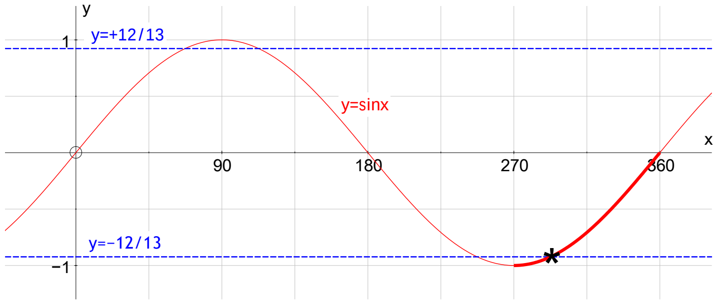
$\Rightarrow \quad \sin^2 A = 1 - \left(\frac{5}{13}\right)^2 = \frac{144}{169}$

$\Rightarrow \quad \sin A = \pm\frac{12}{13}.$

But $270^\circ < A < 360^\circ$

$\Rightarrow \quad \sin A$ is negative

$\Rightarrow \quad \sin A = -\frac{12}{13}.$



Also $\tan A = \dfrac{\sin A}{\cos A}$

$\Rightarrow \quad \tan A = \dfrac{-12/13}{5/13} = \dfrac{-12}{5} = -2{\cdot}4.$

Example: Solve $2\sin^2 x + \sin x - \cos^2 x = 1$

Solution: Rewriting $\cos^2 x$ in terms of $\sin x$ will make life easier

Using $\sin^2 x + \cos^2 x \equiv 1$

$\Rightarrow \quad \cos^2 x = 1 - \sin^2 x$

$2\sin^2 x + \sin x - \cos^2 x = 1$

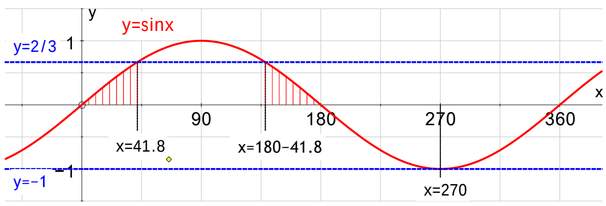
$\Rightarrow \quad 2\sin^2 x + \sin x - (1 - \sin^2 x) = 1$

$\Rightarrow \quad 3\sin^2 x + \sin x - 2 = 0$

$\Rightarrow \quad (3\sin x - 2)(\sin x + 1) = 0$

$\Rightarrow \quad \sin x = \frac{2}{3} \quad \Rightarrow \quad x = 41{\cdot}8^\circ,$
$138{\cdot}2^\circ,$

or $\quad \sin x = -1 \quad \Rightarrow \quad x = 270^\circ.$



N.B. If you are asked to give answers in radians, you are allowed to work in degrees as above and then convert to radians by multiplying by $\frac{\pi}{180}$

So the answers in radians would be

$x = 41{\cdot}8103 \times {}^{\pi}/_{180} = 0{\cdot}730$, or $138{\cdot}1897 \times {}^{\pi}/_{180} = 2{\cdot}41$, or $270 \times {}^{\pi}/_{180} = {}^{3\pi}/_{2}$.

Under no circumstances should you use the

S	A
T	C

diagram.

You need to understand the graphs and their symmetries, so get used to using them.

3 Coordinate Geometry

Mid point

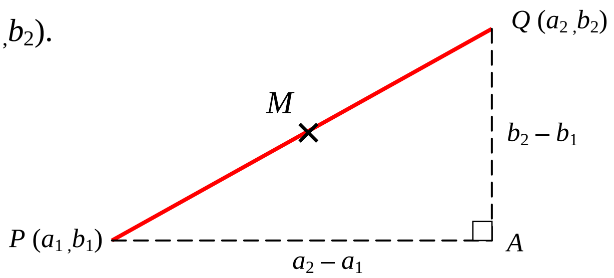
The mid point, M, of the line joining $P\ (a_1, b_1)$ and $Q\ (a_2, b_2)$ is $\left(\frac{1}{2}(a_1 + a_2),\ \frac{1}{2}(b_1 + b_2)\right)$.

Distance between two points

Let P and Q be the points (a_1, b_1) and (a_2, b_2).

Using Pythagoras's theorem

$$PQ = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$$



Perpendicular lines

Two lines with gradients m_1 and m_2 are perpendicular $\Leftrightarrow$ $m_1 \times m_2 = -1$

Example: Find the equation of the line through $(1, -5)$ which is perpendicular to the line with equation $y = 2x - 3$

Solution: The gradient of $y = 2x - 3$ is 2

$\Rightarrow$ Gradient of perpendicular line is $-\frac{1}{2}$

$\Rightarrow$ equation of perpendicular line is $y - -5 = -\frac{1}{2}(x - 1)$ using $y - y_1 = m(x - x_1)$

$\Rightarrow$ $x + 2y + 9 = 0$

Circles

Centre at the origin

Take any point, P, on a circle centre the origin and radius 5.

Suppose that P has coordinates (x, y)

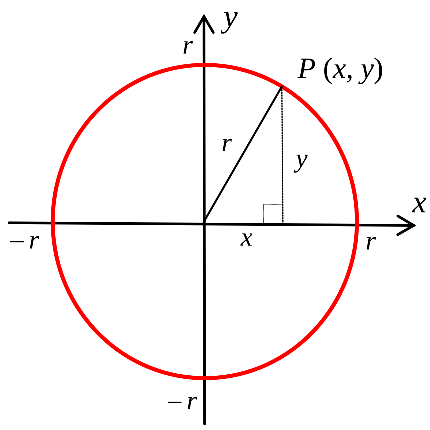
Using Pythagoras' Theorem we have

$x^2 + y^2 = 5^2 \quad \Rightarrow \quad x^2 + y^2 = 25$

which is the equation of the circle.

and in general the equation of a circle centre $(0, 0)$ and radius r is

$x^2 + y^2 = r^2$.



General equation

In the circle shown the centre is C, (a, b), and the radius is r.

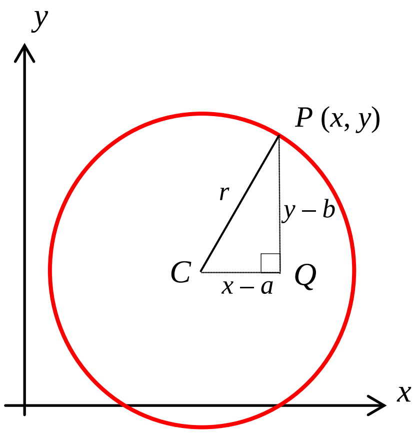
$CQ = x - a$ and $PQ = y - b$

and, using Pythagoras

$\Rightarrow \ CQ^2 + PQ^2 = r^2$

$\Rightarrow \ (x-a)^2 + (y-b)^2 = r^2$,

which is the general equation of a circle.



Example: Find the centre and radius of the circle whose equation is

$x^2 + y^2 - 4x + 6y - 12 = 0$.

Solution: First complete the square in both x and y to give

$x^2 - 4x + 4 + y^2 + 6y + 9 = 12 + 4 + 9 = 25$

$\Rightarrow \quad (x-2)^2 + (y+3)^2 = 5^2$

which is the equation of a circle with centre (2, –3) and radius 5.

Example: Find the equation of the circle which has diameter AB, where A is (3, 5), and B is (8, –7).

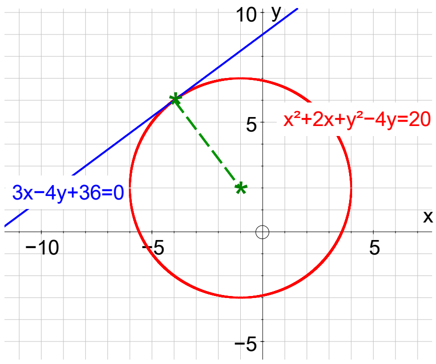
Solution: The centre is the mid point of AB is $\left(\frac{1}{2}(3+8),\ \frac{1}{2}(5-7)\right) = (5\frac{1}{2}, -1)$

and the radius is $\frac{1}{2}AB = \frac{1}{2}\sqrt{(8-3)^2 + (-7-5)^2} = 6{\cdot}5$

$\Rightarrow \quad$ equation is $(x - 5{\cdot}5)^2 + (y+1)^2 = 6{\cdot}5^2$.

Equation of tangent

Example: Find the equation of the tangent to the circle $x^2 + 2x + y^2 - 4y = 20$ at the point $(-4, 6)$.



Solution: First complete the square in x and in y

$\Rightarrow$ $x^2 + 2x + 1 + y^2 - 4y + 4 = 20 + 1 + 4$

$(x + 1)^2 + (y - 2)^2 = 25.$

Second find the gradient of the radius from the centre $(-1, 2)$ to the point $(-4, 6)$

gradient of radius $= \frac{6-2}{-4--1} = -\frac{4}{3}$

$\Rightarrow$ gradient of the tangent at that point is $\frac{3}{4}$, since the tangent is perpendicular to the radius and product of gradients of perpendicular lines is $-1 = -\frac{4}{3} \times \frac{3}{4}$

$\Rightarrow$ equation of the tangent is $y - 6 = \frac{3}{4}(x - -4)$

$\Rightarrow$ $3x - 4y + 36 = 0.$

Intersection of line and circle

Example: Find the intersection of the line $y = 2x + 4$ with the circle $x^2 + y^2 = 5$.

Solution: Put $y = 2x + 4$ in $x^2 + y^2 = 5$ to give $x^2 + (2x + 4)^2 = 5$

$\Rightarrow$ $x^2 + 4x^2 + 16x + 16 = 5$

$\Rightarrow$ $5x^2 + 16x + 11 = 0$

$\Rightarrow$ $(5x + 11)(x + 1) = 0$

$\Rightarrow$ $x = -2{\cdot}2$ or -1

$\Rightarrow$ $y = -0{\cdot}4$ or 2

$\Rightarrow$ the line and the circle intersect at $(-2{\cdot}2, -0{\cdot}4)$ and $(-1, 2)$

Showing a line is a tangent to a circle

If the two points of intersection are the ***same point*** then the line is a ***tangent***.

Example: Show that the line $3x + 4y - 10 = 0$ is a tangent to the circle $x^2 + 2x + y^2 + 6y = 15$.

Solution: Find the intersection of the line and circle

$3x + 4y - 10 = 0 \quad \Rightarrow \quad x = \frac{10-4y}{3}$

Substituting in the equation of the circle

$\Rightarrow \quad \left(\frac{10-4y}{3}\right)^2 + 2\left(\frac{10-4y}{3}\right) + y^2 + 6y = 15$

$\Rightarrow \quad 100 - 80y + 16y^2 + 60 - 24y + 9y^2 + 54y = 135$

$\Rightarrow \quad 25y^2 - 50y + 25 = 0$

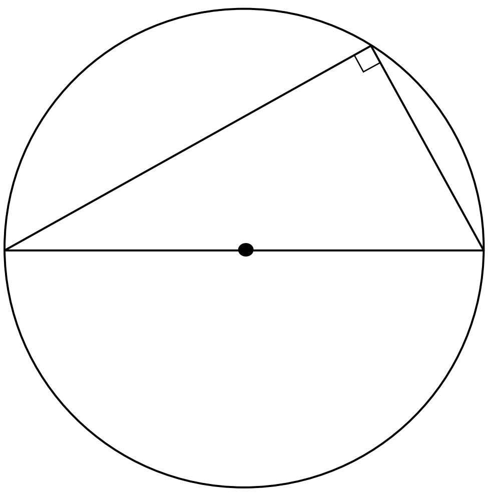
$\Rightarrow \quad y^2 - 2y + 1 = 0$

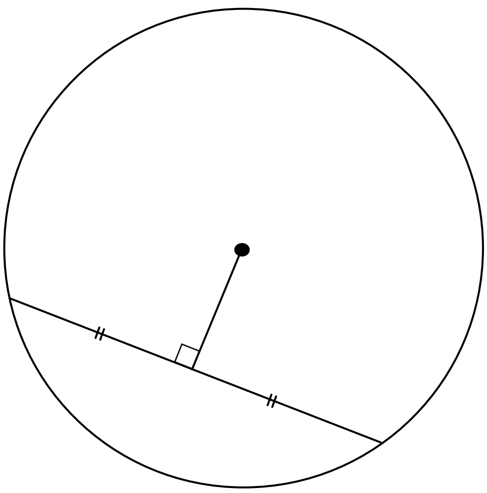
$\Rightarrow \quad (y - 1)^2 = 0$

$\Rightarrow \quad y = 1$ **only**, $\Rightarrow x = 2$

$\Rightarrow$ line is a tangent at (2, 1), since there is only one point of intersection

Note. You should know that the angle in a semi-circle is a right angle and that the perpendicular from the centre to a chord bisects the chord (cuts it exactly in half).





4 Sequences and series

Geometric series

Finite geometric series

A *geometric series* is a series in which each term is a constant amount times the previous term: this *constant amount* is called the *common ratio.*

The common ratio can be any non-zero real number.

Examples: 2, 6, 18, 54, 162, 486, ... with common ratio 3,

40, 20, 10, 5, 2½, 1¼, ... with common ratio ½,

½, –2, 8, –32, 128, –512, ... with common ratio –4.

Generally a geometric series can be written as

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1}, \quad \text{up to } n \text{ terms}$$

where a is the first term and r is the common ratio.

The nth term is $u_n = ar^{n-1}$.

The sum of the first n terms of the above geometric series is

$$S_n = a\frac{(1-r^n)}{1-r} = a\frac{(r^n-1)}{r-1}.$$

Proof of the formula for the sum of a geometric series

You **must** know this proof.

$$S_n = a + ar + ar^2 + ar^3 + \dots ar^{n-2} + ar^{n-1}$$ multiply through by r

$$\Rightarrow \quad r \times S_n = ar + ar^2 + ar^3 + \dots ar^{n-2} + ar^{n-1} + ar^n$$ subtract

$$\Rightarrow S_n - r \times S_n = a + 0 + 0 + 0 + \dots 0 + 0 - ar^n$$

$$\Rightarrow \quad (1-r)\,S_n = a - ar^n = a(1-r^n)$$

$$\Rightarrow \quad S_n = a\frac{(1-r^n)}{1-r} = a\frac{(r^n-1)}{r-1}.$$

For an *infinite* series, if $-1 < r < +1 \Leftrightarrow |r| < 1$ then $r^n \to 0$ as $n \to \infty$, and

$$S_n \to S_\infty = \frac{a}{1-r}.$$

Example: Find the n^{th} term and the sum of the first 11 terms of the geometric series whose 3^{rd} term is 2 and whose 6^{th} term is –16.

Solution: $x_6 = x_3 \times r^3$ multiply by r 3 times to go from the 3^{rd} term to the 6^{th} term

$\Rightarrow \quad -16 = 2 \times r^3$

$\Rightarrow \quad r^3 = -8$

$\Rightarrow \quad r = -2$

Now $x_3 = x_1 \times r^2$

$\Rightarrow \quad x_1 = x_3 \div r^2 = 2 \div (-2)^2$

$\Rightarrow \quad x_1 = \frac{1}{2}$

$\Rightarrow \quad n^{th}$ term, $x_n = ar^{n-1} = \frac{1}{2} \times (-2)^{n-1}$

and the sum of the first 11 terms is

$$S_{11} = \frac{1}{2} \times \frac{(-2)^{11}-1}{-2-1} = \frac{-2049}{-6}$$

$\Rightarrow \quad S_{11} = 341\frac{1}{2}$

Infinite geometric series

When the common ratio is between –1 and +1 the series converges to a limit.

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + \dots \text{ up to } n \text{ terms}$$

$$S_n = a\frac{(1-r^n)}{1-r} .$$

Since $|r| < 1$, $r^n \to 0$ as $n \to \infty$ and so

$$S_n \to S_\infty = \frac{a}{1-r}$$

Example: Show that the geometric series

$$S = 16 + 12 + 9 + 6\frac{3}{4} + \dots$$

converges to a limit and find its sum to infinity.

Solution: Firstly the common ratio is $\frac{12}{16} = \frac{3}{4}$ which lies between –1 and +1 therefore the sum converges to a limit.

The sum to infinity $S_\infty = \dfrac{a}{1-r} = \dfrac{16}{1-\frac{3}{4}}$

$\Rightarrow \quad S_\infty = 64$

Binomial series for positive integral index

Pascal's triangle

When using Pascal's triangle we think of the top row as **row 0.**

row 0							1						
row 1						1		1					
row 2					1		2		1				
row 3				1		3		3		1			
row 4			1		4		6		4		1		
row 5		1		5		10		10		5		1	
row 6	**1**		**6**		**15**		**20**		**15**		**6**		**1**

To expand $(a + b)^{\mathbf{6}}$ we first write out all the terms of '**degree 6**' in order of decreasing powers of a to give

$$\ldots a^6 + \ldots a^5 b + \ldots a^4 b^2 + \ldots a^3 b^3 + \ldots a^2 b^4 + \ldots a b^5 + \ldots b^6$$

and then fill in the coefficients using row **6** of the triangle to give

$$\mathbf{1}a^6 + \mathbf{6}a^5 b + \mathbf{15}a^4 b^2 + \mathbf{20}a^3 b^3 + \mathbf{15}a^2 b^4 + \mathbf{6}ab^5 + \mathbf{1}b^6$$

$$= a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6ab^5 + b^6$$

Factorials

Factorial n, written as $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$.

So $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Binomial coefficients or ${}^nC_r$ or $\binom{n}{r}$

If we think of row 6 in Pascal's triangle starting with the 0th term we use the following notation

0^{th} term	1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	6^{th} term
1	6	15	20	15	6	1
${}^6C_0$	${}^6C_1$	${}^6C_2$	${}^6C_3$	${}^6C_4$	${}^6C_5$	${}^6C_6$
$\binom{6}{0}$	$\binom{6}{1}$	$\binom{6}{2}$	$\binom{6}{3}$	$\binom{6}{4}$	$\binom{6}{5}$	$\binom{6}{6}$

where the *binomial coefficients* ${}^nC_r$ or $\binom{n}{r}$ are defined by

$${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

or $${}^nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)(n-3)\times\ldots \text{ up to } r \text{ numbers}}{r!}$$

This is particularly useful for calculating the numbers further down in Pascal's triangle.

Example: The 'fourth' number in row 15 is

$${}^{15}C_4 = \binom{15}{4} = \frac{15!}{(15-4)!4!} = \frac{15!}{11!\times 4!} = \frac{15\times14\times13\times12}{4\times3\times2\times1} = 1365.$$

You can also use ${}^nC_r$ button on your calculator.

Example: Find the coefficient of x^3 in the expansion of $(3-2x)^5$.

Solution: The term in x^3 is ${}^5C_3 \times 3^2 \times (-2x)^3$ since ${}^5C_3 = 10$

is $10 \times 9 \times (-8x^3) = -720x^3$

so the coefficient of x^3 is -720.

For more ideas on using the binomial coefficients, see the appendix.

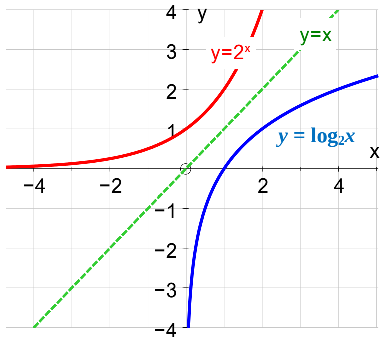
5 Exponentials and logarithms

Graphs of exponentials and logarithms

$y = 2^x$ is an *exponential* function

and its inverse is the *logarithm* function

$y = \log_2 x$.



Remember that the graph of an inverse function is the reflection of the original graph in $y = x$.

Rules of logarithms

$\log_a x = y \quad \Leftrightarrow \quad x = a^y$

$\log_a xy = \log_a x + \log_a y$

$\log_a (x \div y) = \log_a x - \log_a y$

$\log_a x^n = n \log_a x$

$\log_a 1 = 0$

$\log_a a = 1$

Example: Find $\log_3 81$.

Solution: Write $\log_3 81 = y$

$\Rightarrow \quad 81 = 3^y \quad \Rightarrow \quad y = 4 \quad \Rightarrow \quad \log_3 81 = 4.$

To solve 'log' equations we can either use the rules of logarithms to end with

$\log_a ✹ = \log_a ✸ \quad \Rightarrow \quad ✹ = ✸$

or $\log_a ✹ = ✸ \quad \Rightarrow \quad ✹ = a^{✸}$

Example: Solve $\log_a 40 - 3 \log_a x = \log_a 5$

Solution: $\log_a 40 - 3 \log_a x = \log_a 5$

$\Rightarrow \quad \log_a 40 - \log_a x^3 = x \log_a 5$

$\Rightarrow \quad \log_a (40 \div x^3) = \log_a 5$

$\Rightarrow \quad \frac{40}{x^3} = 5$

$\Rightarrow \quad x^3 = 8$

$\Rightarrow \quad x = 2.$

Example: Solve $\log_2 x + \log_2(x+6) = 3 + \log_2(x+1)$.

Solution: $\log_2 \frac{x(x+6)}{(x+1)} = 3 \Rightarrow \frac{x(x+6)}{(x+1)} = 2^3 = 8$

$\Rightarrow \quad x^2 + 6x = 8x + 8 \quad \Rightarrow \quad x^2 - 2x - 8 = 0$

$\Rightarrow \quad (x-4)(x+2) = 0 \quad \Rightarrow \quad x = 4 \text{ or } -2$

But x cannot be negative (you cannot have $\log_2 x$ when $x \le 0$)

$\Rightarrow \quad x = 4$ only

Changing the base of a logarithm

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Example: Find $\log_4 29$.

Solution: $\log_4 29 = \dfrac{\log_{10} 29}{\log_{10} 4} = \dfrac{1 \cdot 4624}{0 \cdot 6021} = 2 \cdot 43$.

A particular case

$\log_a b = \dfrac{\log_b b}{\log_b a} = \dfrac{1}{\log_b a}$ This gives a source of exam questions.

Example: Solve $\log_4 x - 6 \log_x 4 = 1$

Solution: $\Rightarrow \quad \log_4 x - \dfrac{6}{\log_4 x} = 1 \quad \Rightarrow \quad (\log_4 x)^2 - \log_4 x - 6 = 0$

$\Rightarrow \quad (\log_4 x - 3)(\log_4 x + 2) = 0$

$\Rightarrow \quad \log_4 x = 3 \text{ or } -2$

$\Rightarrow \quad x = 4^3 \text{ or } 4^{-2} \quad \Rightarrow \quad x = 64 \text{ or } \frac{1}{16}$.

Equations of the form $a^x = b$

Example: Solve $5^x = 13$

Solution: Take logs of both sides

$\Rightarrow \quad \log_{10} 5^x = \log_{10} 13$

$\Rightarrow \quad x \log_{10} 5 = \log_{10} 13$

$\Rightarrow \quad x = \dfrac{\log_{10} 13}{\log_{10} 5} = \dfrac{1 \cdot 1139}{0 \cdot 6990} = 1 \cdot 59$.

6 Differentiation

Increasing and decreasing functions

y is an *in*creasing function if its gradient is *positive*, $\frac{dy}{dx} > 0$;

y is a *de*creasing function if its gradient is *negative*, $\frac{dy}{dx} < 0$

Example: For what values of x is $y = f(x) = x^3 - x^2 - x + 7$ an increasing function.

Solution: $y = f(x) = x^3 - x^2 - x + 7$

$\Rightarrow \quad \frac{dy}{dx} = f'(x) = 3x^2 - 2x - 1$

For an increasing function we want values of x for which $f'(x) = 3x^2 - 2x - 1 > 0$

Find solutions of $f'(x) = 3x^2 - 2x - 1 = 0$

$\Rightarrow \quad (3x + 1)(x - 1) = 0$

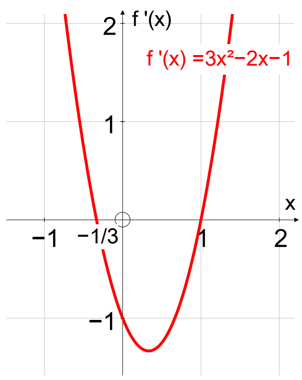
$\Rightarrow \quad x = {}^{-1}/_3$ or 1

so graph of $3x^2 - 2x - 1$ meets x-axis at ${}^{-1}/_3$ and 1

and is above x-axis for

$x < {}^{-1}/_3$ or $x > 1$

$\Rightarrow \quad f'(x) > 0$ for $x < {}^{-1}/_3$ or $x > 1$



So $y = x^3 - x^2 - x + 7$ is an increasing function for $x < {}^{-1}/_3$ or $x > 1$.

Stationary points and local maxima and minima (turning points).

Any point where the gradient is zero is called a *stationary point*.

Local maxima and minima are called *turning points.*

The gradient at a local maximum or minimum is 0.

minimum maximum Stationary points of inflection

Turning points

Therefore to find *max* and *min*

first – differentiate and find the values of x which give gradient, $\frac{dy}{dx}$, equal to zero:

second – find second derivative $\frac{d^2y}{dx^2}$ and substitute value of x found above –

second derivative positive $\Rightarrow$ minimum, and
second derivative negative $\Rightarrow$ maximum:

N.B. If $\frac{d^2y}{dx^2} = 0$, it does not help! In this case you will need to **find the gradient just before and just after** the value of x.

Be careful: you might have a stationary point of inflection

third – substitute x to find the value of y and give both coordinates in your answer.

Using second derivative

Example:

Find the local maxima and minima of the curve with equation $y = x^4 + 4x^3 - 8x^2 - 7$.

Solution:

$y = x^4 + 4x^3 - 8x^2 - 7$.

First find $\frac{dy}{dx} = 4x^3 + 12x^2 - 16x$.

At maxima and minima the gradient = $\frac{dy}{dx} = 0$

$\Rightarrow 4x^3 + 12x^2 - 16x = 0 \Rightarrow x^3 + 3x^2 - 4x = 0 \Rightarrow x(x^2 + 3x - 4) = 0$

$\Rightarrow x(x + 4)(x - 1) = 0 \Rightarrow x = -4, 0$ or 1.

Second find $\frac{d^2y}{dx^2} = 12x^2 + 24x - 16$

When $x = -4$, $\frac{d^2y}{dx^2} = 12 \times 16 - 24 \times 4 - 16 = 80$, *positive* $\Rightarrow$ min at $x = -4$

When $x = 0$, $\frac{d^2y}{dx^2} = -16$, *negative*, $\Rightarrow$ max at $x = 0$

When $x = 1$, $\frac{d^2y}{dx^2} = 12 + 24 - 16 = 20$, *positive*, $\Rightarrow$ min at $x = 1$.

Third find y–values: when $x = -4, 0$ or 1 $\Rightarrow$ $y = -135, -7$ or -10

$\Rightarrow$ **Maximum** at $(0, -7)$ and **Minimums** at $(-4, -135)$ and $(1, -10)$.

N.B. If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, it does not help! You can have any of max, min or stationary point of inflection.

Using gradients before and after

Example: Find the stationary points of $y = 3x^4 - 8x^3 + 6x^2 + 7$.

Solution: $y = 3x^4 - 8x^3 + 6x^2 + 7$

$\frac{dy}{dx} = 12x^3 - 24x^2 + 12x = 0$ for stationary points

$x(x^2 - 2x + 1) = 0 \Rightarrow x(x-1)^2 = 0 \Rightarrow x = 0$ or 1.

$\frac{d^2y}{dx^2} = 36x^2 - 48x + 12$

which is 12 (positive) when $x = 0$ $\Rightarrow$ minimum at (0, 7)

and which is 0 when $x = 1$, so we must look at gradients before and after.

x	=	0.9	1	1.1
$\frac{dy}{dx}$	=	+0·108	0	+0·132

$\Rightarrow$ stationary point of inflection at (1, 2)

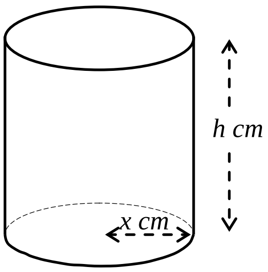
N.B. We could have *max, min or stationary point of inflection* when the second derivative is zero, so we **must** look at gradients before and after.

Maximum and minimum problems

Example:

A manufacturer of cans for baked beans wishes to use as little metal as possible in the manufacture of these cans. The cans must have a volume of 500 cm^3: how should he design the cans?

Solution:



We need to find the radius and height needed to make cans of volume 500 cm^3 using the minimum possible amount of metal.

Suppose that the radius is x cm and that the height is h cm.

The area of top and bottom together is $2 \times \pi x^2$ cm^2 and the area of the curved surface is $2\pi x h$ cm^2

$\Rightarrow$ the total surface area $A = 2\pi x^2 + 2\pi x h$ cm^2. **I**

We have a problem here: A is a function not only of x, but also of h.

But the volume is 500 cm^3 and the volume can also be written as $V = \pi x^2 h$ cm^3

$\Rightarrow \pi x^2 h = 500 \quad \Rightarrow \quad h = \frac{500}{\pi x^2}$

and so **I** can be written $A = 2\pi x^2 + 2\pi x \times \frac{500}{\pi x^2}$

$\Rightarrow A = 2\pi x^2 + \frac{1000}{x} = 2\pi x^2 + 1000\, x^{-1}$

$\Rightarrow \frac{dA}{dx} = 4\pi x - 1000x^{-2} = 4\pi x - \frac{1000}{x^2}$.

For stationary values of A, the area, $\frac{dA}{dx} = 0 \quad \Rightarrow 4\pi x = \frac{1000}{x^2}$

$\Rightarrow \quad 4\pi x^3 = 1000 \quad \Rightarrow \quad x^3 = \frac{1000}{4\pi} = 79 \cdot 57747155 \quad \Rightarrow \quad x = 4 \cdot 301270069$

$\Rightarrow x = 4{\cdot}30$ to 3 S.F. $\quad \Rightarrow \quad h = \frac{500}{\pi x^2} = 8.60$

We do not know whether this value gives a maximum or a minimum value of A or a stationary point of inflection

so we must find $\frac{d^2A}{dx^2} = 4\pi + 2000x^{-3} = 4\pi + \frac{2000}{x^3}$

Clearly this is positive when $x = 4{\cdot}30$ and thus this gives a *minimum* of A

$\Rightarrow$ minimum area of metal is 349 cm^2

when the radius is 4·30 cm and the height is 8·60 cm.

7 Integration

Definite integrals

When limits of integration are given.

Example: Find $\int_1^3 6x^2 - 8x + 1 \ dx$

Solution: $\int_1^3 6x^2 - 8x + 1 \ dx = \left[2x^3 - 4x^2 + x\right]_1^3$ no need for $+C$ as it cancels out

$= [2 \times 3^3 - 4 \times 3^2 + 3] - [2 \times 1^3 - 4 \times 1^2 + 1]$ put top limit in first

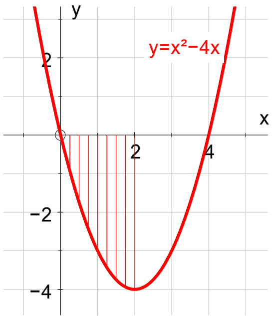
$= [21] - [-1] = 22.$

Area under curve

The integral is the area between the curve and the x–axis, **but** areas **above** the axis are **positive and** areas **below** the axis are **negative**.

Example: Find the area between the x–axis, $x = 0$, $x = 2$ and $y = x^2 - 4x$.

Solution:



$\int_0^2 x^2 - 4x \ dx$

$= \left[\frac{x^3}{3} - 2x^2\right]_0^2 = \left[\frac{8}{3} - 8\right] - [0 - 0] = \frac{-16}{3}$ which is negative

since the area is below the x–axis

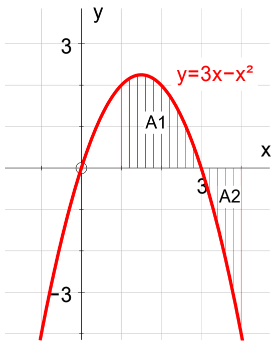
$\Rightarrow$ required area is $\frac{+16}{3}$

Example: Find the area between the x-axis, $x = 1$, $x = 4$ and $y = 3x - x^2$.

Solution: First sketch the curve to see which bits are above (positive) and which bits are below (negative).

$y = 3x - x^2 = x(3 - x)$

$\Rightarrow$ meets x-axis at 0 and 3.



Area A_1, between 1 and 3, is above axis:

area A_2, between 3 and 4, is below axis

so we must find these areas separately.

$A_1 = \int_1^3 3x - x^2 \; dx$

$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^3 = [4\cdot5] - [1\tfrac{1}{6}] = 3^1/_3.$

and $\int_3^4 3x - x^2 \; dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_3^4 = [2\tfrac{2}{3}] - [4\cdot5] = -1\tfrac{5}{6}$

and so area A_2 (areas are positive) $= +1^5/_6$

so total area $= A_1 + A_2 = 3^1/_3 + 1^5/_6 = 5^1/_6$.

Note that $\int_1^4 3x - x^2 \; dx \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^4 = [2\tfrac{2}{3}] - [1\tfrac{1}{6}] = 1\tfrac{1}{2}$

which is $A_1 - A_2$ $(= 3^1/_3 - 1^5/_6 = 1^1/_2)$.

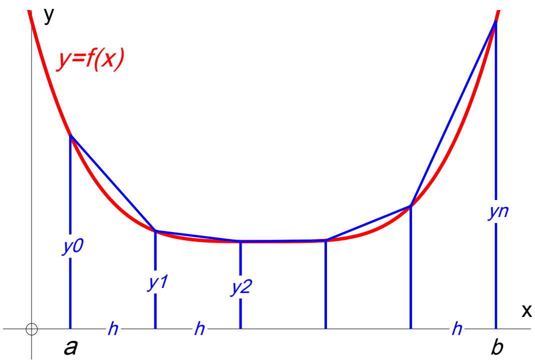
Numerical integration: the trapezium rule

Many functions can **not** be 'anti–differentiated' and the trapezium rule is a way of estimating the area under the curve.

Divide the area under $y = f(x)$ into n strips, each of width h.

Join the top of each strip with a straight line to form a trapezium.

Then the area under the curve
≈ sum of the areas of the trapezia



$$\Rightarrow \int_a^b f(x) \; dx \approx \tfrac{1}{2}h(y_0 + y_1) + \tfrac{1}{2}h(y_1 + y_2) + \tfrac{1}{2}h(y_2 + y_3) + \; ... \; + \tfrac{1}{2}h(y_{n-1} + y_n)$$

$$\Rightarrow \int_a^b f(x) \; dx \approx \tfrac{1}{2}h(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + y_3 \; ... \; + y_{n-1} + y_{n-1} + y_n)$$

$$\Rightarrow \int_a^b f(x) \; dx \approx \tfrac{1}{2}h(y_0 + y_n + 2(y_1 + y_2 + y_3 + \; ... \; + y_{n-1}))$$

⇒ area under curve ≈ ½ width of each strip × ('ends' + 2 × 'middles').

8 Appendix

Binomial coefficients, ^nC_r

Choosing *r* objects from *n*

If we have n objects, the number of ways we can choose r of these objects is $^n\mathrm{C}_r$.

$^nC_r = {}^nC_{n-r}$

Every time r objects from n must, therefore, be the same as the number of ways of leaving $n - r$ behind. are chosen from n, there are $n - r$ objects left behind; the number of ways of choosing r objects

$\Rightarrow \quad {}^nC_r = {}^nC_{n-r}$.

This can be proved algebraically.

$$^nC_{n-r} = \frac{n!}{(n-(n-r))!(n-r)!} = \frac{n!}{(n-n+r)!\,(n-r)!} = \frac{n!}{r!\,(n-r)!} = {}^nC_r$$

$(a + b)^n$

In the expansion of $(a + b)^n = (a + b)\,(a + b)\,(a + b)\,(a + b)\,(a + b)\ldots\,(a + b)\,(a + b)$,

where there are n brackets,

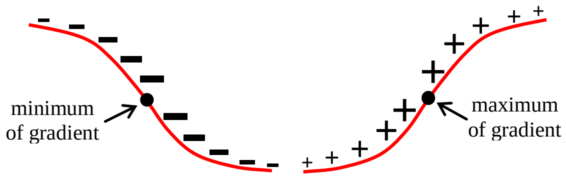
we can think of forming the term $a^{n-r}b^r$ by choosing the r letter bs from the n brackets in ^nC_r ways.

Thus the coefficient of $a^{n-r}b^r$ is ^nC_r.

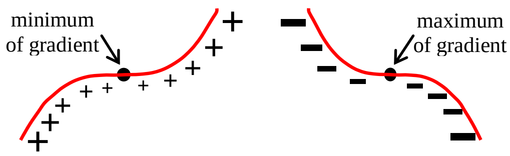
Points of inflexion

A point of inflexion is a *maximum* or *minimum* of the gradient.

When the gradient is also zero, in which case we have a *stationary point of inflexion*, otherwise we have an *oblique (sloping) point of inflexion*.



Oblique points of inflexion



Stationary points of inflexion

To find a point of inflexion

1. Find the value(s) of x for which $\frac{d^2y}{dx^2} = 0$, $x = \alpha, \beta, \ldots$

2. *Either* show that $\frac{d^3y}{dx^3} \neq 0$ for these values of x

   or show that

   either $x = \alpha^- \Rightarrow \frac{d^2y}{dx^2}$ is +ve *and* $x = \alpha^+ \Rightarrow \frac{d^2y}{dx^2}$ is –ve

   or $x = \alpha^- \Rightarrow \frac{d^2y}{dx^2}$ is –ve *and* $x = \alpha^+ \Rightarrow \frac{d^2y}{dx^2}$ is +ve

   $\Leftrightarrow$ $\frac{d^2y}{dx^2}$ changes sign from $x = \alpha^-$ to $x = \alpha^+$.

Example: Find the point(s) of inflexion on the graph of $y = x^4 - x^3 - 3x^2 + 5x + 1$.

Solution: $y = x^4 - x^3 - 3x^2 + 5x + 1$

$$\Rightarrow \quad \frac{dy}{dx} = 4x^3 - 3x^2 - 6x + 5$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 12x^2 - 6x - 6$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow 6(2x^2 - x - 1) = 6(2x + 1)(x - 1) = 0$$

$$\Rightarrow \quad x = -\tfrac{1}{2} \text{ or } 1.$$

$$\frac{d^3y}{dx^3} = 24x - 6$$

$$x = -\tfrac{1}{2} \Rightarrow \frac{d^3y}{dx^3} = -18 \neq 0, \text{ and } x = 1 \Rightarrow \frac{d^3y}{dx^3} = 18 \neq 0$$

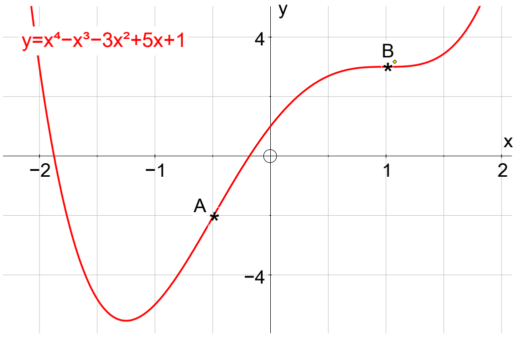
$\Rightarrow$ points of inflexion at A, $(-\frac{1}{2}, -2\frac{1}{16})$, and B, (1, 3).

Notice that $\frac{dy}{dx} = 0$ when $x = 1$,

but $\frac{dy}{dx} = 1\frac{3}{4} \neq 0$ when $x = -\frac{1}{2}$

$\Rightarrow$ A, $(-\frac{1}{2}, -2\frac{1}{16})$, is an oblique point of inflexion, and

B, (1, 3), is a stationary point of inflexion.

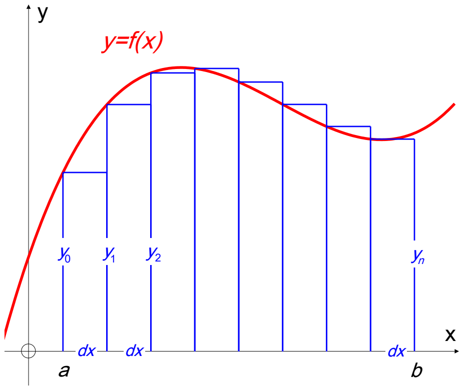


Integration

Area under graph – sum of rectangles

In any continuous graph, $y = f(x)$, we can divide the area between $x = a$ and $x = b$ into n strips, each of width δx.

The area under the graph (between the graph, the x-axis and the lines $x = a$ and $x = b$) is approximately the area of the n rectangles, as shown.



$\Rightarrow$ the area under the graph

$$A \cong \sum_{i=1}^{n} y_i \delta x, \quad \text{and} \quad \text{as } \delta x \to 0, \quad A = \int_a^b y \; dx$$

Integration as 'anti-differentiation'

A = area under the curve from $x = a$ to x

δA = increase in area from x to $x + \delta x$

$\delta A \cong$ area of the rectangle shown

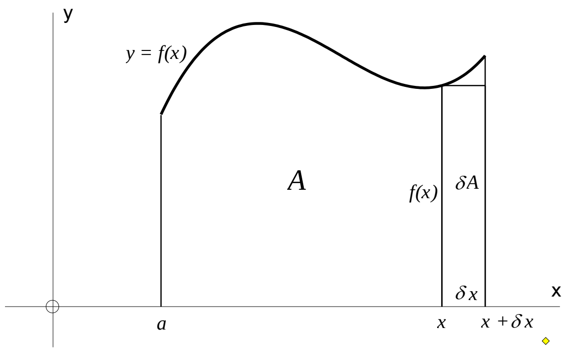
$\Rightarrow \; \delta A \approx f(x) \times \delta x$

$\Rightarrow \; \frac{\delta A}{\delta x} \approx f(x)$

As $\delta x \to 0$

we have $\frac{dA}{dx} = f(x)$

$\Rightarrow$ to find the integral we 'anti-differentiate' $f(x)$.



Index