

Solutionbank C2

Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise A, Question 1

Question:

Write down the expansion of:

- (a) $(x + y)^4$
- (b) $(p + q)^5$
- (c) $(a - b)^3$
- (d) $(x + 4)^3$
- (e) $(2x - 3)^4$
- (f) $(a + 2)^5$
- (g) $(3x - 4)^4$
- (h) $(2x - 3y)^4$

Solution:

- (a) $(x + y)^4$ would have coefficients and terms

$$\begin{array}{ccccccc} 1 & 4 & 6 & 4 & 1 & & \\ x^4 & x^3y & x^2y^2 & xy^3 & y^4 & & \end{array}$$

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

- (b) $(p + q)^5$ would have coefficients and terms

$$\begin{array}{ccccccc} 1 & 5 & 10 & 10 & 5 & 1 & \\ p^5 & p^4q & p^3q^2 & p^2q^3 & pq^4 & q^5 & \end{array}$$

$$(p + q)^5 = 1p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + 1q^5$$

- (c) $(a - b)^3$ would have coefficients and terms

$$\begin{array}{ccccccc} 1 & 3 & & 3 & & 1 & \\ a^3 & a^2(-b) & & a(-b)^2 & & (-b)^3 & \end{array}$$

$$(a - b)^3 = 1a^3 - 3a^2b + 3ab^2 - 1b^3$$

- (d) $(x + 4)^3$ would have coefficients and terms

$$\begin{array}{cccc} 1 & 3 & 3 & 1 \\ x^3 & x^2 4 & x 4^2 & 4^3 \end{array}$$

$$(x + 4)^3 = 1x^3 + 12x^2 + 48x + 64$$

(e) $(2x - 3)^4$ would have coefficients and terms

$$\begin{array}{ccccccc} 1 & & 4 & & 6 & & 4 & & 1 \\ (2x)^4 & (2x)^3 & (-3) & (2x)^2 & (-3)^2 & (2x) & (-3)^3 & (-3)^4 \end{array}$$

$$\begin{aligned} (2x - 3)^4 &= 1(2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + 1(-3)^4 \\ (2x - 3)^4 &= 16x^4 - 96x^3 + 216x^2 - 216x + 81 \end{aligned}$$

(f) $(a + 2)^5$ would have coefficients and terms

$$\begin{array}{ccccccc} 1 & 5 & 10 & 10 & 5 & 1 \\ a^5 & a^4 2 & a^3 2^2 & a^2 2^3 & a 2^4 & 2^5 \end{array}$$

$$(a + 2)^5 = 1a^5 + 10a^4 + 40a^3 + 80a^2 + 80a + 32$$

(g) $(3x - 4)^4$ would have coefficients and terms

$$\begin{array}{ccccccc} 1 & & 4 & & 6 & & 4 & & 1 \\ (3x)^4 & (3x)^3 & (-4) & (3x)^2 & (-4)^2 & (3x) & (-4)^3 & (-4)^4 \end{array}$$

$$\begin{aligned} (3x - 4)^4 &= 1(3x)^4 + 4(3x)^3(-4) + 6(3x)^2(-4)^2 + 4(3x)(-4)^3 + 1(-4)^4 \\ (3x - 4)^4 &= 81x^4 - 432x^3 + 864x^2 - 768x + 256 \end{aligned}$$

(h) $(2x - 3y)^4$ would have coefficients and terms

$$\begin{array}{ccccccc} 1 & & 4 & & 6 & & 4 & & 1 \\ (2x)^4 & (2x)^3 & (-3y) & (2x)^2 & (-3y)^2 & (2x) & (-3y)^3 & (-3y)^4 \end{array}$$

$$\begin{aligned} (2x - 3y)^4 &= 1(2x)^4 + 4(2x)^3(-3y) + 6(2x)^2(-3y)^2 + 4(2x)(-3y)^3 + 1(-3y)^4 \\ (2x - 3y)^4 &= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4 \end{aligned}$$

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Exercise A, Question 2

Question:

Find the coefficient of x^3 in the expansion of:

(a) $(4 + x)^4$

(b) $(1 - x)^5$

(c) $(3 + 2x)^3$

(d) $(4 + 2x)^5$

(e) $(2 + x)^6$

(f) $\left(4 - \frac{1}{2}x\right)^4$

(g) $(x + 2)^5$

(h) $(3 - 2x)^4$

Solution:

(a) $(4 + x)^4$ would have coefficients 1 4 6 ④ 1
 The circled number is the coefficient of the term 4^1x^3 .
 Term is $4 \times 4^1x^3 = 16x^3$
 Coefficient = 16

(b) $(1 - x)^5$ would have coefficients 1 5 10 ⑩ 5 1
 The circled number is the coefficient of the term $1^2(-x)^3$.
 Term is $10 \times 1^2(-x)^3 = -10x^3$
 Coefficient = -10

(c) $(3 + 2x)^3$ would have coefficients 1 3 3 ①
 The circled number is the coefficient of the term $(2x)^3$.
 Term is $1 \times (2x)^3 = 8x^3$
 Coefficient = 8

(d) $(4 + 2x)^5$ would have coefficients 1 5 10 ⑩ 5 1
 The circled number is the coefficient of the term $4^2(2x)^3$.
 Term is $10 \times 4^2(2x)^3 = 1280x^3$
 Coefficient = 1280

(e) $(2 + x)^6$ would have coefficients 1 6 15 ⑳ 15 6 1
 The circled number is the coefficient of the term 2^3x^3 .
 Term is $20 \times 2^3x^3 = 160x^3$
 Coefficient = 160

(f) $\left(4 - \frac{1}{2}x\right)^4$ would have coefficients 1 4 6 ④ 1

The circled number is the coefficients of the term $4 \left(-\frac{1}{2}x\right)^3$.

Term is $4 \times 4 \left(-\frac{1}{2}x\right)^3 = -2x^3$

Coefficient = -2

(g) $(x + 2)^5$ would have coefficients 1 5 ⑩ 10 5 1

The circled number is the coefficient of the term $x^3 2^2$.

Term is $10 \times x^3 2^2 = 40x^3$

Coefficient = 40

(h) $(3 - 2x)^4$ would have coefficients 1 4 6 ④ 1

The circled number is the coefficient of the term $3^1 (-2x)^3$.

Term is $4 \times 3^1 (-2x)^3 = -96x^3$

Coefficient = -96

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Exercise A, Question 3

Question:

Fully expand the expression $(1 + 3x)(1 + 2x)^3$.

Solution:

$(1 + 2x)^3$ has coefficients and terms

$$\begin{matrix} 1 & 3 & 3 & 1 \\ 1^3 & 1^2(2x) & 1(2x)^2 & (2x)^3 \end{matrix}$$

Hence $(1 + 2x)^3 = 1 + 6x + 12x^2 + 8x^3$

$$\begin{aligned} & (1 + 3x)(1 + 2x)^3 \\ &= (1 + 3x)(1 + 6x + 12x^2 + 8x^3) \\ &= 1(1 + 6x + 12x^2 + 8x^3) + 3x(1 + 6x + 12x^2 + 8x^3) \\ &= 1 + 6x + 12x^2 + 8x^3 + 3x + 18x^2 + 36x^3 + 24x^4 \\ &= 1 + 9x + 30x^2 + 44x^3 + 24x^4 \end{aligned}$$

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Exercise A, Question 4

Question:

Expand $(2 + y)^3$. Hence or otherwise, write down the expansion of $(2 + x - x^2)^3$ in ascending powers of x .

Solution:

$(2 + y)^3$ has coefficients and terms

$$\begin{array}{cccc} 1 & 3 & 3 & 1 \\ 2^3 & 2^2y & 2y^2 & y^3 \end{array}$$

Therefore, $(2 + y)^3 = 8 + 12y + 6y^2 + 1y^3$

Substitute $y = x - x^2$

$$\Rightarrow (2 + x - x^2)^3 = 8 + 12(x - x^2) + 6(x - x^2)^2 + 1(x - x^2)^3$$

$$\Rightarrow (2 + x - x^2)^3 = 8 + 12x(1 - x) + 6x^2(1 - x)^2 + x^3(1 - x)^3$$

Now

$$(1 - x)^2 = (1 - x)(1 - x) = 1 - 2x + x^2$$

and

$$(1 - x)^3 = (1 - x)(1 - x)^2$$

$$(1 - x)^3 = (1 - x)(1 - 2x + x^2)$$

$$(1 - x)^3 = 1 - 2x + x^2 - x + 2x^2 - x^3$$

$$(1 - x)^3 = 1 - 3x + 3x^2 - x^3$$

Or, using Pascal's Triangles

$$(1 - x)^3 = 1(1)^3 + 3(1)^2(-x) + 3(1)(-x)^2 + 1(-x)^3$$

$$(1 - x)^3 = 1 - 3x + 3x^2 - x^3$$

So $(2 + x - x^2)^3 = 8 + 12x(1 - x) + 6x^2(1 - 2x + x^2) + x^3(1 - 3x + 3x^2 - x^3)$

$$\Rightarrow (2 + x - x^2)^3 = 8 + 12x - 12x^2 + 6x^2 - 12x^3 + 6x^4 + x^3 - 3x^4 + 3x^5 - x^6$$

$$\Rightarrow (2 + x - x^2)^3 = 8 + 12x - 6x^2 - 11x^3 + 3x^4 + 3x^5 - x^6$$

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Exercise A, Question 5

Question:

Find the coefficient of the term in x^3 in the expansion of $(2 + 3x)^3 (5 - x)^3$.

Solution:

$(2 + 3x)^3$ would have coefficients and terms

$$\begin{array}{cccc} 1 & 3 & 3 & 1 \\ 2^3 & 2^2 & (3x) & 2 & (3x)^2 & (3x)^3 \\ (2 + 3x)^3 = 1 \times 2^3 + 3 \times 2^2 & (3x) & + 3 \times 2 & (3x)^2 & + 1 \times & (3x)^3 \\ (2 + 3x)^3 = 8 + 36x + 54x^2 + 27x^3 \end{array}$$

$(5 - x)^3$ would have coefficients and terms

$$\begin{array}{cccc} 1 & 3 & 3 & 1 \\ 5^3 & 5^2 & (-x) & 5 & (-x)^2 & (-x)^3 \\ (5 - x)^3 = 1 \times 5^3 + 3 \times 5^2 & (-x) & + 3 \times 5 & (-x)^2 & + 1 \times & (-x)^3 \\ (5 - x)^3 = 125 - 75x + 15x^2 - x^3 \end{array}$$

$$(2 + 3x)^3 (5 - x)^3 = \underbrace{(8 + 36x + 54x^2 + 27x^3)(125 - 75x + 15x^2 - x^3)}$$

Term in x^3 is

$$\begin{aligned} & 8 \times (-x^3) + 36x \times 15x^2 + 54x^2 \times (-75x) + 27x^3 \times 125 \\ & = -8x^3 + 540x^3 - 4050x^3 + 3375x^3 \\ & = -143x^3 \end{aligned}$$

Coefficient of $x^3 = -143$

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Exercise A, Question 6

Question:

The coefficient of x^2 in the expansion of $(2 + ax)^3$ is 54. Find the possible values of the constant a .

Solution:

$(2 + ax)^3$ has coefficients 1 3 ③ 1

The circled number is the coefficient of the term $2^1 (ax)^2$.

Term in x^2 is $3 \times 2^1 \times (ax)^2 = 6a^2x^2$

Coefficient of x^2 is $6a^2$.

Hence

$$6a^2 = 54 \quad (\div 6)$$

$$a^2 = 9$$

$$a = \pm 3$$

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Exercise A, Question 7

Question:

The coefficient of x^2 in the expansion of $(2 - x)(3 + bx)^3$ is 45. Find possible values of the constant b .

Solution:

$(3 + bx)^3$ has coefficients and terms

$$1 \quad 3 \quad 3 \quad 1$$

$$3^3 \quad 3^2 (bx) \quad 3 (bx)^2 \quad (bx)^3$$

$$(3 + bx)^3 = 1 \times 3^3 + 3 \times 3^2 bx + 3 \times 3 (bx)^2 + 1 \times (bx)^3$$

$$(3 + bx)^3 = 27 + 27bx + 9b^2x^2 + b^3x^3$$

$$\text{So } (2 - x)(3 + bx)^3 = (2 - x)(27 + 27bx + 9b^2x^2 + b^3x^3)$$

$$\text{Term in } x^2 \text{ is } 2 \times 9b^2x^2 - x \times 27bx = 18b^2x^2 - 27bx^2$$

$$\text{Coefficient of } x^2 \text{ is } 18b^2 - 27b$$

Hence

$$18b^2 - 27b = 45 \quad (\div 9)$$

$$2b^2 - 3b = 5$$

$$2b^2 - 3b - 5 = 0$$

$$(2b - 5)(b + 1) = 0$$

$$b = \frac{5}{2}, -1$$

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Exercise A, Question 8

Question:

Find the term independent of x in the expansion of $\left(x^2 - \frac{1}{2x}\right)^3$.

Solution:

$\left(x^2 - \frac{1}{2x}\right)^3$ has coefficients and terms

$$\begin{array}{cccc} 1 & 3 & 3 & 1 \\ (x^2)^3 & (x^2)^2\left(-\frac{1}{2x}\right)^1 & (x^2)\left(-\frac{1}{2x}\right)^2 & \left(-\frac{1}{2x}\right)^3 \\ & & \uparrow & \end{array}$$

This term would be independent of x as the x 's cancel.

$$\text{Term independent of } x \text{ is } 3 \binom{3}{1} \binom{2}{2} \left(-\frac{1}{2x}\right)^2 = 3 \cdot x^0 \times \frac{1}{4x^2} = \frac{3}{4}$$

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Exercise B, Question 1

Question:

Find the values of the following:

(a) $4!$

(b) $6!$

(c) $\frac{8!}{6!}$

(d) $\frac{10!}{9!}$

(e) 4C_2

(f) 8C_6

(g) 5C_2

(h) 6C_3

(i) ${}^{10}C_9$

(j) 6C_2

(k) 8C_5

(l) nC_3

Solution:

(a) $4! = 4 \times 3 \times 2 \times 1 = 24$

(b) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

(c) $\frac{8!}{6!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 8 \times 7 = 56$

(d) $\frac{10!}{9!} = \frac{10 \times 9!}{9!} = 10$

(e) ${}^4C_2 = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = \frac{12}{2} = 6$

$$(f) {}^8C_6 = \frac{8!}{(8-6)!6!} = \frac{8!}{2!6!} = \frac{8 \times 7 \times 6!}{2!6!} = \frac{56}{2} = 28$$

$$(g) {}^5C_2 = \frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5 \times 4 \times 3!}{3!2!} = \frac{20}{2} = 10$$

$$(h) {}^6C_3 = \frac{6!}{(6-3)!3!} = \frac{6!}{3!3!} = \frac{6 \times 5 \times 4 \times 3!}{3! \times 3!} = \frac{6 \times 5 \times 4}{6} = 20$$

$$(i) {}^{10}C_9 = \frac{10!}{(10-9)!9!} = \frac{10!}{1!9!} = \frac{10 \times 9!}{1!9!} = \frac{10}{1} = 10$$

$$(j) {}^6C_2 = \frac{6!}{(6-2)!2!} = \frac{6!}{4!2!} = \frac{6 \times 5 \times 4!}{4!2!} = \frac{30}{2} = 15$$

$$(k) {}^8C_5 = \frac{8!}{(8-5)!5!} = \frac{8 \times 7 \times 6 \times 5!}{3!5!} = \frac{8 \times 7 \times 6}{6} = 56$$

$$(l) {}^nC_3 = \frac{n!}{(n-3)!3!} = \frac{n \times (n-1) \times (n-2) \times \cancel{(n-3)!}}{\cancel{(n-3)!} 3!} = \frac{n(n-1)(n-2)}{6}$$

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Exercise B, Question 2

Question:

Calculate:

(a) 4C_0

(b) $\binom{4}{1}$

(c) 4C_2

(d) $\binom{4}{3}$

(e) $\binom{4}{4}$

Now look at line 4 of Pascal's Triangle. Can you find any connection?

Solution:

$$(a) {}^4C_0 = \frac{4!}{(4-0)!0!} = \frac{4!}{4!0!} = 1$$

$$(b) \binom{4}{1} = \frac{4!}{(4-1)!1!} = \frac{4!}{3!1!} = \frac{4 \times 3!}{3!1!} = 4$$

$$(c) {}^4C_2 = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{4 \times 3 \times 2!}{2!2!} = \frac{12}{2} = 6$$

$$(d) \binom{4}{3} = \frac{4!}{(4-3)!3!} = \frac{4!}{1!3!} = \frac{4 \times 3!}{1!3!} = \frac{4}{1} = 4$$

$$(e) \binom{4}{4} = \frac{4!}{(4-4)!4!} = \frac{4!}{0!4!} = \frac{1}{0!} = 1$$

The numbers 1, 4, 6, 4, 1 form the fourth line of Pascal's Triangle.

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Exercise B, Question 3

Question:

Write using combination notation:

- (a) Line 3 of Pascal's Triangle.
- (b) Line 5 of Pascal's Triangle.

Solution:

(a) Line 3 of Pascal's Triangle is

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

(b) Line 5 of Pascal's Triangle is

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

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Exercise B, Question 4

Question:

Why is 6C_2 equal to $\binom{6}{4}$?

- (a) Answer using ideas on choosing from a group.
 (b) Answer by calculating both quantities.

Solution:

(a) 6C_2 or $\binom{6}{2}$ is the number of ways of choosing 2 items from a group of 6 items.

$\binom{6}{4}$ or 6C_4 is the number of ways of choosing 4 items from a group of 6 items.

These have to be the same.

For example, if you have a group of six people and want to pick a team of four, you have automatically selected a team of two.

$$(b) {}^6C_2 = \frac{6!}{(6-2)!2!} = \frac{6!}{4!2!} = \frac{6 \times 5 \times 4!}{4! 2!} = 15$$

$$\binom{6}{4} = \frac{6!}{(6-4)!4!} = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4!}{2! 4!} = 15$$

$$\text{Hence } {}^6C_2 = \binom{6}{4}$$

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Exercise C, Question 1

Question:

Write down the expansion of the following:

(a) $(2x + y)^4$

(b) $(p - q)^5$

(c) $(1 + 2x)^4$

(d) $(3 + x)^4$

(e) $\left(1 - \frac{1}{2}x\right)^4$

(f) $(4 - x)^4$

(g) $(2x + 3y)^5$

(h) $(x + 2)^6$

Solution:

(a) $(2x + y)^4$
 $= {}^4C_0 (2x)^4 + {}^4C_1 (2x)^3 (y) + {}^4C_2 (2x)^2 (y)^2 + {}^4C_3 (2x)^1 (y)^3 + {}^4C_4 (y)^4$
 $= 1 \times 16x^4 + 4 \times 8x^3y + 6 \times 4x^2y^2 + 4 \times 2xy^3 + 1 \times y^4$
 $= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

(b) $(p - q)^5$
 $= {}^5C_0 p^5 + {}^5C_1 p^4 (-q) + {}^5C_2 p^3 (-q)^2 + {}^5C_3 p^2 (-q)^3 + {}^5C_4 p (-q)^4 + {}^5C_5 (-q)^5$
 $= 1 \times p^5 + 5 \times (-p^4q) + 10 \times p^3q^2 + 10 \times (-p^2q^3) + 5 \times pq^4 + 1 \times (-q^5)$
 $= p^5 - 5p^4q + 10p^3q^2 - 10p^2q^3 + 5pq^4 - q^5$

(c) $(1 + 2x)^4$
 $= {}^4C_0 (1)^4 + {}^4C_1 (1)^3 (2x)^1 + {}^4C_2 (1)^2 (2x)^2 + {}^4C_3 (1) (2x)^3 + {}^4C_4 (2x)^4$
 $= 1 \times 1 + 4 \times 2x + 6 \times 4x^2 + 4 \times 8x^3 + 1 \times 16x^4$
 $= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

(d) $(3 + x)^4$
 $= {}^4C_0 (3)^4 + {}^4C_1 (3)^3 (x) + {}^4C_2 (3)^2 (x)^2 + {}^4C_3 (3) (x)^3 + {}^4C_4 (x)^4$
 $= 1 \times 81 + 4 \times 27x + 6 \times 9x^2 + 4 \times 3x^3 + 1 \times x^4$
 $= 81 + 108x + 54x^2 + 12x^3 + x^4$

(e) $\left(1 - \frac{1}{2}x\right)^4$

$$\begin{aligned}
&= {}^4C_0(1)^4 + {}^4C_1(1)^3 \left(-\frac{1}{2}x\right) + {}^4C_2(1)^2 \left(-\frac{1}{2}x\right)^2 + {}^4C_3 \binom{1}{1} \left(-\frac{1}{2}x\right)^3 + {}^4C_4 \left(-x\right)^4 \\
&= 1 \times 1 + 4 \times \left(-\frac{1}{2}x\right) + 6 \times \frac{1}{4}x^2 + 4 \times \left(-\frac{1}{8}x^3\right) + 1 \times \frac{1}{16}x^4 \\
&= 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4
\end{aligned}$$

$$\begin{aligned}
\text{(f)} \quad &(4-x)^4 \\
&= {}^4C_0(4)^4 + {}^4C_1(4)^3(-x) + {}^4C_2(4)^2(-x)^2 + {}^4C_3(4)^1(-x)^3 + {}^4C_4(-x)^4 \\
&= 1 \times 256 + 4 \times (-64x) + 6 \times 16x^2 + 4 \times (-4x^3) + 1 \times x^4 \\
&= 256 - 256x + 96x^2 - 16x^3 + x^4
\end{aligned}$$

$$\begin{aligned}
\text{(g)} \quad &(2x+3y)^5 \\
&= {}^5C_0(2x)^5 + {}^5C_1(2x)^4(3y) + {}^5C_2(2x)^3(3y)^2 + {}^5C_3(2x)^2(3y)^3 + {}^5C_4(2x)(3y)^4 + {}^5C_5(3y)^5 \\
&= 1 \times 32x^5 + 5 \times 48x^4y + 10 \times 72x^3y^2 + 10 \times 108x^2y^3 + 5 \times 162xy^4 + 1 \times 243y^5 \\
&= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5
\end{aligned}$$

$$\begin{aligned}
\text{(h)} \quad &(x+2)^6 \\
&= {}^6C_0(x)^6 + {}^6C_1(x)^5 \cdot 2 + {}^6C_2(x)^4 \cdot 2^2 + {}^6C_3(x)^3 \cdot 2^3 + {}^6C_4(x)^2 \cdot 2^4 + {}^6C_5(x) \cdot 2^5 + {}^6C_6 \cdot 2^6 \\
&= 1 \times x^6 + 6 \times 2x^5 + 15 \times 4x^4 + 20 \times 8x^3 + 15 \times 16x^2 + 6 \times 32x + 1 \times 64 \\
&= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64
\end{aligned}$$

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Exercise C, Question 2

Question:

Find the term in x^3 of the following expansions:

(a) $(3 + x)^5$

(b) $(2x + y)^5$

(c) $(1 - x)^6$

(d) $(3 + 2x)^5$

(e) $(1 + x)^{10}$

(f) $(3 - 2x)^6$

(g) $(1 + x)^{20}$

(h) $(4 - 3x)^7$

Solution:

(a) $(3 + x)^5$

Term in x^3 is ${}^5C_3 (3)^2 (x)^3 = 10 \times 9x^3 = 90x^3$

(b) $(2x + y)^5$

Term in x^3 is ${}^5C_2 (2x)^3 (y)^2 = 10 \times 8x^3y^2 = 80x^3y^2$

(c) $(1 - x)^6$

Term in x^3 is ${}^6C_3 (1)^3 (-x)^3 = 20 \times (-1x^3) = -20x^3$

(d) $(3 + 2x)^5$

Term in x^3 is ${}^5C_3 (3)^2 (2x)^3 = 10 \times 72x^3 = 720x^3$

(e) $(1 + x)^{10}$

Term in x^3 is ${}^{10}C_3 (1)^7 (x)^3 = 120 \times 1x^3 = 120x^3$

(f) $(3 - 2x)^6$

Term in x^3 is ${}^6C_3 (3)^3 (-2x)^3 = 20 \times (-216x^3) = -4320x^3$

(g) $(1 + x)^{20}$

Term in x^3 is ${}^{20}C_3 (1)^{17} (x)^3 = 1140 \times 1x^3 = 1140x^3$

(h) $(4 - 3x)^7$

Term in x^3 is ${}^7C_3 (4)^4 (-3x)^3 = 35 \times (-6912x^3) = -241\,920x^3$

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise C, Question 3

Question:

Use the binomial theorem to find the first four terms in the expansion of:

(a) $(1 + x)^{10}$

(b) $(1 - 2x)^5$

(c) $(1 + 3x)^6$

(d) $(2 - x)^8$

(e) $\left(2 - \frac{1}{2}x\right)^{10}$

(f) $(3 - x)^7$

(g) $(x + 2y)^8$

(h) $(2x - 3y)^9$

Solution:

(a) $(1 + x)^{10}$
 $= {}^{10}C_0 1^{10} + {}^{10}C_1 1^9 x^1 + {}^{10}C_2 1^8 x^2 + {}^{10}C_3 1^7 x^3 + \dots$
 $= 1 + 10 \times 1x + 45 \times 1x^2 + 120 \times 1x^3 + \dots$
 $= 1 + 10x + 45x^2 + 120x^3 + \dots$

(b) $(1 - 2x)^5$
 $= {}^5C_0 1^5 + {}^5C_1 1^4 (-2x)^1 + {}^5C_2 1^3 (-2x)^2 + {}^5C_3 1^2 (-2x)^3 + \dots$
 $= 1 \times 1 + 5 \times (-2x) + 10 \times 4x^2 + 10 \times (-8x^3) + \dots$
 $= 1 - 10x + 40x^2 - 80x^3 + \dots$

(c) $(1 + 3x)^6$
 $= {}^6C_0 1^6 + {}^6C_1 1^5 (3x)^1 + {}^6C_2 1^4 (3x)^2 + {}^6C_3 1^3 (3x)^3 + \dots$
 $= 1 \times 1 + 6 \times 3x + 15 \times 9x^2 + 20 \times 27x^3 + \dots$
 $= 1 + 18x + 135x^2 + 540x^3 + \dots$

(d) $(2 - x)^8$
 $= {}^8C_0 2^8 + {}^8C_1 2^7 (-x)^1 + {}^8C_2 2^6 (-x)^2 + {}^8C_3 2^5 (-x)^3 + \dots$
 $= 1 \times 256 + 8 \times (-128x) + 28 \times 64x^2 + 56 \times (-32x^3) + \dots$
 $= 256 - 1024x + 1792x^2 - 1792x^3 + \dots$

(e) $\left(2 - \frac{1}{2}x\right)^{10}$

$$\begin{aligned}
&= {}^{10}C_0 2^{10} + {}^{10}C_1 2^9 \left(-\frac{1}{2}x \right)^1 + {}^{10}C_2 2^8 \left(-\frac{1}{2}x \right)^2 + {}^{10}C_3 2^7 \left(-\frac{1}{2}x \right)^3 + \dots \\
&= 1 \times 1024 + 10 \times (-256x) + 45 \times 64x^2 + 120 \times (-16x^3) + \dots \\
&= 1024 - 2560x + 2880x^2 - 1920x^3 + \dots
\end{aligned}$$

$$\begin{aligned}
\text{(f)} \quad &(3-x)^7 \\
&= {}^7C_0 3^7 + {}^7C_1 3^6 (-x)^1 + {}^7C_2 3^5 (-x)^2 + {}^7C_3 3^4 (-x)^3 + \dots \\
&= 1 \times 2187 + 7 \times (-729x) + 21 \times 243x^2 + 35 \times (-81x^3) + \dots \\
&= 2187 - 5103x + 5103x^2 - 2835x^3 + \dots
\end{aligned}$$

$$\begin{aligned}
\text{(g)} \quad &(x+2y)^8 \\
&= {}^8C_0 x^8 + {}^8C_1 x^7 (2y)^1 + {}^8C_2 x^6 (2y)^2 + {}^8C_3 x^5 (2y)^3 + \dots \\
&= 1 \times x^8 + 8 \times 2x^7y + 28 \times 4x^6y^2 + 56 \times 8x^5y^3 + \dots \\
&= x^8 + 16x^7y + 112x^6y^2 + 448x^5y^3 + \dots
\end{aligned}$$

$$\begin{aligned}
\text{(h)} \quad &(2x-3y)^9 \\
&= {}^9C_0 (2x)^9 + {}^9C_1 (2x)^8 (-3y)^1 + {}^9C_2 (2x)^7 (-3y)^2 + {}^9C_3 (2x)^6 (-3y)^3 + \dots \\
&= 1 \times 512x^9 + 9 \times (-768x^8y) + 36 \times 1152x^7y^2 + 84 \times (-1728x^6y^3) + \dots \\
&= 512x^9 - 6912x^8y + 41472x^7y^2 - 145152x^6y^3 + \dots
\end{aligned}$$

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The binomial expansion

Exercise C, Question 4

Question:

The coefficient of x^2 in the expansion of $(2 + ax)^6$ is 60.
Find possible values of the constant a .

Solution:

$(2 + ax)^6$
Term in x^2 is ${}^6C_2 2^4 (ax)^2 = 15 \times 16a^2x^2 = 240a^2x^2$

Coefficient of x^2 is $240a^2$.

If this is equal to 60 then

$$240a^2 = 60 \quad (\div 240)$$

$$a^2 = \frac{1}{4} \quad \left(\sqrt{\quad} \right)$$

$$a = \pm \frac{1}{2}$$

Therefore $a = \pm \frac{1}{2}$.

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The binomial expansion

Exercise C, Question 5

Question:

The coefficient of x^3 in the expansion of $(3 + bx)^5$ is -720 .
Find the value of the constant b .

Solution:

$(3 + bx)^5$
Term in x^3 is ${}^5C_3 3^2 (bx)^3 = 10 \times 9b^3x^3 = 90b^3x^3$
Coefficient of x^3 is $90b^3$.
If this is equal to -720 then
 $90b^3 = -720$ ($\div 90$)
 $b^3 = -8$ ($\sqrt{\quad}$)
 $b = -2$
Hence $b = -2$.

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The binomial expansion

Exercise C, Question 6

Question:

The coefficient of x^3 in the expansion of $(2 + x)(3 - ax)^4$ is 30.
Find the values of the constant a .

Solution:

$$\begin{aligned} & (3 - ax)^4 \\ &= {}^4C_0 3^4 + {}^4C_1 3^3 (-ax) + {}^4C_2 3^2 (-ax)^2 + {}^4C_3 3^1 (-ax)^3 + {}^4C_4 (-ax)^4 \\ &= 1 \times 81 + 4 \times (-27ax) + 6 \times 9a^2x^2 + 4 \times (-3a^3x^3) + 1 \times a^4x^4 \\ &= 81 - 108ax + 54a^2x^2 - 12a^3x^3 + a^4x^4 \end{aligned}$$

$$(2 + x)(3 - ax)^4 =$$

$$(2+x)(81-108ax + 54a^2x^2 - 12a^3x^3 + a^4x^4)$$

Term in x^3 is $2 \times (-12a^3x^3) + x \times 54a^2x^2 = -24a^3x^3 + 54a^2x^3$

Hence

$$-24a^3 + 54a^2 = 30 \quad (\div 6)$$

$$-4a^3 + 9a^2 = 5$$

$$0 = 4a^3 - 9a^2 + 5 \quad (4 \times 1^3 - 9 \times 1^2 + 5 = 0 \Rightarrow a = 1 \text{ is a root})$$

$$0 = (a - 1)(4a^2 - 5a - 5)$$

So $a = 1$ and

$$4a^2 - 5a - 5 = 0$$

Using the formula for roots,

$$a = \frac{5 \pm \sqrt{25 + 80}}{8} = \frac{5 \pm \sqrt{105}}{8}$$

Possible values of a are 1, $\frac{5 + \sqrt{105}}{8}$ and $\frac{5 - \sqrt{105}}{8}$

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise C, Question 7

Question:

Write down the first four terms in the expansion of $\left(1 - \frac{x}{10}\right)^6$.

By substituting an appropriate value for x , find an approximate value to $(0.99)^6$. Use your calculator to find the degree of accuracy of your approximation.

Solution:

$$\begin{aligned} & \left(1 - \frac{x}{10}\right)^6 \\ &= {}^6C_0 1^6 + {}^6C_1 1^5 \left(-\frac{x}{10}\right) + {}^6C_2 1^4 \left(-\frac{x}{10}\right)^2 + {}^6C_3 1^3 \left(-\frac{x}{10}\right)^3 + \dots \\ &= 1 \times 1 + 6 \times \left(-\frac{x}{10}\right) + 15 \times \frac{x^2}{100} + 20 \times \left(-\frac{x^3}{1000}\right) + \dots \\ &= 1 - 0.6x + 0.15x^2 - 0.02x^3 + \dots \end{aligned}$$

We need to find $(0.99)^6$

$$\text{So } 1 - \frac{x}{10} = 0.99$$

$$\Rightarrow \frac{x}{10} = 0.01$$

$$\Rightarrow x = 0.1$$

Substitute $x = 0.1$ into our expansion for $\left(1 - \frac{x}{10}\right)^6$

$$\Rightarrow \left(1 - \frac{0.1}{10}\right)^6 = 1 - 0.6 \times 0.1 + 0.15 \times (0.1)^2 - 0.02 \times (0.1)^3 + \dots$$

$$\Rightarrow (0.99)^6 = 0.94148$$

From a calculator $(0.99)^6 = 0.941480149$

Accurate to 5 decimal places.

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The binomial expansion

Exercise C, Question 8

Question:

Write down the first four terms in the expansion of $\left(2 + \frac{x}{5}\right)^{10}$.

By substituting an appropriate value for x , find an approximate value to $(2.1)^{10}$. Use your calculator to find the degree of accuracy of your approximation.

Solution:

$$\begin{aligned} & \left(2 + \frac{x}{5}\right)^{10} \\ & {}^{10}C_0 2^{10} + {}^{10}C_1 2^9 \left(\frac{x}{5}\right)^1 + {}^{10}C_2 2^8 \left(\frac{x}{5}\right)^2 + {}^{10}C_3 2^7 \left(\frac{x}{5}\right)^3 + \dots \\ & = 1 \times 1024 + 10 \times \frac{512x}{5} + 45 \times \frac{256x^2}{25} + 120 \times \frac{128x^3}{125} + \dots \\ & = 1024 + 1024x + 460.8x^2 + 122.88x^3 + \dots \end{aligned}$$

If we want to find $(2.1)^{10}$ we need

$$2 + \frac{x}{5} = 2.1$$

$$\Rightarrow \frac{x}{5} = 0.1$$

$$\Rightarrow x = 0.5$$

Substitute $x = 0.5$ into the expansion for $\left(2 + \frac{x}{5}\right)^{10}$

$$(2.1)^{10} = 1024 + 1024 \times 0.5 + 460.8 \times (0.5)^2 + 122.88 \times (0.5)^3 + \dots$$

$$(2.1)^{10} = 1024 + 512 + 115.2 + 15.36 + \dots$$

$$(2.1)^{10} = 1666.56$$

From a calculator

$$(2.1)^{10} = 1667.988 \dots$$

Approximation is correct to 3 s.f. (both 1670).

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The binomial expansion

Exercise D, Question 1

Question:

Use the binomial expansion to find the first four terms of

(a) $(1 + x)^8$

(b) $(1 - 2x)^6$

(c) $\left(1 + \frac{x}{2}\right)^{10}$

(d) $(1 - 3x)^5$

(e) $(2 + x)^7$

(f) $(3 - 2x)^3$

(g) $(2 - 3x)^6$

(h) $(4 + x)^4$

(i) $(2 + 5x)^7$

Solution:

(a) Here $n = 8$ and $x = x$

$$(1 + x)^8 = 1 + 8x + \frac{8 \times 7}{2!}x^2 + \frac{8 \times 7 \times 6}{3!}x^3 + \dots$$

$$(1 + x)^8 = 1 + 8x + 28x^2 + 56x^3 + \dots$$

(b) Here $n = 6$ and $x = -2x$

$$(1 - 2x)^6 = 1 + 6 \binom{6}{1}(-2x) + \frac{6 \times 5}{2!}(-2x)^2 + \frac{6 \times 5 \times 4}{3!}(-2x)^3 + \dots$$

$$(1 - 2x)^6 = 1 - 12x + 60x^2 - 160x^3 + \dots$$

(c) Here $n = 10$ and $x = \frac{x}{2}$

$$\left(1 + \frac{x}{2}\right)^{10} = 1 + 10 \binom{10}{1}\left(\frac{x}{2}\right) + \frac{10 \times 9}{2!}\left(\frac{x}{2}\right)^2 + \frac{10 \times 9 \times 8}{3!}\left(\frac{x}{2}\right)^3 + \dots$$

$$\left(1 + \frac{x}{2}\right)^{10} = 1 + 5x + \frac{45}{4}x^2 + 15x^3 + \dots$$

(d) Here $n = 5$ and $x = -3x$

$$(1 - 3x)^5 = 1 + 5 \binom{5}{1}(-3x) + \frac{5 \times 4}{2!}(-3x)^2 + \frac{5 \times 4 \times 3}{3!}(-3x)^3 + \dots$$

$$(1 - 3x)^5 = 1 - 15x + 90x^2 - 270x^3 + \dots$$

$$(e) (2 + x)^7 = \left[2 \left(1 + \frac{x}{2} \right) \right]^7 = 2^7 \left(1 + \frac{x}{2} \right)^7$$

Here $n = 7$ and $x = \frac{x}{2}$, so

$$(2 + x)^7 = 128 \left[1 + 7 \left(\frac{x}{2} \right) + \frac{7 \times 6}{2!} \left(\frac{x}{2} \right)^2 + \frac{7 \times 6 \times 5}{3!} \left(\frac{x}{2} \right)^3 + \dots \right]$$

$$(2 + x)^7 = 128 \left(1 + \frac{7}{2}x + \frac{21}{4}x^2 + \frac{35}{8}x^3 + \dots \right)$$

$$(2 + x)^7 = 128 + 448x + 672x^2 + 560x^3 + \dots$$

$$(f) (3 - 2x)^3 = \left[3 \left(1 - \frac{2x}{3} \right) \right]^3 = 3^3 \left(1 - \frac{2x}{3} \right)^3$$

Here $n = 3$ and $x = \frac{-2x}{3}$, so

$$(3 - 2x)^3 = 27 \left[1 + 3 \left(-\frac{2x}{3} \right) + \frac{3 \times 2}{2!} \left(-\frac{2x}{3} \right)^2 + \frac{3 \times 2 \times 1}{3!} \left(-\frac{2x}{3} \right)^3 \right]$$

$$(3 - 2x)^3 = 27 \left(1 - 2x + \frac{4}{3}x^2 - \frac{8}{27}x^3 \right)$$

$$(3 - 2x)^3 = 27 - 54x + 36x^2 - 8x^3$$

$$(g) (2 - 3x)^6 = \left[2 \left(1 - \frac{3x}{2} \right) \right]^6 = 2^6 \left(1 - \frac{3x}{2} \right)^6$$

Here $n = 6$ and $x = -\frac{3x}{2}$, so

$$(2 - 3x)^6 = 64 \left[1 + 6 \left(-\frac{3x}{2} \right) + \frac{6 \times 5}{2!} \left(-\frac{3x}{2} \right)^2 + \frac{6 \times 5 \times 4}{3!} \left(-\frac{3x}{2} \right)^3 + \dots \right]$$

$$(2 - 3x)^6 = 64 \left(1 - 9x + \frac{135}{4}x^2 - \frac{135}{2}x^3 + \dots \right)$$

$$(2 - 3x)^6 = 64 - 576x + 2160x^2 - 4320x^3 + \dots$$

$$(h) (4 + x)^4 = \left[4 \left(1 + \frac{x}{4} \right) \right]^4 = 4^4 \left(1 + \frac{x}{4} \right)^4$$

Here $n = 4$ and $x = \frac{x}{4}$, so

$$(4 + x)^4 = 256 \left[1 + 4 \left(\frac{x}{4} \right) + \frac{4 \times 3}{2!} \left(\frac{x}{4} \right)^2 + \frac{4 \times 3 \times 2}{3!} \left(\frac{x}{4} \right)^3 + \dots \right]$$

$$(4 + x)^4 = 256 \left(1 + x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \dots \right)$$

$$(4 + x)^4 = 256 + 256x + 96x^2 + 16x^3 + \dots$$

$$(i) (2 + 5x)^7 = \left[2 \left(1 + \frac{5x}{2} \right) \right]^7 = 2^7 \left(1 + \frac{5x}{2} \right)^7$$

Here $n = 7$ and $x = \frac{5x}{2}$, so

$$(2 + 5x)^7 = 128 \left[1 + 7 \left(\frac{5x}{2} \right) + \frac{7 \times 6}{2!} \left(\frac{5x}{2} \right)^2 + \frac{7 \times 6 \times 5}{3!} \left(\frac{5x}{2} \right)^3 + \dots \right]$$

$$(2 + 5x)^7 = 128 \left(1 + \frac{35}{2}x + \frac{525}{4}x^2 + \frac{4375}{8}x^3 + \dots \right)$$

$$(2 + 5x)^7 = 128 + 2240x + 16800x^2 + 70000x^3 + \dots$$

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise D, Question 2

Question:

If x is so small that terms of x^3 and higher can be ignored, show that:

$$(2 + x)(1 - 3x)^5 \approx 2 - 29x + 165x^2$$

Solution:

$$(1 - 3x)^5 = 1 + 5 \binom{-3x}{1} + \frac{5 \times 4}{2!} (-3x)^2 + \dots$$

$$(1 - 3x)^5 = 1 - 15x + 90x^2 + \dots$$

$$\begin{aligned} (2 + x)(1 - 3x)^5 &= (2 + x)(1 - 15x + 90x^2 + \dots) \\ &= 2 - 30x + 180x^2 + \dots \\ &\quad + x - 15x^2 + \dots \\ &= 2 - 29x + 165x^2 \end{aligned}$$

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The binomial expansion

Exercise D, Question 3

Question:

If x is so small that terms of x^3 and higher can be ignored, and
 $(2 - x)(3 + x)^4 \approx a + bx + cx^2$
 find the values of the constants a , b and c .

Solution:

$$\begin{aligned} & (3 + x)^4 \\ &= \left[3 \left(1 + \frac{x}{3} \right) \right]^4 \\ &= 3^4 \left(1 + \frac{x}{3} \right)^4 \\ &= 81 \left[1 + 4 \left(\frac{x}{3} \right) + \frac{4 \times 3}{2!} \left(\frac{x}{3} \right)^2 + \frac{4 \times 3 \times 2}{3!} \left(\frac{x}{3} \right)^3 + \dots \right] \\ &= 81 \left(1 + \frac{4}{3}x + \frac{2}{3}x^2 + \frac{4}{27}x^3 + \dots \right) \\ &= 81 + 108x + 54x^2 + 12x^3 + \dots \end{aligned}$$

$$\begin{aligned} & (2 - x)(3 + x)^4 \\ &= (2 - x)(81 + 108x + 54x^2 + 12x^3 + \dots) \\ &= 162 + 216x + 108x^2 + \dots \\ &\quad - 81x - 108x^2 + \dots \\ &= 162 + 135x + 0x^2 + \dots \end{aligned}$$

Therefore $a = 162$, $b = 135$, $c = 0$

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise D, Question 4

Question:

When $(1 - 2x)^p$ is expanded, the coefficient of x^2 is 40. Given that $p > 0$, use this information to find:

- (a) The value of the constant p .
- (b) The coefficient of x .
- (c) The coefficient of x^3 .

Solution:

$$(1 - 2x)^p$$

$$= 1 + p \binom{-2x}{1} + \frac{p(p-1)}{2!} (-2x)^2 + \dots$$

$$= 1 - 2px + 2p(p-1)x^2 + \dots$$

$$\text{Coefficient of } x^2 \text{ is } 2p(p-1) = 40$$

$$\Rightarrow p(p-1) = 20$$

$$\Rightarrow p^2 - p - 20 = 0$$

$$\Rightarrow (p-5)(p+4) = 0$$

$$\Rightarrow p = 5$$

- (a) Value of p is 5.
- (b) Coefficient of x is $-2p = -10$.

$$(c) \text{ Term in } x^3 = \frac{p(p-1)(p-2)}{3!} (-2x)^3 = \frac{5 \times 4 \times 3}{3!} \binom{-8x^3}{1} = -80x^3$$

$$\text{Coefficient of } x^3 \text{ is } -80.$$

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The binomial expansion

Exercise D, Question 5

Question:

Write down the first four terms in the expansion of $(1 + 2x)^8$. By substituting an appropriate value of x (which should be stated), find an approximate value of 1.02^8 . State the degree of accuracy of your answer.

Solution:

$$\begin{aligned}(1 + 2x)^8 &= 1 + 8 \times 2x + \frac{8 \times 7}{2!} (2x)^2 + \frac{8 \times 7 \times 6}{3!} (2x)^3 + \dots \\ &= 1 + 16x + 112x^2 + 448x^3 + \dots\end{aligned}$$

If we want an approximate value to $(1.02)^8$ we require

$$1 + 2x = 1.02$$

$$2x = 0.02$$

$$x = 0.01$$

Substitute $x = 0.01$ into our approximation for $(1 + 2x)^8$

$$\begin{aligned}(1.02)^8 &= 1 + 16 \times 0.01 + 112 \times (0.01)^2 + 448 \times (0.01)^3 \\ &= 1 + 0.16 + 0.0112 + 0.000448 \\ &= 1.171648\end{aligned}$$

By using a calculator

$$(1.02)^8 = 1.171659$$

Approximation is correct to 4 s.f. (1.172 for both solutions)

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise E, Question 1

Question:

When $\left(1 - \frac{3}{2}x\right)^p$ is expanded in ascending powers of x , the coefficient of x is -24 .

- (a) Find the value of p .
 (b) Find the coefficient of x^2 in the expansion.
 (c) Find the coefficient of x^3 in the expansion.

[E]

Solution:

$$\left(1 - \frac{3x}{2}\right)^p = 1 + p \left(-\frac{3x}{2}\right) + \frac{p(p-1)}{2!} \left(-\frac{3x}{2}\right)^2 + \frac{p(p-1)(p-2)}{3!} \left(-\frac{3x}{2}\right)^3 + \dots$$

- (a) Coefficient of x is $-\frac{3p}{2}$

We are given its value is -24

$$\Rightarrow -\frac{3p}{2} = -24$$

$$\Rightarrow p = 16$$

- (b) Coefficient of x^2 is $\frac{p(p-1)}{2} \times \frac{9}{4} = \frac{16 \times 15}{2} \times \frac{9}{4} = 270$

- (c) Coefficient of x^3 is $-\frac{p(p-1)(p-2)}{3!} \times \frac{27}{8} = -\frac{16 \times 15 \times 14}{3!} \times \frac{27}{8} = -1890$

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The binomial expansion

Exercise E, Question 2

Question:

Given that:

$$(2 - x)^{13} \equiv A + Bx + Cx^2 + \dots$$

Find the values of the integers A , B and C .

[E]

Solution:

$$\begin{aligned} (2 - x)^{13} &= 2^{13} + {}^{13}C_1 2^{12} (-x) + {}^{13}C_2 2^{11} (-x)^2 + \dots \\ &= 8192 + 13 \times (-4096x) + 78 \times 2048x^2 + \dots \\ &= 8192 - 53248x + 159744x^2 + \dots \\ &\equiv A + Bx + Cx^2 + \dots \end{aligned}$$

So $A = 8192$, $B = -53248$, $C = 159744$

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The binomial expansion

Exercise E, Question 3

Question:

(a) Expand $(1 - 2x)^{10}$ in ascending powers of x up to and including the term in x^3 , simplifying each coefficient in the expansion.

(b) Use your expansion to find an approximation to $(0.98)^{10}$, stating clearly the substitution which you have used for x .

[E]

Solution:

$$\begin{aligned}
 \text{(a) } (1 - 2x)^{10} &= 1 + 10 \binom{10}{1} (-2x) + \frac{10 \times 9}{2!} (-2x)^2 + \frac{10 \times 9 \times 8}{3!} (-2x)^3 + \dots \\
 &= 1 + 10 \times (-2x) + 45 \times 4x^2 + 120 \times (-8x^3) + \dots \\
 &= 1 - 20x + 180x^2 - 960x^3 + \dots
 \end{aligned}$$

(b) We need $(1 - 2x) = 0.98$

$$\Rightarrow 2x = 0.02$$

$$\Rightarrow x = 0.01$$

Substitute $x = 0.01$ into our expansion for $(1 - 2x)^{10}$

$$(1 - 2 \times 0.01)^{10} = 1 - 20 \times 0.01 + 180 \times 0.01^2 - 960 \times 0.01^3 + \dots$$

$$(0.98)^{10} = 1 - 0.2 + 0.018 - 0.00096 + \dots$$

$$(0.98)^{10} = 0.81704 + \dots$$

$$\text{So } (0.98)^{10} \approx 0.81704$$

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The binomial expansion

Exercise E, Question 4

Question:

(a) Use the binomial series to expand $(2 - 3x)^{10}$ in ascending powers of x up to and including the term in x^3 , giving each coefficient as an integer.

(b) Use your series expansion, with a suitable value for x , to obtain an estimate for 1.97^{10} , giving your answer to 2 decimal places.

[E]

Solution:

$$\begin{aligned}
 \text{(a)} \quad (2 - 3x)^{10} &= 2^{10} + {}^{10}C_1 2^9 (-3x)^1 + {}^{10}C_2 2^8 (-3x)^2 + {}^{10}C_3 2^7 (-3x)^3 + \dots \\
 &= 1024 + 10 \times (-1536x) + 45 \times 2304x^2 + 120 \times (-3456x^3) + \dots \\
 &= 1024 - 15360x + 103680x^2 - 414720x^3 + \dots
 \end{aligned}$$

(b) We require $2 - 3x = 1.97$

$$\Rightarrow 3x = 0.03$$

$$\Rightarrow x = 0.01$$

Substitute $x = 0.01$ in both sides of our expansion of $(2 - 3x)^{10}$

$$\begin{aligned}
 (2 - 3 \times 0.01)^{10} &= 1024 - 15360 \times 0.01 + 103680 \times 0.01^2 - 414720 \times 0.01^3 + \dots \\
 (1.97)^{10} &\approx 1024 - 153.6 + 10.368 - 0.41472 = 880.35328 = 880.35 \text{ (2 d.p.)}
 \end{aligned}$$

Solutionbank C2

Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise E, Question 5

Question:

- (a) Expand $(3 + 2x)^4$ in ascending powers of x , giving each coefficient as an integer.
- (b) Hence, or otherwise, write down the expansion of $(3 - 2x)^4$ in ascending powers of x .
- (c) Hence by choosing a suitable value for x show that $(3 + 2\sqrt{2})^4 + (3 - 2\sqrt{2})^4$ is an integer and state its value.

[E]

Solution:

- (a) $(3 + 2x)^4$ has coefficients and terms

$$\begin{array}{ccccccc} 1 & 4 & & 6 & & 4 & & 1 \\ 3^4 & 3^3 & (2x) & 3^2 & (2x)^2 & 3 & (2x)^3 & (2x)^4 \end{array}$$

Putting these together gives

$$\begin{aligned} (3 + 2x)^4 &= 1 \times 3^4 + 4 \times 3^3 \times 2x + 6 \times 3^2 \times (2x)^2 + 4 \times 3 \times (2x)^3 + 1 \times (2x)^4 \\ (3 + 2x)^4 &= 81 + 216x + 216x^2 + 96x^3 + 16x^4 \end{aligned}$$

$$\begin{aligned} (b) \quad (3 - 2x)^4 &= 1 \times 3^4 + 4 \times 3^3 \times (-2x) + 6 \times 3^2 \times (-2x)^2 + 4 \times 3 \times (-2x)^3 + 1 \times (-2x)^4 \\ (3 - 2x)^4 &= 81 - 216x + 216x^2 - 96x^3 + 16x^4 \end{aligned}$$

- (c) Using parts (a) and (b)

$$(3 + 2x)^4 + (3 - 2x)^4 =$$

$$\begin{array}{r} 81 + 216x + 216x^2 + 96x^3 + 16x^4 \\ + 81 - 216x + 216x^2 - 96x^3 + 16x^4 \\ \hline 162 \qquad \qquad + 432x^2 \qquad \qquad + 32x^4 \end{array}$$

Substituting $x = \sqrt{2}$ into both sides of this expansion gives

$$\begin{aligned} (3 + 2\sqrt{2})^4 + (3 - 2\sqrt{2})^4 &= 162 + 432(\sqrt{2})^2 + 32(\sqrt{2})^4 \\ &= 162 + 432 \times 2 + 32 \times 4 = 162 + 864 + 128 = 1154 \end{aligned}$$

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise E, Question 6

Question:

The coefficient of x^2 in the binomial expansion of $\left(1 + \frac{x}{2}\right)^n$, where n is a positive integer, is 7.

(a) Find the value of n .

(b) Using the value of n found in part (a), find the coefficient of x^4 .

[E]

Solution:

$$\left(1 + \frac{x}{2}\right)^n = 1 + n \left(\frac{x}{2}\right) + \frac{n(n-1)}{2!} \left(\frac{x}{2}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{x}{2}\right)^3 + \frac{n(n-1)(n-2)(n-3)}{4!} \left(\frac{x}{2}\right)^4 + \dots$$

(a) We are told the coefficient of x^2 is 7

$$\Rightarrow \frac{n(n-1)}{2} \times \frac{1}{4} = 7$$

$$\Rightarrow n(n-1) = 56$$

$$\Rightarrow n^2 - n - 56 = 0$$

$$\Rightarrow (n-8)(n+7) = 0$$

$$\Rightarrow n = 8$$

(b) Coefficient of x^4 is

$$\frac{n(n-1)(n-2)(n-3)}{4!} \times \frac{1}{2^4} = \frac{8 \times 7 \times \overset{2}{\cancel{6}} \times 5}{\cancel{4} \times \cancel{3} \times 2 \times 1} \times \frac{1}{\cancel{16}} = \frac{35}{8}$$

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise E, Question 7

Question:

- (a) Use the binomial theorem to expand $(3 + 10x)^4$ giving each coefficient as an integer.
- (b) Use your expansion, with an appropriate value for x , to find the exact value of $(1003)^4$. State the value of x which you have used.

[E]

Solution:

$$\begin{aligned}
 \text{(a) } (3 + 10x)^4 &= 3^4 + {}^4C_1 3^3 (10x) + {}^4C_2 (3)^2 (10x)^2 + {}^4C_3 (3)^1 (10x)^3 + (10x)^4 \\
 &= 3^4 + 4 \times 270x + 6 \times 900x^2 + 4 \times 3000x^3 + 10000x^4 \\
 &= 81 + 1080x + 5400x^2 + 12000x^3 + 10000x^4
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) We require } 1003 &= 3 + 10x \\
 \Rightarrow 1000 &= 10x \\
 \Rightarrow 100 &= x
 \end{aligned}$$

Substitute $x = 100$ in both sides of our expansion

$$\begin{aligned}
 (3 + 10 \times 100)^4 &= 81 + 1080 \times 100 + 5400 \times 100^2 + 12000 \times 100^3 + 10000 \times 100^4 \\
 (1003)^4 &= 81 + 108\,000 + 54\,000\,000 + 12\,000\,000\,000 + 1\,000\,000\,000\,000 \\
 (1003)^4 &=
 \end{aligned}$$

$$\begin{array}{r}
 1\,000\,000\,000\,000 \\
 12\,000\,000\,000 \\
 54\,000\,000 \\
 108\,000 \\
 81 \\
 \hline
 1\,012\,054\,108\,081
 \end{array}$$

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The binomial expansion

Exercise E, Question 8

Question:

- (a) Expand $(1 + 2x)^{12}$ in ascending powers of x up to and including the term in x^3 , simplifying each coefficient.
- (b) By substituting a suitable value for x , which must be stated, into your answer to part (a), calculate an approximate value of $(1.02)^{12}$.
- (c) Use your calculator, writing down all the digits in your display, to find a more exact value of $(1.02)^{12}$.
- (d) Calculate, to 3 significant figures, the percentage error of the approximation found in part (b).

[E]

Solution:

$$\begin{aligned} \text{(a)} \quad & (1 + 2x)^{12} \\ &= 1 + 12 \binom{12}{1} (2x)^1 + \frac{12 \times 11}{2!} (2x)^2 + \frac{12 \times 11 \times 10}{3!} (2x)^3 + \dots \\ &= 1 + 24x + 264x^2 + 1760x^3 + \dots \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & \text{We require } 1 + 2x = 1.02 \\ & \Rightarrow 2x = 0.02 \\ & \Rightarrow x = 0.01 \end{aligned}$$

$$\begin{aligned} & \text{Substitute } x = 0.01 \text{ in both sides of expansion} \\ & (1 + 2 \times 0.01)^{12} = 1 + 24 \times 0.01 + 264 \times 0.01^2 + 1760 \times 0.01^3 \\ & (1.02)^{12} = 1 + 0.24 + 0.0264 + 0.00176 \\ & (1.02)^{12} = 1.26816 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad & \text{Using a calculator} \\ & (1.02)^{12} = 1.268241795 \end{aligned}$$

$$\text{(d)} \quad \% \text{ error} = \frac{|\text{Answer b} - \text{Answer c}|}{\text{Answer c}} \times 100$$

$$\begin{aligned} \% \text{ error} &= 0.006449479 \\ \% \text{ error} &= 0.00645 \% \end{aligned}$$

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The binomial expansion

Exercise E, Question 9

Question:

Expand $\left(x - \frac{1}{x}\right)^5$, simplifying the coefficients.

[E]

Solution:

$\left(x - \frac{1}{x}\right)^5$ has coefficients and terms

$${}^1_5 x^5 x^4 \left(-\frac{1}{x}\right) {}^{10}_3 x^3 \left(-\frac{1}{x}\right)^2 {}^{10}_2 x^2 \left(-\frac{1}{x}\right)^3 {}^5_1 x \left(-\frac{1}{x}\right)^4 \left(-\frac{1}{x}\right)^5$$

Putting these together gives

$$\left(x - \frac{1}{x}\right)^5 = 1x^5 + 5x^4 \left(-\frac{1}{x}\right) + 10x^3 \left(-\frac{1}{x}\right)^2 + 10x^2 \left(-\frac{1}{x}\right)^3 + 5x \left(-\frac{1}{x}\right)^4 + 1 \left(-\frac{1}{x}\right)^5$$

$$\left(x - \frac{1}{x}\right)^5 = x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$$

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise E, Question 10

Question:

In the binomial expansion of $(2k + x)^n$, where k is a constant and n is a positive integer, the coefficient of x^2 is equal to the coefficient of x^3 .

(a) Prove that $n = 6k + 2$.

(b) Given also that $k = \frac{2}{3}$, expand $(2k + x)^n$ in ascending powers of x up to and including the term in x^3 , giving each coefficient as an exact fraction in its simplest form.

[E]

Solution:

$$(a) (2k + x)^n = (2k)^n + {}^n C_1 (2k)^{n-1} x + {}^n C_2 (2k)^{n-2} x^2 + {}^n C_3 (2k)^{n-3} x^3 + \dots$$

Coefficient of $x^2 =$ coefficient of x^3

$${}^n C_2 (2k)^{n-2} = {}^n C_3 (2k)^{n-3}$$

$$\frac{n!}{(n-2)!2!} (2k)^{n-2} = \frac{n!}{(n-3)!3!} (2k)^{n-3}$$

$$\frac{(2k)^{n-2}}{(2k)^{n-3}} = \frac{(n-2)!2!}{(n-3)!3!} \quad (\text{Use laws of indices})$$

$$(2k)^1 = \frac{(n-2)!2!}{(n-3)!3!} \left[(n-2)! = (n-2) \times (n-3)! \right]$$

$$2k = \frac{(n-2) \times 2}{3}$$

$$3 \times 2k = n - 2$$

$$6k = n - 2$$

$$n = 6k + 2$$

(b) If $k = \frac{2}{3}$ then $n = 6 \times \frac{2}{3} + 2 = 6$

Expression is

$$\left(2 \times \frac{2}{3} + x \right)^6$$

$$= \left(\frac{4}{3} + x \right)^6$$

$$= \left(\frac{4}{3} \right)^6 + {}^6 C_1 \left(\frac{4}{3} \right)^5 x^1 + {}^6 C_2 \left(\frac{4}{3} \right)^4 x^2 + {}^6 C_3 \left(\frac{4}{3} \right)^3 x^3 + \dots$$

$$= \frac{4096}{729} + \frac{2048}{81} x + \frac{1280}{27} x^2 + \frac{1280}{27} x^3 + \dots$$

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise E, Question 11

Question:

(a) Expand $(2 + x)^6$ as a binomial series in ascending powers of x , giving each coefficient as an integer.

(b) By making suitable substitutions for x in your answer to part (a), show that $(2 + \sqrt{3})^6 - (2 - \sqrt{3})^6$ can be simplified to the form $k\sqrt{3}$, stating the value of the integer k .

[E]

Solution:

$$\begin{aligned} \text{(a) } (2 + x)^6 &= 2^6 + {}^6C_1 2^5 x + {}^6C_2 2^4 x^2 + {}^6C_3 2^3 x^3 + {}^6C_4 2^2 x^4 + {}^6C_5 2 x^5 + x^6 \\ &= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6 \end{aligned}$$

(b) With $x = \sqrt{3}$

$$(2 + \sqrt{3})^6 = 64 + 192\sqrt{3} + 240(\sqrt{3})^2 + 160(\sqrt{3})^3 + 60(\sqrt{3})^4 + 12(\sqrt{3})^5 + (\sqrt{3})^6 \quad \textcircled{1}$$

with $x = -\sqrt{3}$

$$(2 - \sqrt{3})^6 = 64 + 192(-\sqrt{3}) + 240(-\sqrt{3})^2 + 160(-\sqrt{3})^3 + 60(-\sqrt{3})^4 + 12(-\sqrt{3})^5 + (-\sqrt{3})^6$$

$$(2 - \sqrt{3})^6 = 64 - 192\sqrt{3} + 240(\sqrt{3})^2 - 160(\sqrt{3})^3 + 60(\sqrt{3})^4 - 12(\sqrt{3})^5 + (\sqrt{3})^6 \quad \textcircled{2}$$

$\textcircled{1} - \textcircled{2}$ gives

$$\begin{aligned} (2 + \sqrt{3})^6 - (2 - \sqrt{3})^6 &= 384\sqrt{3} + 320(\sqrt{3})^3 + 24(\sqrt{3})^5 \\ &= 384\sqrt{3} + 320 \times 3\sqrt{3} + 24 \times 3 \times 3\sqrt{3} \\ &= 384\sqrt{3} + 960\sqrt{3} + 216\sqrt{3} \\ &= 1560\sqrt{3} \end{aligned}$$

Hence $k = 1560$

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The binomial expansion

Exercise E, Question 12

Question:

The coefficient of x^2 in the binomial expansion of $(2 + kx)^8$, where k is a positive constant, is 2800.

- (a) Use algebra to calculate the value of k .
- (b) Use your value of k to find the coefficient of x^3 in the expansion.

[E]

Solution:

(a) The term in x^2 of $(2 + kx)^8$ is
 ${}^8C_2 2^6 (kx)^2 = 28 \times 64k^2x^2 = 1792k^2x^2$

Hence $1792k^2 = 2800$

$$k^2 = 1.5625$$

$$k = \pm 1.25$$

Since k is positive $k = 1.25$.

(b) Term in x^3 of $(2 + kx)^8$ is

$${}^8C_3 2^5 (kx)^3 = 56 \times 32k^3x^3$$

Coefficient of x^3 term is $1792k^3 = 1792 \times 1.25^3 = 3500$

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise E, Question 13

Question:

(a) Given that

$$(2+x)^5 + (2-x)^5 \equiv A + Bx^2 + Cx^4,$$

find the value of the constants A , B and C .

(b) Using the substitution $y = x^2$ and your answers to part (a), solve

$$(2+x)^5 + (2-x)^5 = 349.$$

[E]

Solution:

(a) $(2+x)^5$ will have coefficients and terms

$$\begin{array}{cccccc} 1 & 5 & 10 & 10 & 5 & 1 \\ 2^5 & 2^4x & 2^3x^2 & 2^2x^3 & 2x^4 & x^5 \end{array}$$

Putting these together we get

$$(2+x)^5 = 1 \times 2^5 + 5 \times 2^4x + 10 \times 2^3x^2 + 10 \times 2^2x^3 + 5 \times 2x^4 + 1 \times x^5$$

$$(2+x)^5 = 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$$

Therefore

$$(2-x)^5 = 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$$

$$\text{Adding } (2+x)^5 + (2-x)^5 = 64 + 160x^2 + 20x^4$$

$$\text{So } A = 64, B = 160, C = 20$$

(b) $(2+x)^5 + (2-x)^5 = 349$

$$64 + 160x^2 + 20x^4 = 349$$

$$20x^4 + 160x^2 - 285 = 0 \quad (\div 5)$$

$$4x^4 + 32x^2 - 57 = 0$$

Substitute $y = x^2$

$$4y^2 + 32y - 57 = 0$$

$$(2y - 3)(2y + 19) = 0$$

$$y = \frac{3}{2}, -\frac{19}{2}$$

$$\text{But } y = x^2, \text{ so } x^2 = \frac{3}{2} \Rightarrow x = \pm \sqrt{\frac{3}{2}}$$

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Edexcel Modular Mathematics for AS and A-Level

The binomial expansion

Exercise E, Question 14

Question:

In the binomial expansion of $(2 + px)^5$, where p is a constant, the coefficient of x^3 is 135. Calculate:

- (a) The value of p ,
- (b) The value of the coefficient of x^4 in the expansion.

[E]

Solution:

(a) The term in x^3 in the expansion of $(2 + px)^5$ is

$${}^5C_3 2^2 (px)^3 = 10 \times 4p^3 x^3 = 40p^3 x^3$$

We are given the coefficient is 135 so

$$40p^3 = 135 \quad (\div 40)$$

$$p^3 = 3.375 \quad \left(\sqrt[3]{\quad} \right)$$

$$p = 1.5$$

(b) The term in x^4 in the expansion of $(2 + px)^5$ is

$${}^5C_4 2^1 (px)^4 = 5 \times 2p^4 x^4 = 5 \times 2(1.5)^4 x^4 = 50.625x^4$$

Coefficient of x^4 is 50.625