# Pure Core 1

# Revision Notes

May 2016

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# 1 Algebra

## *Indices*

**Rules of indices**

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$a^0 = 1$

$a^{-n} = \dfrac{1}{a^n}$

$a^{1/n} = \sqrt[n]{a}$

$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

*Examples:*

(i) $5^{-3} \times 5^4 = 5^{-3+4} = 5^1 = 5$.

$7^{-4} \times 7^{-2} = 7^{-4+-2} = 7^{-4-2} = 7^{-6} = \dfrac{1}{7^6}$.

(ii) $3^5 \div 3^{-2} = 3^{5--2} = 3^{5+2} = 3^7$.

$9^{-4} \div 9^6 = 9^{-4-6} = 9^{-10} = \dfrac{1}{9^{10}}$

$11^{-3} \div 11^{-5} = 11^{-3--5} = 11^{-3+5} = 11^2 = 121$

(iii) $(6^{-3})^4 = 6^{-3\times 4} = 6^{-12} = \dfrac{1}{6^{12}}$.

(iv) $64^{2/3} = \left(64^{1/3}\right)^2 = (4)^2 = 16$

(v) $125^{-2/3} = \dfrac{1}{125^{2/3}}$ since *minus* means *turn upside down*

$= \dfrac{1}{5^2}$, since 3 on bottom of fraction is cube root, $\sqrt[3]{125} = 5$

$= \dfrac{1}{25}$

*Example:* Express $(16^a) \div (8^b)$ as power of 2.

*Solution:* $(16^a) \div (8^b) = (2^4)^a \div (2^3)^b = 2^{4a} \div 2^{3b} = 2^{4a-3b}$.

*Example:* Find $x$ if $9^{2x} = 27^{x+1}$.

*Solution:* First notice that $9 = 3^2$ and $27 = 3^3$ and so

$9^{2x} = 27^{x+1} \quad \Rightarrow \quad (3^2)^{2x} = (3^3)^{x+1}$

$\Rightarrow \quad 3^{4x} = 3^{3x+3}$

$\Rightarrow \quad 4x = 3x + 3 \quad \Rightarrow \quad x = 3.$

## *Surds*

A surd is a 'nasty' root – i.e. a root which is not rational.

Thus $\sqrt{64} = 8$, $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$, $\sqrt[5]{-243} = -3$ are rational and not surds

and $\sqrt{5}$, $\sqrt[5]{45}$, $\sqrt[3]{-72}$ are irrational and **are surds**.

### Simplifying surds

*Example:* To simplify $\sqrt{50}$ we notice that $50 = 25 \times 2 = 5^2 \times 2$

$\Rightarrow \quad \sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$.

*Example:* To simplify $\sqrt[3]{40}$ we notice that $40 = 8 \times 5 = 2^3 \times 5$

$\Rightarrow \quad \sqrt[3]{40} = \sqrt[3]{8 \times 5} = \sqrt[3]{8} \times \sqrt[3]{5} = 2 \times \sqrt[3]{5}$.

### Rationalising the denominator

*Rationalising* means getting rid of surds.

We remember that multiplying $(a + b)$ by $(a - b)$ gives $a^2 - b^2$ which has the effect of squaring **both** $a$ and $b$ at the same time!!

*Example:* Rationalise the denominator of $\frac{2+3\sqrt{5}}{3-\sqrt{5}}$ .

*Solution:* $\frac{2+3\sqrt{5}}{3-\sqrt{5}} = \frac{2+3\sqrt{5}}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$

$= \frac{6+3\sqrt{5}\sqrt{5}+ 9\sqrt{5}+2\sqrt{5}}{3^2-\left(\sqrt{5}\right)^2}$

$= \frac{21+11\sqrt{5}}{4}$.

# 2 Quadratic functions

A quadratic function is a function $ax^2 + bx + c$, where $a$, $b$ and $c$ are constants and the highest power of $x$ is 2.

The numbers $a$ and $b$ are called the *coefficients* of $x^2$ and $x$, and $c$ is the *constant term*.

**Completing the square.**

*a*) The coefficient of $x^2$ must be +1.

*b*) Halve the coefficient of $x$, square it then add it and subtract it.

*Example:* Complete the square in $x^2 - 6x + 7$.

*Solution:* *a*) The coefficient of $x^2$ is already +1,

*b*) the coefficient of $x$ is $-6$, halve it to give $-3$ then square to give 9 which is then added and subtracted

$$x^2 - 6x + 7 = x^2 - 6x + (-3)^2 - 9 + 7$$
$$= (x - 3)^2 - 2.$$

*Notice* the *min*imum value of $x^2 - 6x + 7$ is $-2$ when $x = 3$,

since the *min*imum value of $(x - 3)^2$ is 0

$\Rightarrow$ the *vertex* of the graph of $y = x^2 - 6x + 7$ (parabola) is at $(3, -2)$.

*Example:* Complete the square in $-3x^2 - 24x + 5$.

*Solution:* *a*) The coefficient of $x^2$ is not +1, so we must take out a factor of $-3$ first, and then go on to step *b*).

$$-3x^2 - 24x + 5$$
$$= -3(x^2 + 8x) + 5$$

*b*)
$$= -3(x^2 + 8x + 4^2 - 16) + 5$$
$$= -3(x + 4)^2 + 48 + 5$$
$$= -3(x + 4)^2 + 53$$

*Notice* the *max*imum value of $-3x^2 - 24x + 5$ is $+53$ when $x = -4$,

since the *max*imum value of $(x + 4)^2$ is 0

$\Rightarrow$ the *vertex* of the graph of $y = -3x^2 - 24x + 5$ (parabola) is at $(-4, 53)$.

**Factorising quadratics**

If all the coefficients are integers (whole numbers) then if we can find factors with rational (fraction) coefficients, then we could have found factors with integer coefficients. There is no point in looking for factors with fractions as coefficients.

*Example:* We could factorise $x^2 - 5x + 6$ using fractions as $(2x-4)\left(\frac{1}{2}x - \frac{3}{2}\right)$,

BUT $(2x-4)\left(\frac{1}{2}x - \frac{3}{2}\right) = 2(x-2)\left(\frac{1}{2}x - \frac{3}{2}\right) = (x-2)\left(2 \times \frac{1}{2}x - 2 \times \frac{3}{2}\right)$

$= (x-2)(x-3)$, factors with integer coefficients, so no point in using fractions.

**More complicated expressions**

*Example:* Factorise $10x^2 + 11x - 6$.

*Solution:* Looking at the $10x^2$ and the $-6$ we see that possible factors are

| | | | |
|---|---|---|---|
| $(10x \pm 1)$, | $(10x \pm 2)$, | $(10x \pm 3)$, | $(10x \pm 6)$, |
| $(5x \pm 1)$, | $(5x \pm 2)$, | $(5x \pm 3)$, | $(5x \pm 6)$, |
| $(2x \pm 1)$, | $(2x \pm 2)$, | $(2x \pm 3)$, | $(2x \pm 6)$, |
| $(x \pm 1)$, | $(x \pm 2)$, | $(x \pm 3)$, | $(x \pm 6)$, |

Also the $-6$ tells us that the factors must have opposite signs, and by trial and error and the factor theorem, or using common sense

$$10x^2 + 11x - 6 = (2x+3)(5x-2).$$

## *Solving quadratic equations.*

**By factorising.**

*Example:* $3x^2 + 2x - 8 = 0 \Rightarrow (3x-4)(x+2) = 0$

$\Rightarrow$ $3x - 4 = 0$ or $x + 2 = 0$ $\Rightarrow$ $x = \frac{4}{3}$ or $x = -2$.

*Example:* $x^2 + 8x = 0 \Rightarrow x(x+8) = 0$

$\Rightarrow$ $x = 0$ or $x + 8 = 0$

$\Rightarrow$ $x = 0$ or $x = -8$

**N.B.** Do **not** divide through by $x$ first: you will lose the root of $x = 0$.

**By completing the square**

*Example:* $x^2 - 6x - 4 = 0$

$\Rightarrow$ $x^2 - 6x = 4$ coefficient of $x^2$ is $+1$, halve $-6$, square and add 9 to both sides

$\Rightarrow$ $x^2 - 6x + 9 = 4 + 9$

$\Rightarrow$ $(x-3)^2 = 13$

$(x-3) = \pm\sqrt{13}$ $\Rightarrow$ $x = 3 \pm \sqrt{13} = -0{\cdot}61$ or $6{\cdot}61$ to 2 D.P.

**By using the formula**

For a proof of the formula, see the appendix

Always try to factorise first.

$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

*Example:* Solve the equation $3x^2 - x - 5 = 0$

*Solution:* $3x^2 - x - 5$ will not factorise,

so we use the formula with $a = 3$, $b = -1$, $c = -5$

$\Rightarrow \quad x = \frac{-(-1) \pm \sqrt{(-1)^2-4\times3\times(-5)}}{2\times3} = -1{\cdot}135$ or $+1{\cdot}468$ to 3 D.P.

**The discriminant, $b^2 - 4ac$**

The quadratic equation

$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ will have

i) two distinct real roots if $b^2 - 4ac > 0$

ii) only one real root if $b^2 - 4ac = 0$ (or two coincident real roots)

iii) no real roots if $b^2 - 4ac < 0$

*Example:* For what values of $k$ does the equation $3x^2 - kx + 5 = 0$ have

i) two distinct real roots,
ii) exactly one real root
iii) no real roots.

*Solution:* The *discriminant* $b^2 - 4ac = (-k)^2 - 4 \times 3 \times 5 = k^2 - 60$

$\Rightarrow$ i) for two distinct real roots $k^2 - 60 > 0$

$\Rightarrow \quad k^2 > 60 \quad \Rightarrow \quad k < -\sqrt{60}$, or $k > +\sqrt{60}$

and ii) for only one real root $k^2 - 60 = 0$

$\Rightarrow \quad k = \pm\sqrt{60}$

and iii) for no real roots $k^2 - 60 < 0$

$\Rightarrow \quad -\sqrt{60} < k < +\sqrt{60}$.

**Miscellaneous quadratic equations**

*Example:* Solve $3^{2x} - 10 \times 3^{x} + 9 = 0$

*Solution:* Notice that $3^{2x} = (3^{x})^{2}$

so put $y = 3^{x}$ to give

$$y^2 - 10y + 9 = 0$$
$$\Rightarrow \quad (y-9)(y-1) = 0$$
$$\Rightarrow \quad y = 9 \text{ or } y = 1$$
$$\Rightarrow \quad 3^x = 9 \text{ or } 3^x = 1$$
$$\Rightarrow \quad x = 2 \text{ or } x = 0.$$

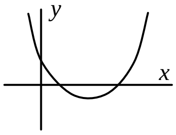
*Example:* Solve $y - 3\sqrt{y} + 2 = 0$.

*Solution:* Put $\sqrt{y} = x$ to give

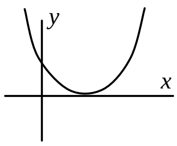
$$x^2 - 3x + 2 = 0$$
$$\Rightarrow \quad (x-2)(x-1)$$
$$\Rightarrow \quad x = 2 \text{ or } 1$$
$$\Rightarrow \quad y = x^2 = 4 \text{ or } 1.$$

## Quadratic graphs

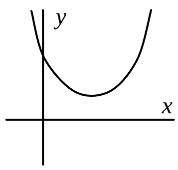
*a*) If $a > 0$ the parabola will be 'the right way up'




$b^2 - 4ac > 0$
2 distinct real roots

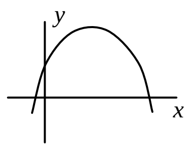



$b^2 - 4ac = 0$
only 1 real root

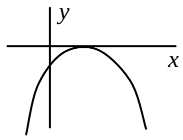



$b^2 - 4ac < 0$
no real roots

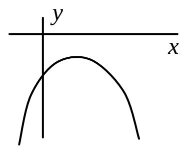
b) If $a < 0$ the parabola will be ‘the wrong way up’




$b^2 - 4ac > 0$
2 distinct real roots




$b^2 - 4ac = 0$
only 1 real root




$b^2 - 4ac < 0$
no real roots

**Note:** When sketching the curve of a quadratic function you should always show the value on the $y$-axis,

if you have factorised you should show the values where it meets the $x$-axis,

if you have completed the square you should give the co-ordinates of the vertex.

## *Simultaneous equations*

**Two linear equations**

*Example:* Solve

$$3x - 2y = 4 \qquad \mathbf{I}$$
$$4x + 7y = 15 \qquad \mathbf{II}$$

*Solution:* Make the coefficients of $x$ (or $y$) equal then add or subtract the equations to eliminate $x$ (or $y$).

$$4 \times \mathbf{I} \quad \Rightarrow \quad 12x - \;\; 8y = 16$$
$$3 \times \mathbf{II} \quad \Rightarrow \quad \underline{12x + 21y = 45} \qquad \text{Subtracting}$$
$$-29y = -29$$
$$\Rightarrow \quad y = 1$$

Put $y = 1$ in **I** $\Rightarrow$ $3x = 4 + 2y = 4 + 2$ $\Rightarrow$ $x = 2$

Check in equation two: L.H.S. = $4 \times 2 + 7 \times 1 = 15$ = R.H.S.

*Answer* : $x = 2$, $y = 1$

## One linear equation and one quadratic.

Find $x$ (or $y$) from the linear equation and substitute in the quadratic equation.

*Example:* Solve $x - 2y = 3$ **I**

$x^2 - 2y^2 - 3y = 5$ **II**

*Solution:* From **I** $x = 2y + 3$

Substitute in **II**

$\Rightarrow \quad (2y + 3)^2 - 2y^2 - 3y = 5$

$\Rightarrow \quad 4y^2 + 12y + 9 - 2y^2 - 3y = 5$

$\Rightarrow \quad 2y^2 + 9y + 4 = 0 \quad \Rightarrow \quad (2y + 1)(y + 4) = 0$

$\Rightarrow \quad y = -½$ or $y = -4$

$\Rightarrow \quad x = 2$ or $x = -5$ using **I** (do not use **II** unless you like doing extra work!)

Check in the quadratic equation

When $x = 2$ and $y = -½$

L.H.S. $= 2^2 - 2(-½)^2 - 3(-½) = 5 =$ R.H.S.

and when $x = -5$ and $y = -4$

L.H.S. $= (-5)^2 - 2(-4)^2 - 3(-4) = 25 - 32 + 12 = 5 =$ R.H.S.

*Answer:* $x = 2$, $y = -½$ or $x = -5$, $y = -4$

## *Inequalities*

### Linear inequalities

Solving algebraic inequalities is just like solving equations, add, subtract, multiply or divide the same number to, from, etc. BOTH SIDES

EXCEPT - if you multiply or divide both sides by a NEGATIVE number then you must TURN THE INEQUALITY SIGN ROUND.

*Example:* Solve $3 + 2x < 8 + 4x$

*Solution:* sub 3 from B.S. $\Rightarrow \quad 2x < 5 + 4x$
sub $4x$ from B.S. $\Rightarrow \quad -2x < 5$
divide B.S. by $-2$ and **turn the inequality sign round**
$\Rightarrow \quad x > -2{\cdot}5$.

### Quadratic inequalities, $x^2 > k$ or $< k$

*Example:* Solve $x^2 > 16$

*Solution:* We must be careful here since the square of a negative number is positive giving the full range of solutions as

$\Rightarrow \qquad x < -4$ **or** $x > +4$.

### Quadratic inequalities, $ax^2 + bx + c > k$ or $< k$

Always sketch a graph and find where the curve meets the $x$-axis

*Example:* Find the values of $x$ which satisfy

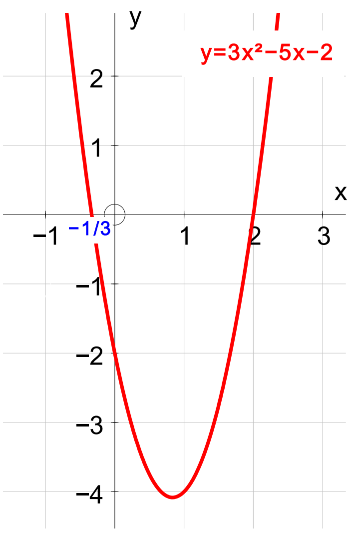
$3x^2 - 5x - 2 \geq 0$.

*Solution:*

solving $\qquad 3x^2 - 5x - 2 = 0$

$\Rightarrow \qquad (3x + 1)(x - 2) = 0$

$\Rightarrow \qquad x = -{}^1/_3$ or $2$




We want the part of the curve $y = 3x^2 - 5x - 2$ which is above or on the $x$-axis

$\Rightarrow \quad x \leq -{}^1/_3$ **or** $x \geq 2$

# 3 Coordinate geometry

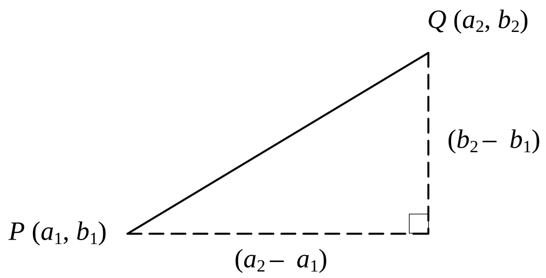
## Distance between two points

Distance between $P\ (a_1, b_1)$ and $Q\ (a_2, b_2)$ is $\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$

## Gradient

Gradient of $PQ$ is $\quad m = \dfrac{b_2 - b_1}{a_2 - a_1}$

## Equation of a line




Equation of the line $PQ$, above, is $y = mx + c$
and use a point to find $c$

or the equation of the line with gradient $m$
through the point $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$

or the equation of the line through the points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(you do not need to know this one!!)}$$

## Parallel and perpendicular lines

Two lines are parallel if they have the same gradient
and they are perpendicular if the product of their gradients is $-1$.

*Example:* Find the equation of the line through $(4\frac{1}{2}, 1)$ and perpendicular to the line joining the points $A\ (3, 7)$ and $B\ (6, -5)$.

*Solution:* Gradient of $AB$ is $\frac{7--5}{3-6} = -4$

$\Rightarrow$ gradient of line perpendicular to $AB$ is $\frac{1}{4}$, product of perpendicular gradients is $-1$

so we want the line through $(4\frac{1}{2}, 1)$ with gradient $\frac{1}{4}$.

Using $y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 1 = \frac{1}{4}(x - 4\frac{1}{2})$

$\Rightarrow \quad 4y - x = -\frac{1}{2} \quad$ or $\quad 2x - 8y - 1 = 0.$

# 4 Sequences and series

A *sequence* is any list of numbers.

## *Definition by a formula $x_n = f(n)$*

*Example:* The definition $x_n = 3n^2 - 5$ gives

$x_1 = 3 \times \mathbf{1}^2 - 5 = -2$, $x_2 = 3 \times \mathbf{2}^2 - 5 = 7$, $x_3 = 3 \times \mathbf{3}^2 - 5 = 22$, … .

## *Definitions of the form $x_{n+1} = f(x_n)$*

These have two parts:–

(i) a starting value (or values)

(ii) a method of obtaining each term from the one(s) before.

*Examples:* (i) The definition $x_1 = 3$ and $x_n = 3x_{n-1} + 2$

defines the sequence 3, 11, 35, 107, . . .

(ii) The definition $x_1 = 1$, $x_2 = 1$, $x_n = x_{n-1} + x_{n-2}$

defines the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 65, . . . .

This is the Fibonacci sequence.

## *Series and Σ notation*

A *series* is the sum of the first so many terms of a sequence.

For a sequence whose $n$th term is $x_n = 2n + 3$ the sum of the first $n$ terms is a series

$S_n = x_1 + x_2 + x_3 + x_4 \ . \ . \ . + x_n = 5 + 7 + 9 + 11 + \ . \ . \ . + (2n + 3)$

This is written in Σ notation as $S_n = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (2i + 3)$ and is a *finite* series of $n$ terms.

An *infinite* series has an infinite number of terms $S_\infty = \sum_{i=1}^{\infty} x_i$ .

## *Arithmetic series*

An *arithmetic series* is a series in which each term is a constant amount bigger (or smaller) than the previous term: this *constant amount* is called the *common difference*.

*Examples:* 3, 7, 11, 15, 19, 23, . . . . with common difference 4

28, 25, 22, 19, 16, 13, . . . with common difference –3.

*Generally* an arithmetic series can be written as

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + \ldots$ upto $n$ terms,

where the first term is $a$ and the common difference is $d$.

The $n$th term $x_n = a + (n - 1)d$

The *sum* of the first $n$ terms of the above arithmetic series is

$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$, or $S_n = \frac{n}{2}\left(a + l\right)$ where $l$ is the **last** term.

**Proof of the formula for the sum of an arithmetic series**

You **must** know this proof.

First write down the general series and then write it down in reverse order

$$S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)$$

$$\Rightarrow S_n = (a + (n - 1)d) + (a + (n - 2)d) + (a + (n - 3)d) + \ldots + a$$

**ADD**

$$\Rightarrow 2S_n = (2a + (n - 1)d) + (2a + (n - 1)d) + (2a + (n - 1)d) + \ldots (2a + (n - 1)d)$$

$$\Rightarrow 2S_n = n(2a + (n - 1)d)$$

$$\Rightarrow S_n = \frac{n}{2} \times (2a + (n - 1)d). \quad \textbf{I}$$

This can be written as

$$S_n = \frac{n}{2} \times \{a + (a + (n - 1)d)\}$$

$$\Rightarrow S_n = \frac{n}{2} \times (a + l), \quad \text{where } l \text{ is the last term.} \quad \textbf{II}$$

You should know both **I** and **II**.

*Example:* Find the $n$th term and the sum of the first 100 terms of the arithmetic series with 3rd term 5 and 7th term 17.

*Solution:* $x_7 = x_3 + 4d$ add $d$ four times to get from the third term to the seventh

$\Rightarrow$ $17 = 5 + 4d$ $\Rightarrow$ $d = 3$

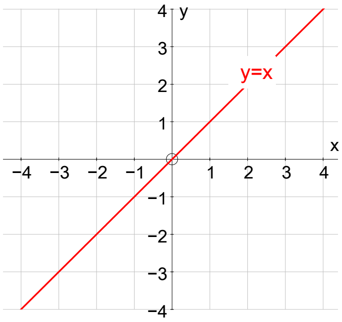
$\Rightarrow$ $x_1 = x_3 - 2d$ $\Rightarrow$ $x_1 = 5 - 6 = -1$

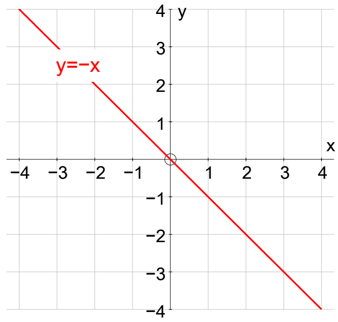
$\Rightarrow$ $n$th term $x_n = a + (n - 1)d = -1 + (n - 1) \times 3$

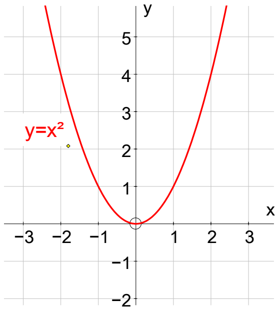
and $\Rightarrow$ $S_{100} = \frac{100}{2} \times (2 \times (-1) + (100 - 1) \times 3) = 14750.$

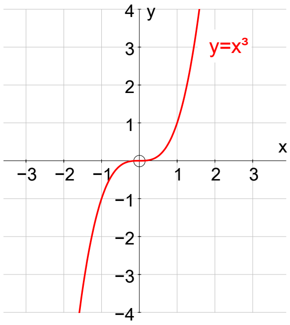
# 5 Curve sketching

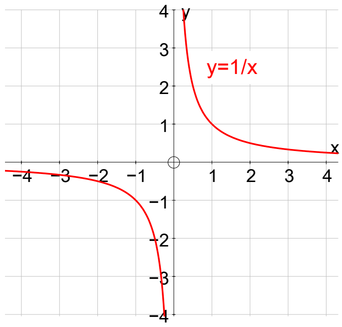
## *Standard graphs*

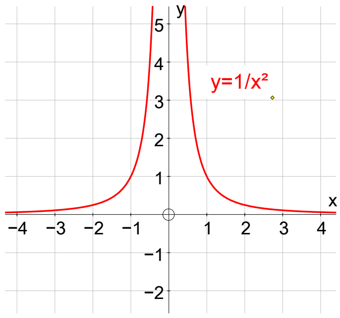


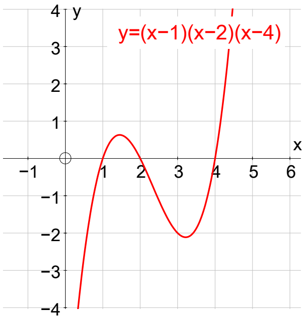



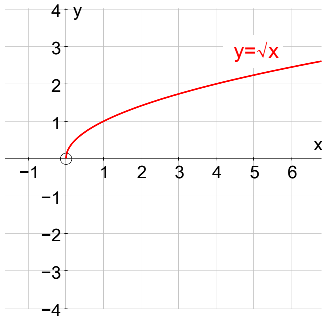



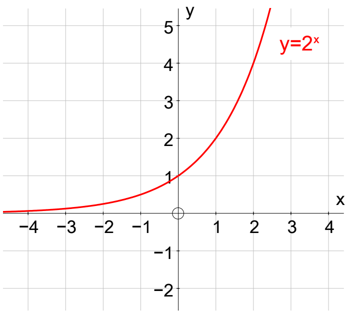



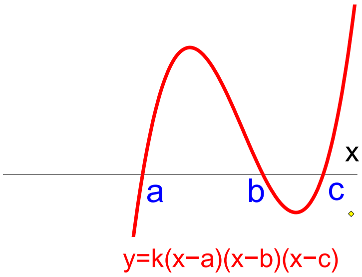



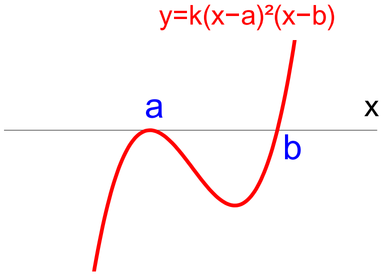

$y = 3x^2$ is like $y = x^2$ but steeper: similarly for $y = 5x^3$ and $y = {}^7/_x$, etc.

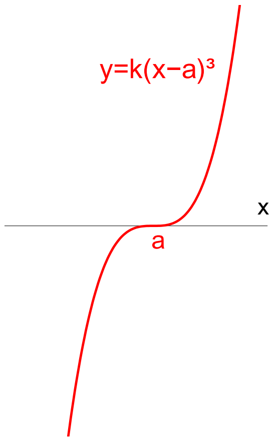
**Cubic graphs**

Some types of cubic graph are shown below:

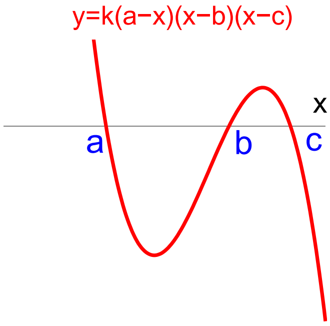
Coefficient of $x^3$ is positive

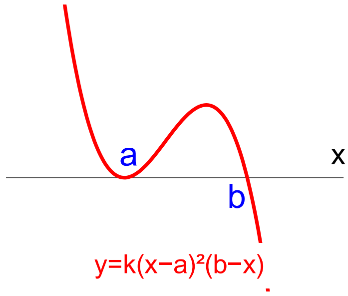


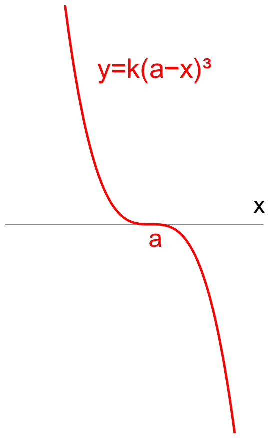




Coefficient of $x^3$ is negative



## *Transformations of graphs*

### Translations

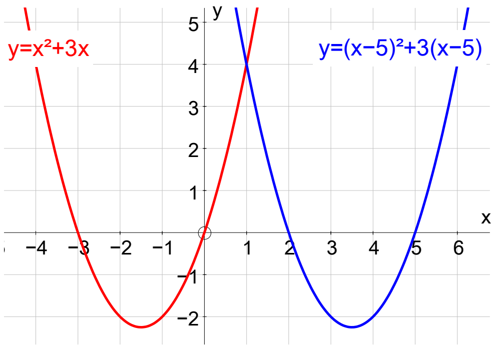
(i) If the graph of $y = x^2 + 3x$ is translated through $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$, +5 in the $y$-direction, the equation of the new graph is

$$y = x^2 + 3x + 5;$$

and, in general, the graph of $y = f(x)$ after a translation through $\begin{pmatrix} 0 \\ b \end{pmatrix}$ has equation $y = f(x) + b$.

(ii) If the graph of $y = x^2 + 3x$ is translated through $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$, +5 in the $x$-direction, the equation of the new graph is

$$y = (x - 5)^2 + 3(x - 5).$$

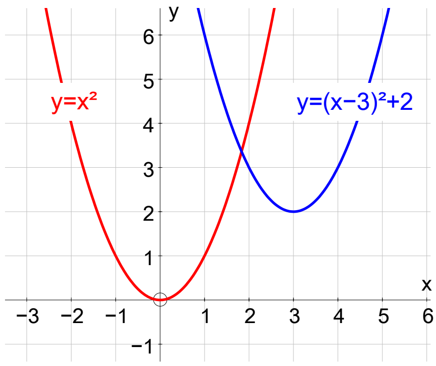



and, in general, the graph of $y = f(x)$ after a translation of $\begin{pmatrix} a \\ 0 \end{pmatrix}$ has equation $y = f(x - a)$.

We replace $x$ by $(x - a)$ everywhere in the formula for $y$: note the minus sign, $-a$, which seems wrong but is correct!

*Example:*

The graph of $y = (x - 3)^2 + 2$ is the graph of $y = x^2$ after a translation of $\begin{pmatrix} +3 \\ +2 \end{pmatrix}$

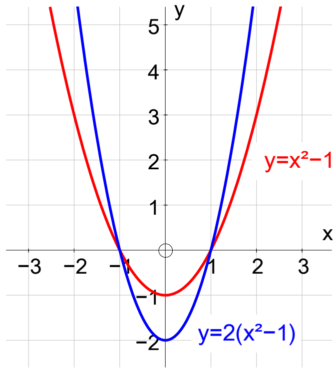



**In general** The equation of the graph of $y = x^2$, or $y = f(x)$, becomes $y = (x - a)^2 + b$, or $y = f(x - a) + b$, after a translation through $\begin{pmatrix} a \\ b \end{pmatrix}$.

## Stretches

(i) If the graph of $y = f(x)$ is stretched by a factor of $+2$ in the $y$-direction then the new equation is $y = 2 \times f(x)$.

*Example:*




The graph of $y = f(x) = x^2 - 1$ becomes

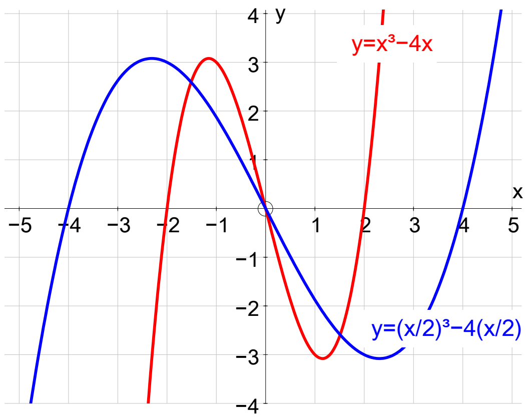
$y = 2f(x) = 2(x^2 - 1)$ after a stretch of factor 2 in the $y$-direction

In general $y = x^2$ or $y = f(x)$ becomes $y = ax^2$ or $y = af(x)$ after a stretch in the $y$-direction of factor $a$.

(ii) If the graph of $y = f(x)$ is stretched by a factor of $+2$ in the $x$-direction then the new equation is $y = f\left(\frac{x}{2}\right)$.

We replace $x$ by $\left(\frac{x}{2}\right)$ everywhere in the formula for $y$.

*Example:*




In this example the graph of

$$y = x^3 - 4x$$

has been stretched by a factor of 2 in the $x$-direction to form a new graph with equation

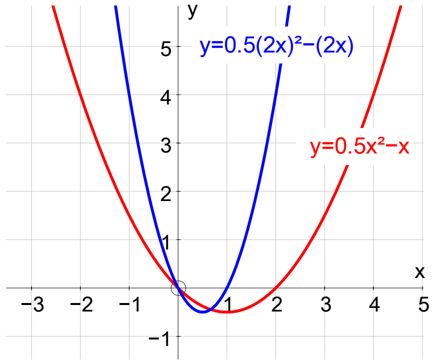
$$y = f(x) \;\rightarrow\; y = f\left(\frac{x}{2}\right)$$

$$\Rightarrow \quad y = \left(\frac{x}{2}\right)^3 - 4\left(\frac{x}{2}\right)$$

(iii) Note that to stretch by a factor of $\frac{1}{2}$ in the $x$-direction we replace $x$ by $\frac{x}{\frac{1}{2}} = 2x$ so that $y = f(x)$ becomes $y = f(2x)$

*Example:*

In this example the graph of $y = 0{\cdot}5x^2 - x$




has been stretched by a factor of $\frac{1}{2}$ in the $x$-direction to form a new graph with equation

$$y = f(x) \quad \to \quad y = f\left(\frac{x}{\frac{1}{2}}\right) = f(2x)$$

$$\Rightarrow \quad y = 0{\cdot}5 \times (2x)^2 - (2x) \;=\; 2x^2 - 2x$$

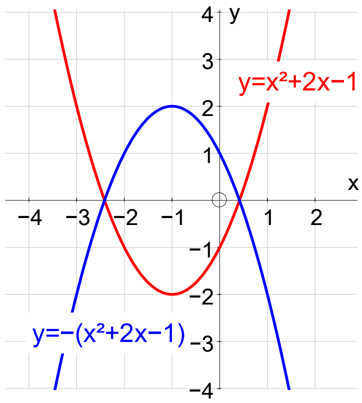
The new equation is formed by replacing $x$ by $\frac{x}{\frac{1}{2}} = 2x$ in the original equation.

**Reflections in the *x*-axis**

When reflecting in the $x$-axis all the positive $y$-coordinates become negative and vice versa

$\Rightarrow$ the image of $y = f(x)$ after reflection in the $x$-axis is $y = -f(x)$.

*Example:* The image of $y = f(x) = x^2 + 2x - 1$ after reflection in the $x$-axis is




$$y = f(x) \quad \to \quad y = -f(x)$$

$$\Rightarrow \quad y = -(x^2 + 2x - 1)$$

$$\Rightarrow \quad y = -x^2 - 2x + 1$$

**Reflections in the *y*–axis**

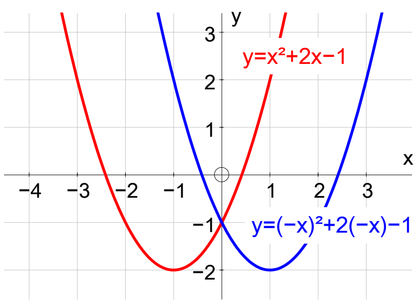
When reflecting in the $y$-axis

the $y$-coordinate for $x = +3$ becomes the $y$-coordinate for $x = -3$ and
the $y$-coordinate for $x = -2$ becomes the $y$-coordinate for $x = +2$.

Thus the equation of the new graph is found by replacing $x$ by $-x$

$\Rightarrow$ the image of $y = f(x)$ after reflection in the $y$–axis is $y = f(-x)$.

*Example:* The image of $y = f(x) = x^2 + 2x - 1$ after reflection in the $x$-axis is




$$y = f(x) \quad \to \quad y = f(-x)$$

$$\Rightarrow \quad y = (-x)^2 + 2(-x) - 1$$

$$\Rightarrow \quad y = x^2 - 2x - 1$$

**Summary of transformations**

| Old equation | Transformation | New equation |
|---|---|---|
| $y = f(x)$ | Translation through $\begin{pmatrix} a \\ b \end{pmatrix}$ | $y = f(x - a) + b$ |
| $y = f(x)$ | Stretch with factor $a$ in the $y$-direction. | $y = a \times f(x)$ |
| $y = f(x)$ | Stretch with factor $a$ in the $x$-direction. | $y = f\left(\frac{x}{a}\right)$ |
| $y = f(x)$ | Enlargement with factor $a$ centre (0, 0) | $y = a \times f\left(\frac{x}{a}\right)$ |
| $y = f(x)$ | Stretch with factor $\frac{1}{a}$ in the $x$-direction. | $y = f(ax)$ |
| $y = f(x)$ | Reflection in the $x$-axis | $y = -f(x)$ |
| $y = f(x)$ | Reflection in the $y$-axis | $y = f(-x)$ |

# 6 Differentiation

## General result

Differentiating is finding the gradient of the curve.

$y = x^n \Rightarrow \dfrac{dy}{dx} = nx^{n-1}$, or $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

*Examples:*

*a*) $y = 3x^2 - 7x + 4$, $\dfrac{dy}{dx} = 6x - 7$

*b*) $f(x) = 7\sqrt{x} = 7x^{1/2}$ $f'(x) = 7 \times \frac{1}{2} x^{\frac{-1}{2}} = \dfrac{7}{2\sqrt{x}}$

*c*) $y = \dfrac{8}{x^3} = 8x^{-3}$ $\dfrac{dy}{dx} = 8 \times (-3x^{-4}) = \dfrac{-24}{x^4}$

*d*) $f(x) = (2x + 1)(x - 3)$

multiply out first

$= 2x^2 - 5x - 3$ $f'(x) = 4x - 5$

*e*) $y = \dfrac{3x^7 - 4x^2}{x^5}$

split up first

$= \dfrac{3x^7}{x^5} - \dfrac{4x^2}{x^5} = 3x^2 - 4x^{-3}$ $\dfrac{dy}{dx} = 3\times 2x - 4\times(-3x^{-4}) = 6x + \dfrac{12}{x^4}$

## Tangents and Normals

**Tangents**

*Example:* Find the equation of the tangent to $y = 3x^2 - 7x + 5$ at the point where $x = 2$.

*Solution:*

We first find the gradient when $x = 2$.

$y = 3x^2 - 7x + 5 \Rightarrow \dfrac{dy}{dx} = 6x - 7$ and when $x = 2$, $\dfrac{dy}{dx} = 6 \times 2 - 7 = 5$.

so the gradient when $x = 2$ is 5.

To find the equation of the line we need the $y$-coordinate of the point where $x = 2$.

When $x = 2$, $y = 3 \times 2^2 - 7 \times 2 + 5 = 3$.

$\Rightarrow$ the equation of the tangent is $y - 3 = 5(x - 2)$ using $y - y_1 = m(x - x_1)$

$\Rightarrow$ $y = 5x - 7$

**Normals**

(The normal to a curve is the line which is perpendicular to the tangent at that point).

We first remember that if two lines with gradients $m_1$ and $m_2$ are perpendicular then $m_1 \times m_2 = -1$.

*Example:*

Find the equation of the normal to the curve $y = x + \frac{2}{x}$ at the point where $x = 2$.

*Solution:*

We first find the gradient of the tangent when $x = 2$.

$$y = x + 2x^{-1} \Rightarrow \frac{dy}{dx} = 1 - 2x^{-2} = 1 - \frac{2}{x^2}$$

$\Rightarrow$ when $x = 2$ gradient of the tangent is $m_T = 1 - \frac{2}{4} = \frac{1}{2}$.

If gradient of the normal is $m_N$ then $m_T \times m_N = -1 \Rightarrow \frac{1}{2} \times m_N = -1 \Rightarrow m_N = -2$

When $x = 2$, $y = 2 + \frac{2}{2} = 3$

$\Rightarrow$ the equation of the normal is $y - 3 = -2(x - 2)$ using $y - y_1 = m(x - x_1)$

$\Rightarrow$ $y = -2x + 7$.

# 7 Integration

## *Indefinite integrals*

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad \text{provided that } n \neq -1$$

**N.B. NEVER FORGET THE ARBITRARY CONSTANT + *C*.**

*Examples:*

*a*) $\int \frac{4}{3x^6} \, dx = \int \frac{4x^{-6}}{3} \, dx$

$= \frac{4}{3} \times \frac{x^{-5}}{-5} + C = \frac{-4}{15x^5} + C.$

*b*) $\int (3x-2)(x+1) \, dx = \int 3x^2 + x - 2 \, dx$ multiply out first

$= x^3 + \frac{1}{2}x^2 - 2x + C.$

*c*) $\int \frac{x^9 + 5x^2}{x^5} \, dx = \int \frac{x^9}{x^5} + \frac{5x^2}{x^5} \, dx$ split up first

$= \int x^4 + 5x^{-3} \, dx$

$= \frac{x^5}{5} + 5\frac{x^{-2}}{-2} + C = \frac{x^5}{5} - \frac{5}{2x^2} + C.$

**Finding the arbitrary constant**

If you know $\frac{dy}{dx}$ and the coordinates of a point on the curve you can find the arbitrary constant, $C$.

*Example:* Solve $\frac{dy}{dx} = 3x^2 - 5$, given that the curve passes through the point (2, 4).

*Solution:* $y = \int 3x^2 - 5 \, dx = x^3 - 5x + C$

$y = 4$ when $x = 2$

$\Rightarrow \quad 4 = 2^3 - 5 \times 2 + C \quad \Rightarrow \quad C = 6$

$\Rightarrow \quad y = x^3 - 5x + 6.$

# Appendix

## Quadratic equation formula proof

$$ax^2 + bx + c = 0$$

$\Rightarrow \quad ax^2 + bx = -c$

$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$ ÷ by $a$ to make coefficient of $x^2 = +1$

$\Rightarrow \quad x^2 + \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$ add (half coefficient of $x$)$^2$ to both sides

$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ complete square

$\Rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \pm\frac{\sqrt{b^2-4ac}}{2a}$ square root, do not forget $\pm$

$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

## Differentiation from first principles

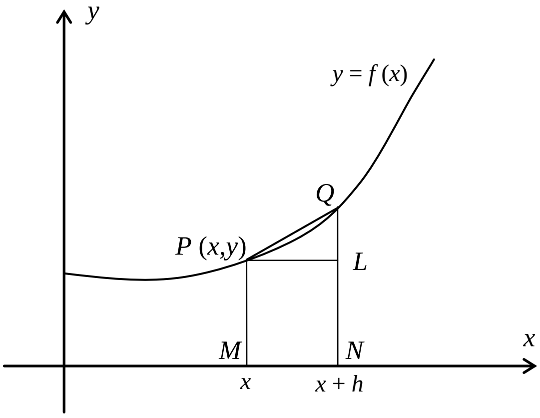
If $P$ and $Q$ are close together, the gradient of $PQ$ will be nearly equal to the gradient of the tangent of the curve at $P$.

Let the $x$-coordinates of $M$ be $x$, and of $N$ be $(x + h)$.

$\Rightarrow \quad PM = f(x)$ and $QN = f(x + h)$

$\Rightarrow \quad QL = QN - PM = f(x + h) - f(x)$

and $PL = h$.




The gradient of $PQ = \frac{\text{increase in } y}{\text{increase in } x} = \frac{\delta y}{\delta x} = \frac{QL}{PL} = \frac{f(x+h) - f(x)}{h}$

As $h \to 0$, $Q$ gets closer and closer to $P$, and the gradient of $PQ$ gets closer and closer to the gradient of the curve at $P$.

We write $f'(x)$ to mean the gradient of the curve (or tangent) at $P$,

$\Rightarrow \quad f'(x) = \underset{h \to 0}{\text{limit}} \frac{f(x+h) - f(x)}{h}$

and we also write the gradient as $\frac{dy}{dx} = \underset{\delta x \to 0}{\text{limit}} \frac{\delta y}{\delta x}$.

*Example:* Find, from first principles, the gradient of $f(x) = x^3$.

*Solution:* $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 \qquad \text{cancelling } h$$

$$\Rightarrow \quad f'(x) = 3x^2 + 0 + 0 \qquad \text{putting } h = 0 \text{ (which we can do after cancelling } h\text{)}$$

$$\Rightarrow \quad f'(x) = 3x^2, \quad \text{or} \quad \frac{dy}{dx} = 3x^2.$$

*Example:* Find, from first principles, the gradient of $f(x) = \frac{5}{x}$.

*Solution:* $f(x) = \frac{5}{x}$

$$f'(x) = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{\frac{5}{x+h} - \frac{5}{x}}{h} = \lim_{h \to 0} \frac{5x - 5(x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{-5h}{hx(x+h)} = \lim_{h \to 0} \frac{-5}{x(x+h)} \qquad \text{cancelling } h$$

$$\Rightarrow \quad f'(x) = \frac{-5}{x(x+0)} \qquad \text{putting } h = 0 \text{ (which we can do after cancelling } h\text{)}$$

$$\Rightarrow \quad f'(x) = \frac{-5}{x^2}, \quad \text{or} \quad \frac{dy}{dx} = \frac{-5}{x^2}.$$

**General formulae**

For any function $f(x)$, the derivative is

$$f'(x) = \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x}\right) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

The second formula comes from taking two points, $Q$ and $Q'$, equally placed on either side of $P$. $Q'$ has $x$-coord $x - h$, and $Q$ has $x$-coord $x + h$.

# Index